Do elementary $(2,1)$-toposes support an internal model of intensional type theory?

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ASSUME 30/07/2026

Outline

Topos theory and type theory

Awodey's conjecture

There is a model of HoTT in any $(\infty, 1)$-topos
(+ a univalent universe)

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What about truncated versions?

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A *subobject classifier* is a monomorphism $\top : \mathbf{1} \to \Omega$ in $\mathscr{C}$ such that for all $A \in \mathscr{C}$ the pullback map

$$\top^* : \mathscr{C}(A, \Omega) \to \mathbf{Sub}(A)$$

is an isomorphism.

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Definition (Weber)

A Weber $(2,1)$-topos is a cartesian closed $(2,1)$-category $\mathscr{K}$ with finite $(2,1)$-limits and a discrete opfibration classifier $p : \mathbb{S}_* \to \mathbb{S}$.

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A *discrete opfibration classifier* is a discrete opfibration $p : \mathbb{S}_* \to \mathbb{S}$ in $\mathscr{K}$ such that for all $\mathbb{A} \in \mathscr{K}$ the pullback map

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is fully faithful.

Example of a Weber $(2,1)$-topos

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GPD the $(2,1)$-category of large groupoids together with discrete opfibration classifier $\mathscr{U} : \mathbf{Set}_*^{\cong} \to \mathbf{Set}^{\cong}$

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- The types are modelled by the cloven isofibrations i.e. $F : \mathbb{Y} \to \mathbb{X}$ such that for any isomorphism $\phi : x \cong F(y)$ in $\mathbb{X}$ there is a chosen lift $\psi : y' \cong y$ in $\mathbb{Y}$ such that $F(\psi) = \phi$.

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- The identity types is given by the space of all possible lifts.
- The discrete opfibrations have *unique* lifts, and are therefore 0-types.
- In general, 0-types are given by the *setoidal* opfibrations.
- Identity types are themselves 0-types.

The question

Question: Given a Weber $(2, 1)$-topos $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$, can we mimic this construction so that when $\mathscr{K} = \mathbf{GPD}$, we recover Hofmann and Streicher's model?

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Question: Given a Weber $(2,1)$-topos $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$, can we mimic this construction so that when $\mathscr{K} = \mathbf{GPD}$, we recover Hofmann and Streicher's model?
Answer: It seems not without some additional assumptions!

Outline

The importance of discreteness

In Hofmann and Streicher's model, the sets and setoids were the non-dependent 0-types which were classified, so the locally discrete sub-$(2,1)$-category

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played an important role.
$X \in \mathbf{GPD}$ is a set iff

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_*^{\cong} \arrow[d, "\mathcal{U}"] \\
\mathbf{1} \arrow[r, dashed, "\exists \chi"'] & \mathbf{Set}^{\cong}
\end{tikzcd}$$

The importance of discreteness

Definition

Let $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ be a Weber $(2,1)$-topos.
We call a discrete opfibration $F : \mathbb{B} \to \mathbb{A}$ *p-small* if it is in the essential image of $p^* : \mathscr{K}(\mathbb{A}, \mathbb{S}) \to \mathbf{Dopfib}(\mathbb{A})$.
We call an object p-small and *discrete* if it is in the essential image of $p^* : \mathscr{K}(\mathbf{1}, \mathbb{S}) \to \mathbf{Dopfib}(\mathbf{1})$.

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We denote the full sub-$(2,1)$-category spanned by p-small discrete objects by $\mathbf{Disc}(\mathscr{K}_\sigma)$.

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Definition (Helfer)

We call a Weber $(2,1)$-topos *pre-plentiful* if $p : \mathbb{S}_* \to \mathbb{S}$ satisfies Seperation and Replacement.

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GPD has the property that it is exact in a 2-dimensional sense and also has *enough discretes*: For all $\mathbb{A} \in$ **GPD** there is a codescent diagram:

$$A_1 \times_{A_0} A_1 \;\substack{\xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow[p_2]{}}\; A_1 \;\substack{\xrightarrow{d_1} \\ \xleftarrow{i} \\ \xrightarrow[d_0]{}}\; A_0 \xrightarrow{q} \mathbb{A}$$

such that A_0, A_1 are discrete and $A_1 \cong q \downarrow q$.

Beyond discretes

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such that A_0, A_1 are discrete and $A_1 \cong q \downarrow q$.
$\mathbb{A}$ is a small groupoid iff A_0 is a set and
$(d_1, d_0) : A_1 \to A_0 \times A_0$ is a small discrete opfibration.

Small objects

Definition

Let $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ be a pre-plentiful Weber $(2,1)$-topos. Call an object $\mathbb{X} \in \mathscr{K}$ *p-small* if there exists a p-small discrete object X_0 and codescent morphism $q : X_0 \to \mathbb{X}$ such that $(d_1, d_0) : q \downarrow q \to X_0 \times X_0$ is a p-small discrete opfibration.

We denote the full sub-$(2,1)$-category of small objects by $\mathscr{K}_\sigma$. There is a $(2,1)$-functor:

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\begin{aligned}
\mathscr{K}_\sigma &\to \mathbf{Gpd}(\mathbf{Disc}(\mathscr{K}_\sigma)) \\
\mathbb{X} &\mapsto \left(\ \ldots q \downarrow q \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \ X_0 \ \right)
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✓ p-small discrete objects are p-small.

Small objects

Theorem

Let $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ be a BO-exact, pre-plentiful Weber $(2,1)$-topos that has BO-projective discretes. Then

$$\mathscr{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathscr{K}_\sigma))$$

Idea: BO-exactness allows us to define a functor

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Example

For $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S}) = (\mathbf{GPD}, \mathscr{U} : \mathbf{Set}_*^{\cong} \to \mathbf{Set}^{\cong})$,

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Note: pre-plentiful Weber $(2,1)$-toposes are not always BO-exact, e.g. $(\mathbf{Set}, \top : \mathbf{1} \to \{\top, \bot\})$.

Some extra conditions

We will assume on top of this the following:

1. We assume $\mathscr{K}$ is extensive.
2. We assume that $\mathbf{0}$ and $\mathbf{1}+\mathbf{1}$ are p-small.
3. We assume p-small maps are closed under coproducts.

Definition

We call $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ a *classic* $(2,1)$-topos if it is a BO-exact, pre-plentiful Weber $(2,1)$-topos that has BO-projective discretes and satisfies (1)-(3).

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The intended model

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- Effectively, this amounts to constructing a monad on $\mathscr{K}_\sigma^{\to}$ whose algebras are exactly the (split/ normal) isofibrations.
- This method allows us to see which extra assumptions on $\mathscr{K}$ give us extra type theoretic properties in the model.

Main Theorem

Theorem

Let $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ *be a classic* $(2,1)$*-topos. Then* $\mathscr{K}_\sigma$ *forms a model of MLTT with* Σ*- and intensional* $\mathtt{Id}$*-types. Moreover,* $p : \mathbb{S}_* \to \mathbb{S}$ *provides a universe for* 0*-types.*

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If we assume additionally:

- that $p : \mathbb{S}_* \to \mathbb{S}$ is exponentiable, then we have Π-types.
- that we have a small NNO, then we have a Natural numbers type.
- If we assume extra exactness conditions, we get quotient types.

Examples

- $(\mathbf{GPD}, \mathscr{U} : \mathbf{Set}_*^{\cong} \to \mathbf{Set}^{\cong})$ is a classic $(2,1)$-topos and its associated model of type theory is given by Hofmann and Streicher's groupoid model for **Gpd**.

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- Let $\mathscr{C}$ be a small 1-category. Then $[\mathscr{C}^{\mathrm{op}}, \mathbf{GPD}]$ together with a classifier given by Mesiti is a classic $(2,1)$-topos with a novel associated model of MLTT given by $[\mathscr{C}^{\mathrm{op}}, \mathbf{Gpd}]$.

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- Let $(2,1)$-category $\mathscr{K}$ be a category of semi-strict stacks together with a classifier given by Mesiti. This is *almost* a classic $(2,1)$-topos, but not quite. Regardless, $\mathbf{Gpd}(\mathbf{Disc}(\mathscr{K}_\sigma))$ is still a sheafy model of MLTT.

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- $\mathbf{Gpd}(\mathbf{Asm}_{\mathsf{A}})$ is a classic $(2,1)$-topos together with a classifier given by Speight.

Outline

Class categories

Let $\mathscr{C}$ be an exact and extensive 1-category and let $\mathscr{S}$ be a collection of maps.

1. Replacement.
2. Stability.
3. $0 \to \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \to \mathbf{1}$ belong to $\mathscr{S}$.
4. Sums.
5. Exponentiality.
6. Representability.
7. Separation.
8. Small NNO.

Any equality is in $\mathscr{S}$ and $\mathscr{S}$ is closed under composition.

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In any pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "G"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "F"] \\ B \arrow[r] & Y \end{tikzcd}$$

If $F \in \mathscr{S}$ then $G \in \mathscr{S}$.

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Class categories

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If $X \to Y$ and $X' \to Y'$ belong to $\mathscr{S}$ then so does $X + X' \to Y + Y'$.

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There exists a classifier for maps in $\mathscr{S}$, which is itself in $\mathscr{S}$.

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Any monomorphism is in $\mathscr{S}$.

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Definition (Joyal-Moerdijk)

If $(\mathscr{C}, \mathscr{S})$ satisfies 1-8, we call it a *class category.*

Examples of class categories

- **Class** together with the class of maps with set-sized fibres. The small objects are sets.

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- For $\mathscr{C}$ a small 1-category, $[\mathscr{C}^{\mathsf{op}}, \textbf{Class}]$ together with the class of maps that pointwise have set-sized fibres. The small objects are set-valued presheaves.
- Categories of sheaves together with maps that pointwise have set-sized fibres. The small objects are the small sheaves.

Class $(2,1)$-categories

Let $\mathscr{K}$ be a BO-exact and extensive $(2,1)$-category and let $\mathscr{S}$ be a collection of discrete opfibrations.

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Definition

If $(\mathscr{K}, \mathscr{S})$ satisfies 1-8, we call it a *class* $(2,1)$*-category.*

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Definition

If $(\mathscr{K}, \mathscr{S})$ satisfies 1-8, we call it a *class* $(2,1)$*-category.*

Theorem

Any class $(2,1)$*-category supports an internal model of intensional type theory, and has a universe for* 0*-types.*

Examples

Example

Let $(\mathscr{K}, p : \mathbb{S}_* \to \mathbb{S})$ be a classic $(2,1)$-topos. Let $\mathscr{S}$ to be the collection of p-small maps. Then $(\mathscr{K}, \mathscr{S})$ is a class $(2,1)$-category.

Outline

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- The universes of higher toposes force you to think about size issues.
- This makes the setting of class categories perhaps more suitable as a notion to categorify.

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