Size in 2-dimensional elementary topos theory

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Outline

Elementary topos theory

Definition (Lawvere-Tierney)

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A *subobject classifier* is a monomorphism $\top : \mathbf{1} \rightarrowtail \Omega$ such that for any other monomorphism $i : A \rightarrowtail B$ there exists a unique $\chi_i : B \to \Omega$ such that the following diagram is a pullback.

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ {\scriptstyle \forall i}\downarrow & & \downarrow{\scriptstyle \top} \\ B & \underset{\exists ! \chi_i}{\longrightarrow} & \Omega \end{array}$$

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$$\mathcal{E}(B, \Omega) \xrightarrow{\cong} \mathrm{Sub}(B)$$

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Key example: **Set**

Elementary topos theory

Definition (Weber)

An *elementary* $(2,1)$*-topos* is a cartesian closed $(2,1)$-category $\mathscr{K}$ with finite limits and a discrete opfibration classifier.

A *discrete opfibration classifier* is a discrete opfibration $p : S_* \rightarrowtail S$ such that for all $X \in \mathscr{K}$, pulling back along $p : S_* \rightarrowtail S$ is fully faithful:

$$\mathscr{K}(X, S) \xrightarrow{\text{f.f.}} \text{DopFib}(X)$$

Example of a Weber $(2,1)$-topos

GPD the $(2,1)$-category of large groupoids

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$\mathbf{GPD}_\mu$ the $(2, 1)$-category of μ-small groupoids for some $\mu > \lambda$

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Gpd the $(2,1)$-category of small groupoids:

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- cartesian closed ✓
- discrete opfib classifier ✓(i.e. $\top \to \{\bot, \top\}$)

Logic in an elementary topos

	1-cats	$(2,1)$-cats
Object	elementary topos	
internal logic	0 dimensional MLTT	
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Von Neumann-Bernays-Gödel class theory:

Class has:

- objects: $\{x \text{ a set} : \phi(x) \text{ is true for } \phi \text{ a formula in FOL}\}$
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We define **GPD** $:=$ **Gpd**(**Class**).

Class categories

	1-cats	$(2,1)$-cats
Object	class categories	
internal logic	small obs: 0D MLTT	
Key example	**Class**	

Definition (Joyal-Moerdijk)

A *class category* is a pair $(\mathscr{C}, \mathscr{S})$ of a category and a class of maps satisfying some axioms.

Class categories

	1-cats	$(2,1)$-cats
Object	class categories	class $(2,1)$-categories
internal logic	small obs: 0D MLTT	small obs: 1D MLTT
Key example	**Class**	**GPD**

Theorem

Let $(\mathscr{K}, \mathscr{S})$ be a class $(2,1)$-category. Then the small objects form a model of 1-dimensional MLTT.

Outline

Cateads

Definition (Bourne-Penon, Bourke)

For a $(2,1)$-category $\mathscr{K}$, a *catead* is

$$C_1 \times_{C_0} C_1 \;\begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\text{comp}} \\ \xrightarrow[p_2]{} \end{array}\; C_1 \;\begin{array}{c} \xrightarrow{\text{src}} \\ \xleftarrow{\text{id}} \\ \xrightarrow[\text{tgt}]{} \end{array}\; C_0$$

such that $(\text{srs}, \text{tgt}) : C_1 \to C_0 \times C_0$ is a discrete opfibration. We call its 2-colimit a *codescent object*.

Codescent morphisms are a $(2,1)$-dimensional analogue of a regular epimorphism in a 1-category.

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  is a catead made of discrete objects. It's codescent object is $\mathbb{X}$.
- In **Gpd**, the codescent morphisms are precisely the bijective-on-objects functors.

Exactness

Given $f : X \to Y$

$$
f \downarrow f \downarrow f \;\; \underset{p_2}{\overset{\overset{p_1}{\longrightarrow}}{\underset{\longrightarrow}{\xrightarrow{m}}}} \;\; f \downarrow f \;\; \underset{d_0}{\overset{\overset{d_1}{\longrightarrow}}{\underset{\longrightarrow}{\xleftarrow{i}}}} \;\; X \xrightarrow{q} \!\!\!\!\rightarrow C
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Definition (Bourke-Garner)

A $(2, 1)$-category with finite limits is called *BO-regular* when codescent objects of cateads exist and codescent morphisms are effective and stable under pullback, and whenever $f : A \twoheadrightarrow B$ is a codescent morphism, so is $\delta_f : A \to A \times_B A$.
It is called *BO-exact* if cateads are effective.

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- **GPD** is BO-exact.
- **Set** thought of as a locally discrete $(2, 1)$-category is not BO-exact (the diagonal of a regular epimorphism is rarely a regular epimorphism).

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$Y \in \mathscr{K}$ is discrete if

$$\mathbb{X} \underset{g}{\overset{f}{\rightrightarrows}} Y \ (\phi : f \Rightarrow g) \quad \Longrightarrow \quad f = g \text{ and } \phi = 1_f$$

Y is called projective if

$$\begin{array}{ccc} & & \mathbb{A} \\ & \overset{\exists !}{\nearrow} & \downarrow\!\!\downarrow \forall \\ Y & \longrightarrow & \mathbb{B} \end{array}$$

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$$P \twoheadrightarrow \mathbb{Y}$$

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Theorem (Carboni-Vitale)

An exact 1-category is an exact completion if and only if it has enough projectives. In this case, it is the exact completion of its projective objects.

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Theorem (Bourke-Garner)

- $\mathscr{E} \mapsto \mathbf{Gpd}(\mathscr{E})$ *is the BO-exact completion of* $\mathscr{E}$.
- *An BO-exact* $(2,1)$*-category is a BO-exact completion of a* 1*-category if and only if it has enough projectives and discrete objects are projective. In this case, it is the BO-exact completion of its discrete objects.*

i.e. $\mathscr{K}$ satisfies (1)-(2) $\iff \mathscr{K} \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathscr{K}))$

Pre-class structure

Let $\mathscr{K}$ be a lextensive $(2,1)$-category and $\mathscr{S}$ a class of discrete opfibrations. We call $(\mathscr{K}, \mathscr{S})$ a *pre-class* $(2,1)$*-category.*

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We call a discrete object $X \in \mathbf{Disc}(\mathscr{K})$ *small* if $! : X \to \mathbf{1}$ is in $\mathscr{S}$.
We call a general object $\mathbb{X} \in \mathscr{K}$ *small* if there exists a small discrete object and a codescent morphism $q : X \twoheadrightarrow \mathbb{X}$, such that $(s, t) : q \downarrow q \to X \times X$ is in $\mathscr{S}$.

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Define the full sub-$(2,1)$-category of small objects by $\mathscr{K}_{\text{small}}$.

Axioms

Let $(\mathscr{K}, \mathscr{S})$ be a pre-class $(2,1)$-category. Consider:

1. Replacement.
2. Stability.
3. $0 \to \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \to \mathbf{1}$ belong to $\mathscr{S}$.
4. Sums.
5. Exponentiality.
6. Representability.
7. Cancellability.
8. Small NNO.
9. Small BO-exactness.
10. Projectivity of small discrete objects.

Any isomorphism is in $\mathscr{S}$ and $\mathscr{S}$ is closed under composition.

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In any strict $(2, 1)$-pullback square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle G}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle F} \\ B & \longrightarrow & Y \end{array}$$

If $F \in \mathscr{S}$ then $G \in \mathscr{S}$.

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If $X \to Y$ and $X' \to Y'$ belong to $\mathscr{S}$ then so does $X + X' \to Y + Y'$.

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Every map in $\mathscr{S}$ is exponentiable

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There is a small discrete opfibration classifier $p : S_* \rightarrowtail S$:

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Let $F : X \to Y$ and $G : Y \to Z$ be discrete opfibrations. If $GF \in \mathscr{S}$ then $F \in \mathscr{S}$.

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Definition

If $(\mathscr{K}, \mathscr{S})$ satisfies 1-10, we call it a *class* $(2, 1)$*-category.*

Examples

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$[\mathscr{A}^{\text{op}}, \mathbf{GPD}]$	$[\mathscr{A}^{\text{op}}, \mathbf{Gpd}]$	pointwise **Set**-sized
$\mathscr{K}$ a stack	$\mathscr{K}_{\text{small}}$ a small stack	as above

Examples II

Groupoids internal to Zwanziger's stratified topoi:

$$\mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \ldots$$

Then
Gpd$(\mathcal{E}_1)$ is a class $(2,1)$-category with $\mathscr{S}$ consisting of those discrete opfibrations that came from $\mathcal{E}_0$.

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This abstracts taking groupoids in $\mathbf{Set}_\lambda \hookrightarrow \mathbf{Set}_\mu$.

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- there exists a small object $\mathbf{2} \odot \mathbf{1}$.
- $\mathscr{K}_{\text{small}}$ is extensive.

Properties

Let $(\mathscr{K}, \mathscr{S})$ be a class $(2,1)$-category.

- for $\mathbb{X} \in \mathscr{K}_{\text{small}}$, $[\mathbb{X}, \mathbb{Y}]$ exists.
- In particular, for $\mathbb{X}$ small, we have presheaves $[\mathbb{X}, S]$
- There is a Yoneda lemma for $\mathscr{K}$ (cf. (Weber, Street)).
- $\mathscr{K}_{\text{small}}$ is cartesian closed.
- there exists a small object $\mathbf{2} \odot \mathbf{1}$.
- $\mathscr{K}_{\text{small}}$ is extensive.
- $\mathscr{K}_{\text{small}}$ is finitely (co)complete.

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Theorem

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The $(2,1)$-equivalence of this follows from the characterisation of $(2,1)$-categories of the form **Gpd**$(\mathscr{E})$.

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Theorem

$\mathscr{K}_{small}$ is a model of 1-dimensional MLTT.

The proof is an internal version of Hofmann and Streicher's Groupoid model of type theory.

Outline

Extra axioms

Let $(\mathscr{K}, \mathscr{S})$ be a class $(2, 1)$-category.

Theorem (cf. Helfer)

Let $(\mathscr{K}, \mathscr{S})$ be a class $(2, 1)$-category with one extra axiom. Then the classifying object S is an internal 1-topos.

Or in a different direction:

Theorem

Let $(\mathscr{K}, \mathscr{S})$ be a class $(2, 1)$-category with three extra axioms. Then $\mathbf{Disc}(\mathscr{K}_{small})$ is a model of the elementary theory of the category of sets and so $\mathscr{K}$ has the same logical power as ZFC.

Further work

- In one dimensions, class categories give a way to see ZF-sets as the algebras for a polynomial endofunctor. Can we do this in 2-dimensions?

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- In one dimensions, class categories give a way to see ZF-sets as the algebras for a polynomial endofunctor. Can we do this in 2-dimensions?
- Properties of $\mathbf{Gpd}(\mathbf{PAsm}_A)$— an $\mathscr{F}_{\mathrm{BO}}$-effective topos?

Further work

- In one dimensions, class categories give a way to see ZF-sets as the algebras for a polynomial endofunctor. Can we do this in 2-dimensions?
- Properties of $\mathbf{Gpd}(\mathbf{PAsm}_A)$— an $\mathscr{F}_{\mathrm{BO}}$-effective topos?
- Bicategorical version... $(\mathscr{K}, \mathscr{S})$ where $\mathscr{K}$ a bicategory and $\mathscr{S}$ is a class of setoidal opfibrations.

Summary

	1-cats	$(2,1)$-cats
Object	class categories	class $(2,1)$-categories
internal logic	small obs: 0D MLTT	small obs: 1D MLTT
Key example	**Class**	**GPD**

Adding axioms to a class $(2,1)$-category, we can give a $(2,1)$-categorical description of a logic which is as powerful as ZFC.

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