

Do elementary $(2, 1)$ -toposes support
an internal model of intensional type
theory?

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Outline

- 1 Motivation
- 2 Additional assumptions
- 3 The model
- 4 Class categories
- 5 Conclusions

Topos theory and type theory

Awodey's conjecture

There is a model of HoTT in any $(\infty, 1)$ -topos
(+ a univalent universe)

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What about truncated versions?

Level 1

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A *subobject classifier* is a monomorphism $\top : \mathbf{1} \rightarrow \Omega$ in $\mathcal{C}$ such that for all $A \in \mathcal{C}$ the pullback map

$$\top^* : \mathcal{C}(A, \Omega) \rightarrow \mathbf{Sub}(A)$$

is an isomorphism.

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Definition (Weber)

A Weber $(2, 1)$ -topos is a cartesian closed $(2, 1)$ -category $\mathcal{K}$ with finite $(2, 1)$ -limits and a discrete opfibration classifier $p : \mathbb{S}_* \rightarrow \mathbb{S}$.

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A *discrete opfibration classifier* is a discrete opfibration $p : \mathbb{S}_* \rightarrow \mathbb{S}$ in $\mathcal{K}$ such that for all $\mathbb{A} \in \mathcal{K}$ the pullback map

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is **fully faithful**.

Example of a Weber $(2, 1)$ -topos

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GPD the $(2, 1)$ -category of large groupoids together with discrete opfibration classifier $\mathcal{U} : \mathbf{Set}_*^{\cong} \rightarrow \mathbf{Set}^{\cong}$

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For any groupoid $\mathbb{A}$, there is an equivalence

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- The types are modelled by the cloven isofibrations
i.e. $F : \mathbb{Y} \rightarrow \mathbb{X}$ such that for any isomorphism
 $\phi : x \cong F(y)$ in $\mathbb{X}$ there is a chosen lift $\psi : y' \cong y$ in $\mathbb{Y}$
such that $F(\psi) = \phi$.

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- The identity types is given by the space of all possible lifts.
- The discrete opfibrations have *unique* lifts, and are therefore 0-types.
- In general, 0-types are given by the *setoidal* opfibrations.
- Identity types are themselves 0-types.

The question

Question: Given a Weber $(2, 1)$ -topos $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S})$, can we mimic this construction so that when $\mathcal{K} = \mathbf{GPD}$, we recover Hofmann and Streicher's model?

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Answer: It seems not without some additional assumptions!

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The importance of discreteness

In Hofmann and Streicher's model, the sets and setoids were the non-dependent 0-types which were classified, so the locally discrete sub- $(2, 1)$ -category

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played an important role.

$X \in \mathbf{GPD}$ is a set iff

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Set}^{\cong}_* \\ \downarrow ! & \lrcorner & \downarrow \mathcal{U} \\ \mathbf{1} & \dashrightarrow_{\exists \chi} & \mathbf{Set}^{\cong} \end{array}$$

The importance of discreteness

Definition

Let $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S})$ be a Weber $(2, 1)$ -topos.

We call a discrete opfibration $F : \mathbb{B} \rightarrow \mathbb{A}$ *p-small* if it is in the essential image of $p^* : \mathcal{K}(\mathbb{A}, \mathbb{S}) \rightarrow \mathbf{Dopfib}(\mathbb{A})$.

We call an object *p-small* and *discrete* if it is in the essential image of $p^* : \mathcal{K}(\mathbf{1}, \mathbb{S}) \rightarrow \mathbf{Dopfib}(\mathbf{1})$.

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We denote the full sub- $(2, 1)$ -category spanned by *p-small discrete* objects by $\mathbf{Disc}(\mathcal{K}_\sigma)$.

Properties of p -small discrete objects

Does **Disc**($\mathcal{K}_\sigma$) have pullbacks?

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We must assume two things about p -small discrete opfibrations.

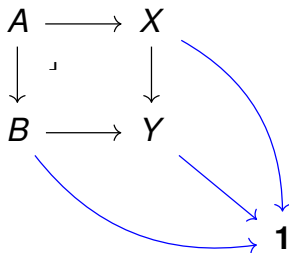
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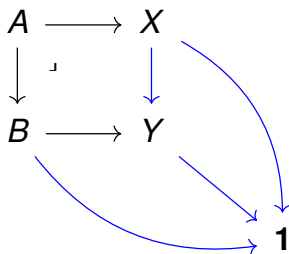


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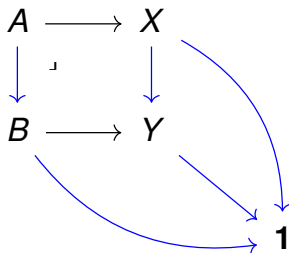


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Definition (Helfer)

We call a Weber $(2, 1)$ -topos *pre-plentiful* if $p : \mathbb{S}_* \rightarrow \mathbb{S}$ satisfies Separation and Replacement.

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$$\begin{array}{ccccc} & \xrightarrow{p_1} & & \xrightarrow{d_1} & \\ A_1 \times_{A_0} A_1 & \xrightarrow{m} & A_1 & \xleftarrow{i} & A_0 \xrightarrow{q} \gg \mathbb{A} \\ & \xrightarrow{p_2} & & \xrightarrow{d_0} & \end{array}$$

such that A_0, A_1 are discrete and $A_1 \cong q \downarrow q$.

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$\mathbb{A}$ is a small groupoid iff A_0 is a set and

$(d_1, d_0) : A_1 \rightarrow A_0 \times A_0$ is a small discrete opfibration.

Small objects

Definition

Let $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S})$ be a pre-plentiful Weber $(2, 1)$ -topos. Call an object $\mathbb{X} \in \mathcal{K}$ *p-small* if there exists a *p*-small discrete object X_0 and codescent morphism $q : X_0 \rightarrow \mathbb{X}$ such that $(d_1, d_0) : q \downarrow q \rightarrow X_0 \times X_0$ is a *p*-small discrete opfibration.

We denote the full sub- $(2, 1)$ -category of small objects by $\mathcal{K}_\sigma$. There is a $(2, 1)$ -functor:

$$\begin{aligned} \mathcal{K}_\sigma &\rightarrow \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma)) \\ \mathbb{X} &\mapsto \left(\dots q \downarrow q \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X_0 \right) \end{aligned}$$

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✓ *p*-small discrete objects are *p*-small.

Theorem

Let $(\mathcal{K}, p : \mathbb{S}_ \rightarrow \mathbb{S})$ be a BO-exact, pre-plentiful Weber $(2, 1)$ -topos that has BO-projective discretes. Then*

$$\mathcal{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$$

Idea: BO-exactness allows us to define a functor

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Example

For $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S}) = (\mathbf{GPD}, \mathcal{U} : \mathbf{Set}_*^{\cong} \rightarrow \mathbf{Set}^{\cong})$,

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Note: pre-plentiful Weber $(2, 1)$ -toposes are not always BO-exact, e.g. $(\mathbf{Set}, \top : \mathbf{1} \rightarrow \{\top, \perp\})$.

Some extra conditions

We will assume on top of this the following:

- 1 We assume $\mathcal{K}$ is extensive.
- 2 We assume that $\mathbf{0}$ and $\mathbf{1} + \mathbf{1}$ are p -small.
- 3 We assume p -small maps are closed under coproducts.

Definition

We call $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S})$ a *classic* $(2, 1)$ -topos if it is a BO-exact, pre-plentiful Weber $(2, 1)$ -topos that has BO-projective discretes and satisfies (1)-(3).

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- Effectively, this amounts to constructing a monad on $\mathcal{K}_\sigma \rightarrow$ whose algebras are exactly the (split/ normal) isofibrations.
- This method allows us to see which extra assumptions on $\mathcal{K}$ give us extra type theoretic properties in the model.

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Let $(\mathcal{K}, p : \mathbb{S}_ \rightarrow \mathbb{S})$ be a classic $(2, 1)$ -topos. Then $\mathcal{K}_\sigma$ forms a model of MLTT with Σ - and intensional Id -types. Moreover, $p : \mathbb{S}_* \rightarrow \mathbb{S}$ provides a universe for 0-types.*

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- that we have a small NNO, then we have a Natural numbers type.
- If we assume extra exactness conditions, we get quotient types.

Examples

- $(\mathbf{GPD}, \mathcal{U} : \mathbf{Set}_*^{\cong} \rightarrow \mathbf{Set}^{\cong})$ is a classic $(2, 1)$ -topos and its associated model of type theory is given by Hofmann and Streicher's groupoid model for **Gpd**.

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- Let $\mathcal{C}$ be a small 1-category. Then $[\mathcal{C}^{\mathrm{op}}, \mathbf{GPD}]$ together with a classifier given by Mesiti is a classic $(2, 1)$ -topos with a novel associated model of MLTT given by $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gpd}]$.

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- Let $(2, 1)$ -category $\mathcal{K}$ be a category of semi-strict stacks together with a classifier given by Mesiti. This is *almost* a classic $(2, 1)$ -topos, but not quite. Regardless, $\mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ is still a sheafy model of MLTT.

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- $\mathbf{Gpd}(\mathbf{Asm}_A)$ is a classic $(2, 1)$ -topos together with a classifier given by Speight.

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Class categories

Let $\mathcal{C}$ be an exact and extensive 1-category and let $\mathcal{S}$ be a collection of maps.

- 1 Replacement.
- 2 Stability.
- 3 $0 \rightarrow \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \rightarrow \mathbf{1}$
belong to $\mathcal{S}$.
- 4 Sums.
- 5 Exponentiality.
- 6 Representability.
- 7 Separation.
- 8 Small NNO.

Any equality is in $\mathcal{S}$ and $\mathcal{S}$
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In any pullback square

$$\begin{array}{ccc} A & \longrightarrow & X \\ G \downarrow & \lrcorner & \downarrow F \\ B & \longrightarrow & Y \end{array}$$

If $F \in \mathcal{S}$ then $G \in \mathcal{S}$.

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- 2 Stability.
- 3 $0 \rightarrow 1$ and $1 + 1 \rightarrow 1$
belong to $\mathcal{S}$.
- 4 Sums.
- 5 Exponentiality.
- 6 Representability.
- 7 Separation.
- 8 Small NNO.

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If $X \rightarrow Y$ and $X' \rightarrow Y'$ belong to $\mathcal{S}$ then so does $X + X' \rightarrow Y + Y'$.

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There exists a classifier for maps in $\mathcal{S}$, which is itself in $\mathcal{S}$.

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Any monomorphism is in $\mathcal{S}$.

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Class categories

Let $\mathcal{C}$ be an exact and extensive 1-category and let $\mathcal{I}$ be a collection of maps.

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- 3 $0 \rightarrow \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \rightarrow \mathbf{1}$ belong to $\mathcal{I}$.
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Definition (Joyal-Moerdijk)

If $(\mathcal{C}, \mathcal{I})$ satisfies 1-8, we call it a *class category*.

Examples of class categories

- **Class** together with the class of maps with set-sized fibres. The small objects are sets.

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- For $\mathcal{C}$ a small 1-category, $[\mathcal{C}^{\text{op}}, \mathbf{Class}]$ together with the class of maps that pointwise have set-sized fibres. The small objects are set-valued presheaves.
- Categories of sheaves together with maps that pointwise have set-sized fibres. The small objects are the small sheaves.

Class $(2, 1)$ -categories

Let $\mathcal{K}$ be a BO-exact and extensive $(2, 1)$ -category and let $\mathcal{S}$ be a collection of discrete opfibrations.

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Definition

If $(\mathcal{K}, \mathcal{S})$ satisfies 1-8, we call it a *class $(2, 1)$ -category*.

Theorem

Any class $(2, 1)$ -category supports an internal model of intensional type theory, and has a universe for 0-types.

Examples

Example

Let $(\mathcal{K}, p : \mathbb{S}_* \rightarrow \mathbb{S})$ be a classic $(2, 1)$ -topos. Let $\mathcal{I}$ to be the collection of p -small maps. Then $(\mathcal{K}, \mathcal{I})$ is a class $(2, 1)$ -category.

Outline

- 1 Motivation
- 2 Additional assumptions
- 3 The model
- 4 Class categories
- 5 Conclusions**

Final thoughts

- We gave a definition of classic $(2, 1)$ -topos that satisfied the conditions to solve a 2-truncated version of Awodey's conjecture.

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- These conditions might be important to the $(\infty, 1)$ -dimensional case.
- The universes of higher toposes force you to think about size issues.
- This makes the setting of class categories perhaps more suitable as a notion to categorify.

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