The Dold-Kan Correspondence

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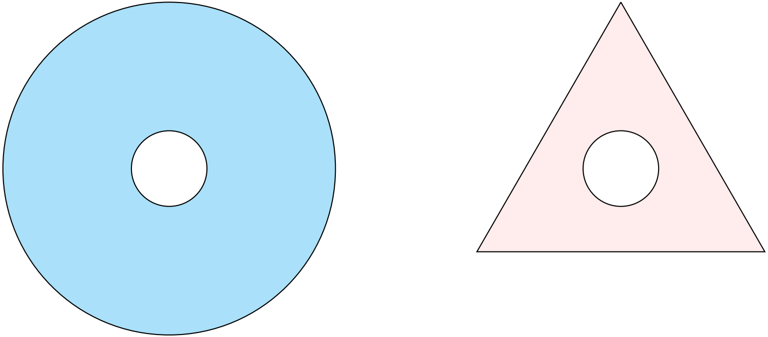
Topological Structures

Are these two shapes the *same*?



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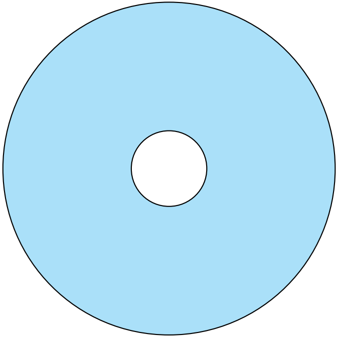

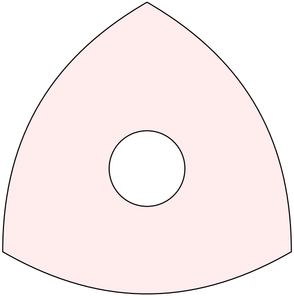

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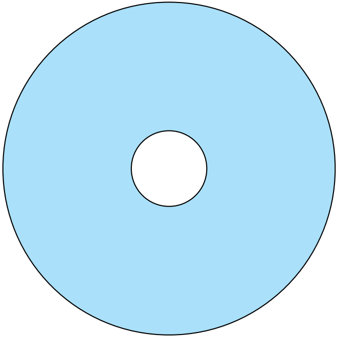
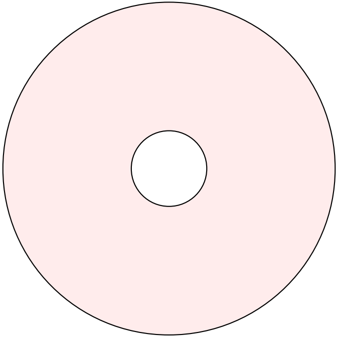
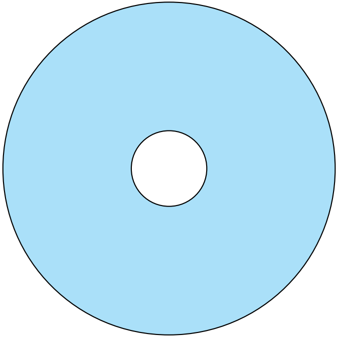
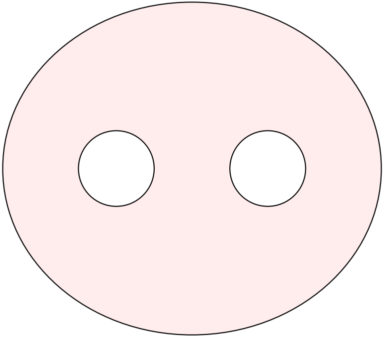




Topological Structures

How about these two?





Topological Structures

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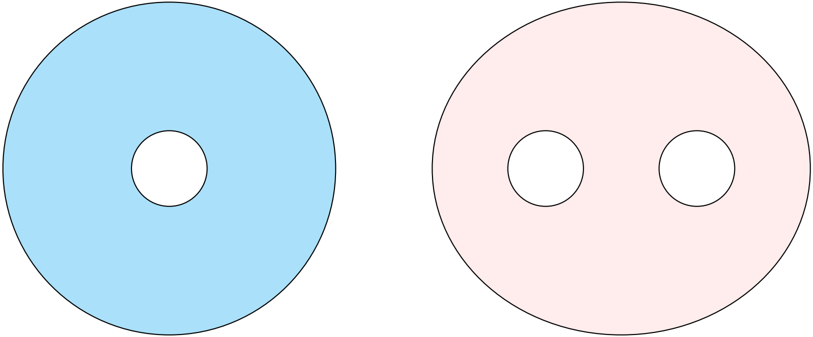
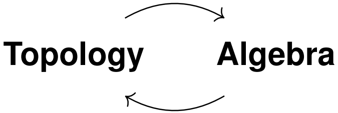


How would we go about *proving* that they are not?

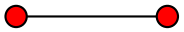
Algebraic Topology

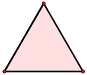
The aim of algebraic topology is to translate topological questions into algebraic ones.



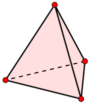
Building Blocks

We use simplices as ‘building blocks’ . These can be thought of as *n*-dimensional triangles.

- A 0-simplex is a point. We call this Δ^0. 
- The 1-simplex is a line. We call this Δ^1. 
- The 2-simplex is a triangle. We call this Δ^2.



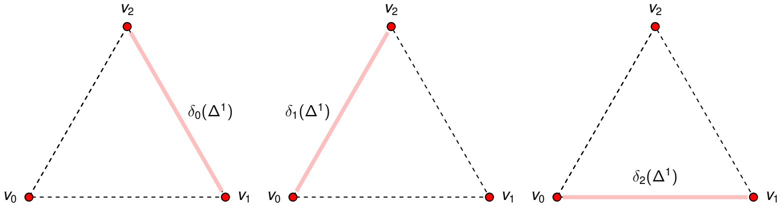
- The 3-simplex is a tetrahedron. We call this Δ^3.

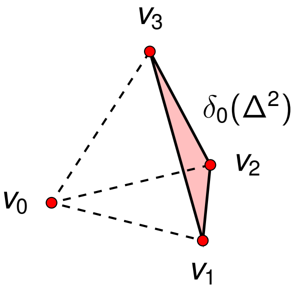


- and so on up to higher dimensions...

Face maps

We can consider maps onto the faces of simplices. These maps help us "glue" the shape back together.

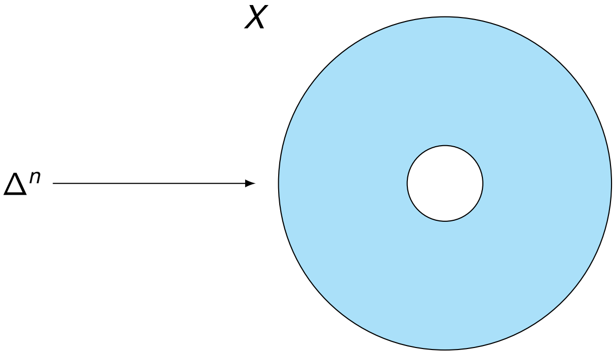




Singular Complex

Let X be a topological space.
Consider the set:

$$S_nX = \{u : \Delta^n \to X : \ u \text{ is continuous}\}$$



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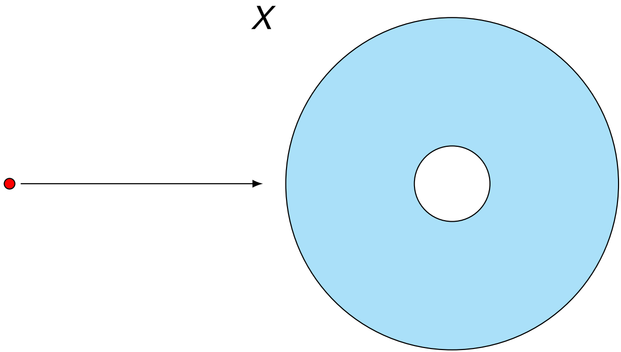

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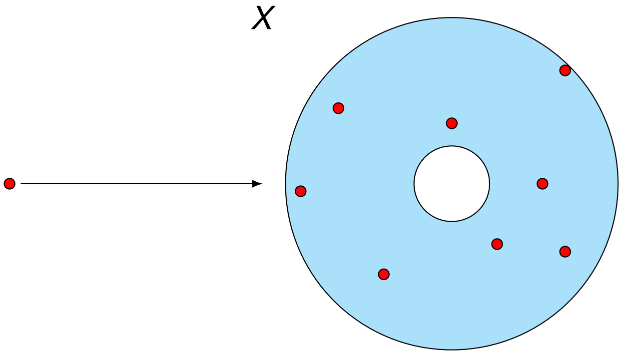

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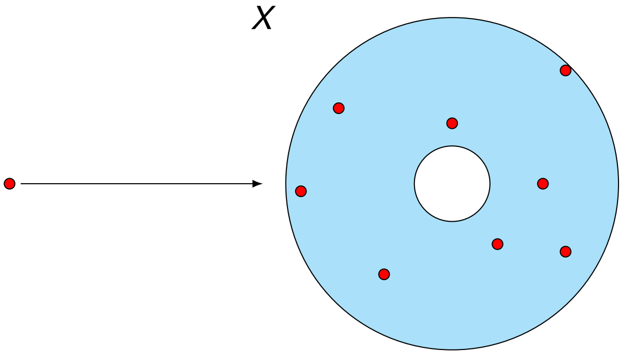

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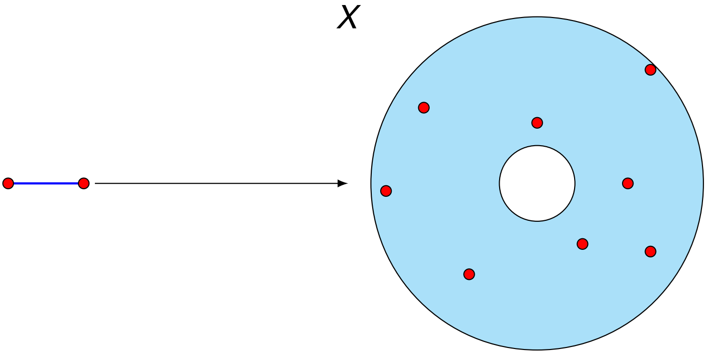

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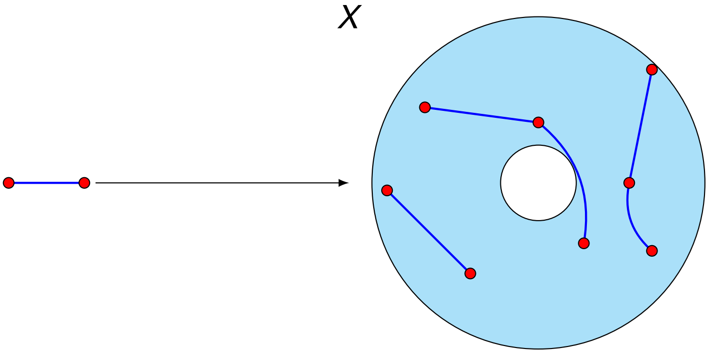

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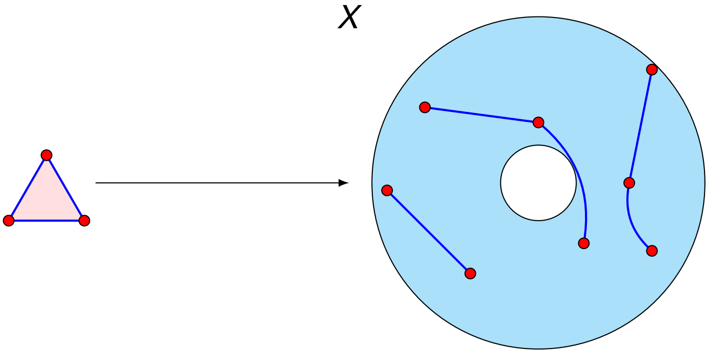
Consider the set:

$$S_2X = \{u : \Delta^2 \to X : \ u \text{ is continuous}\}$$



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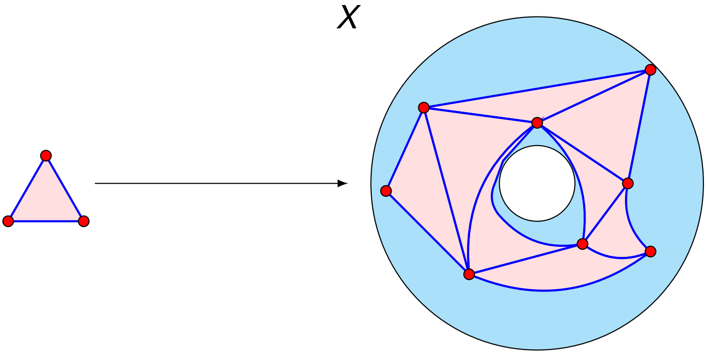

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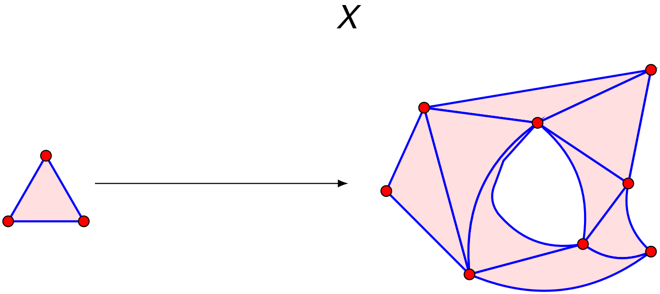

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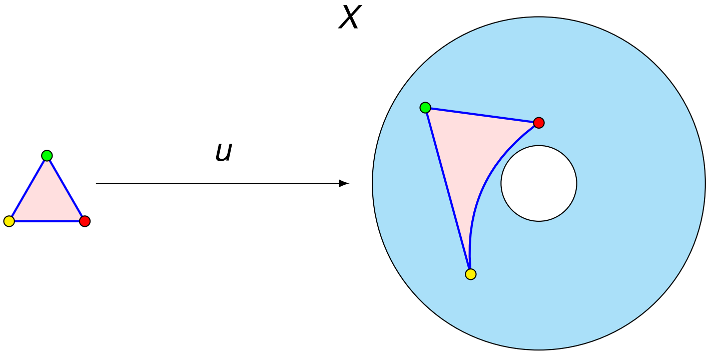

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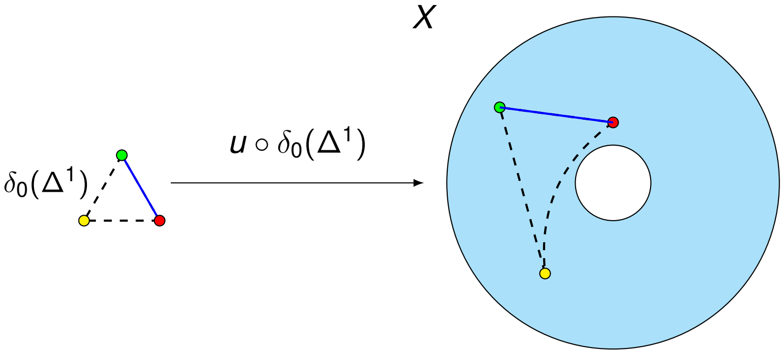

The translation

$$\left\{ \begin{array}{c} \text{Topological} \\ \text{spaces} \end{array} \right\} \underset{|\_|}{\overset{S_*}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Simplicial} \\ \text{Sets} \end{array} \right\}$$

$$X \xmapsto{\quad S_* \quad} S_* X$$

$$u_1, u_2, u_3 \in S_2 X$$

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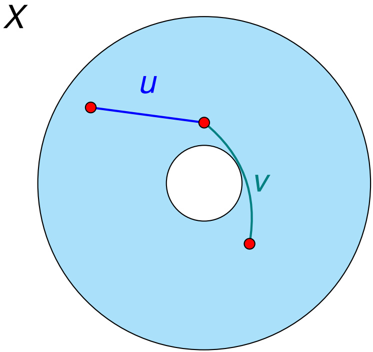
$$\left\{ \begin{array}{c} \text{Topological} \\ \text{spaces} \end{array} \right\} \underset{|\text{-}|}{\overset{S_*}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Simplicial} \\ \text{Sets} \end{array} \right\} \xrightarrow{\mathbb{Z}} \left\{ \begin{array}{c} \text{Simplicial} \\ \text{abelian} \\ \text{groups} \end{array} \right\}$$

$$X \xmapsto{\quad S_* \quad} S_*X \xrightarrow{\quad \mathbb{Z} \quad} \mathbb{Z}S_*X$$

$$u_1, u_2, u_3 \in S_2X$$

$$3u_1 - 4u_2 + 15u_3 \in \mathbb{Z}S_2X$$

Why?



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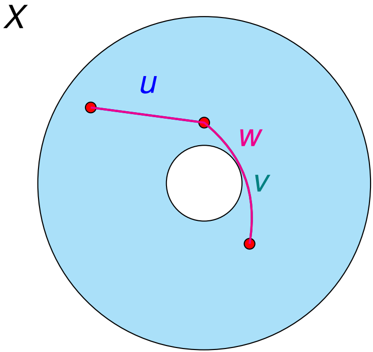

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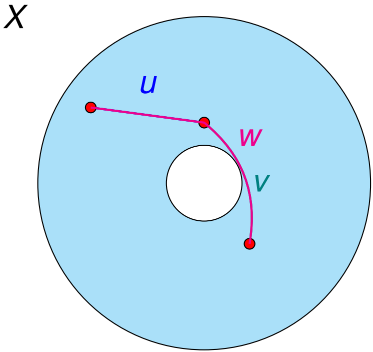

We would like to be able to say something like $u + v = w$.

The Goal of My Project

$$
\begin{array}{c}
\left\{ \begin{array}{c} \text{Topological} \\ \text{spaces} \end{array} \right\} \\
|\_| \uparrow \qquad \downarrow S_* \\
\left\{ \begin{array}{c} \text{Simplicial} \\ \text{Sets} \end{array} \right\} \\
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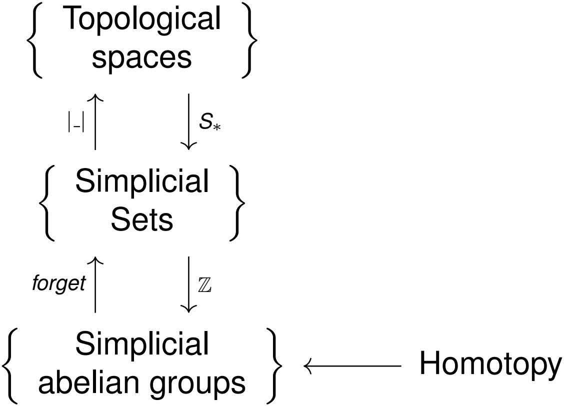

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$$\left\{ \begin{array}{c} \text{Simplicial} \\ \text{abelian groups} \end{array} \right\} \longleftarrow \text{Homotopy}$$

$$K \uparrow \qquad \downarrow N$$

$$\left\{ \begin{array}{c} \text{Non-negatively graded} \\ \text{chain complexes} \end{array} \right\}$$

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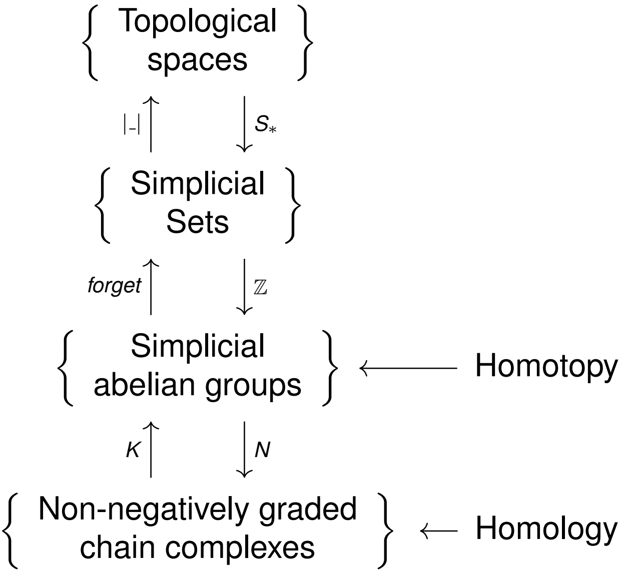

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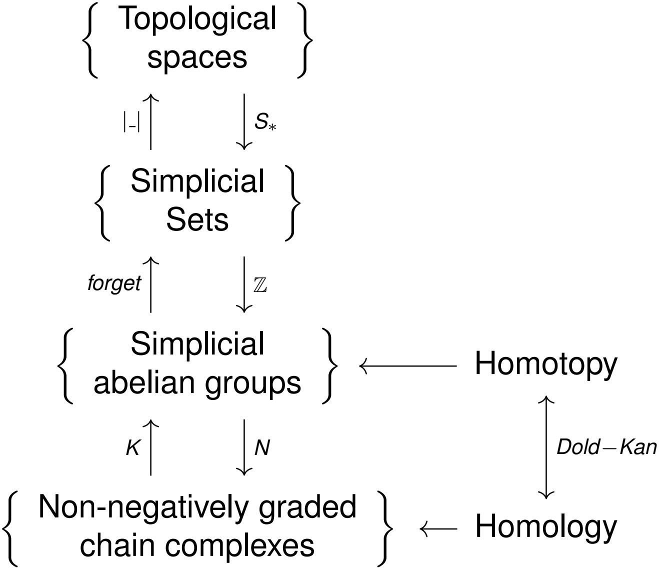

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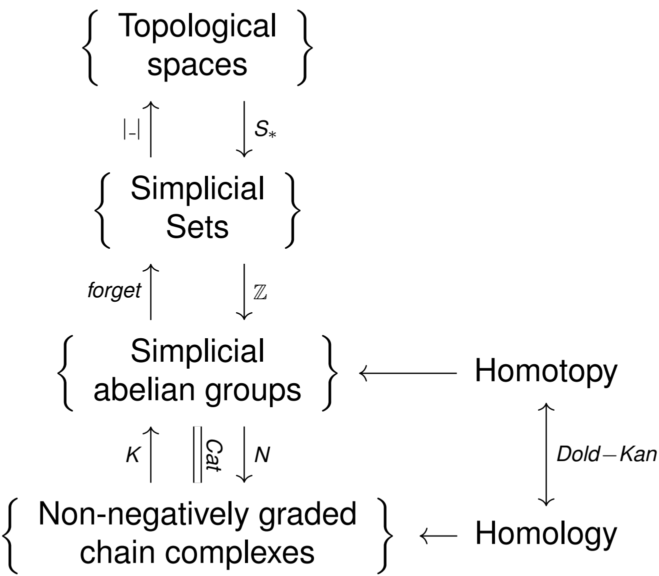

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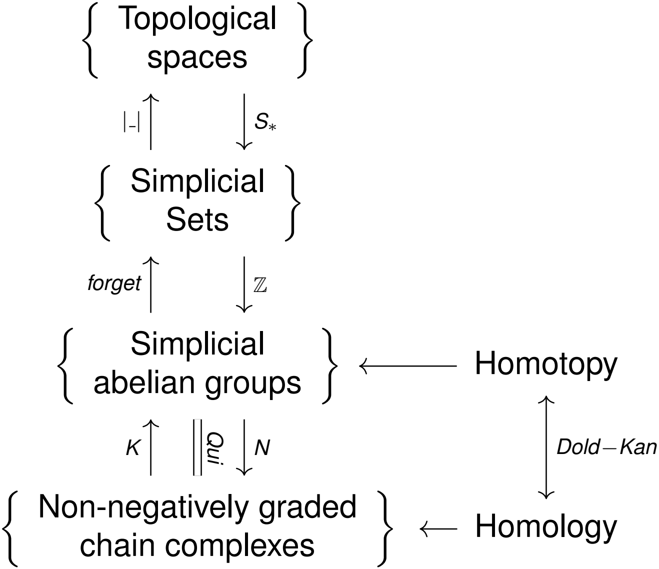

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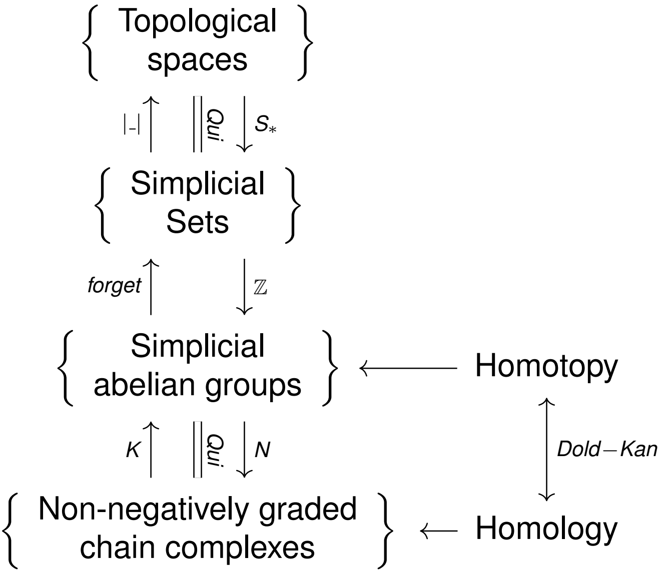
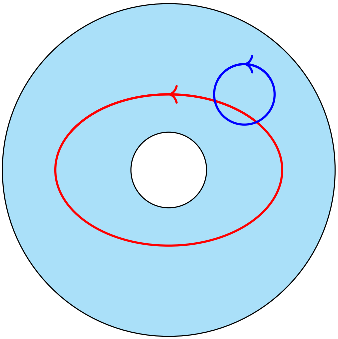


Homology



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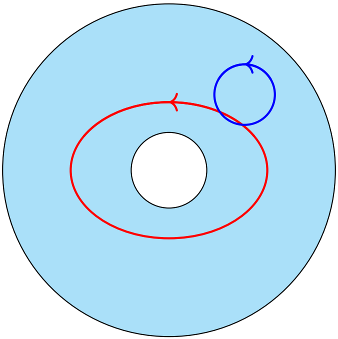

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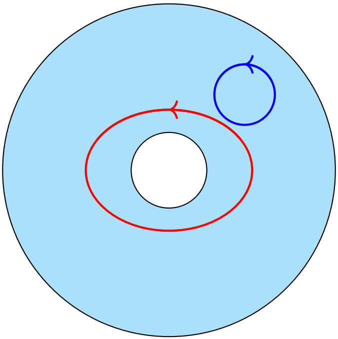

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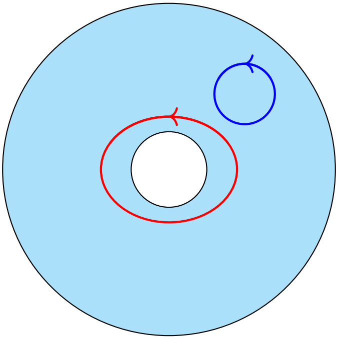

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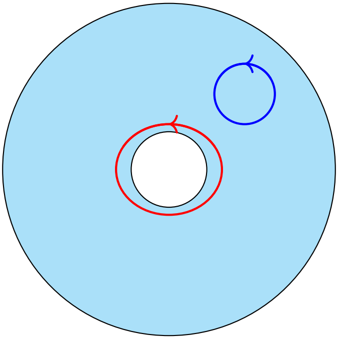

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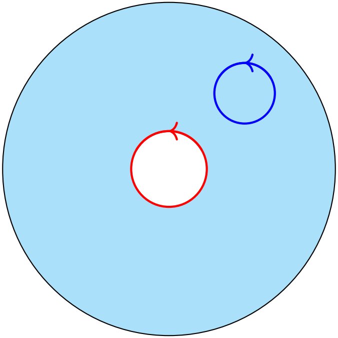

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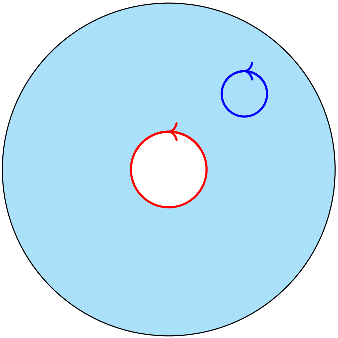

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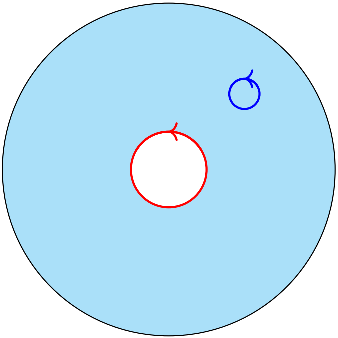

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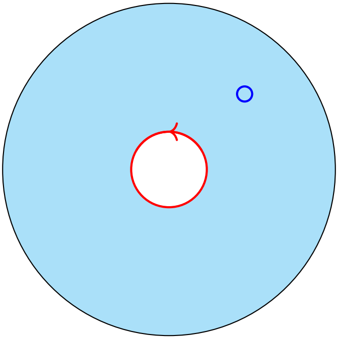

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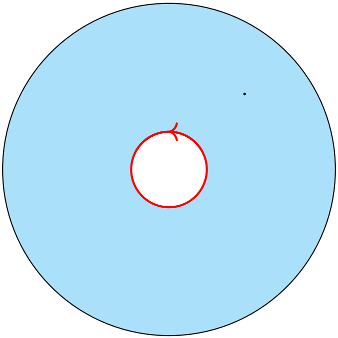

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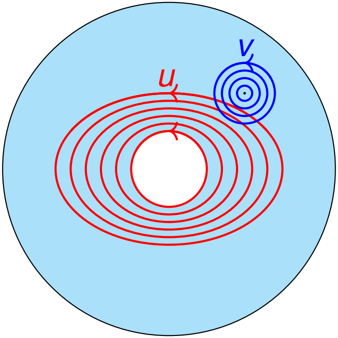
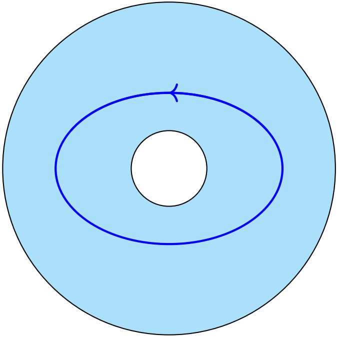


Simplicial Homology



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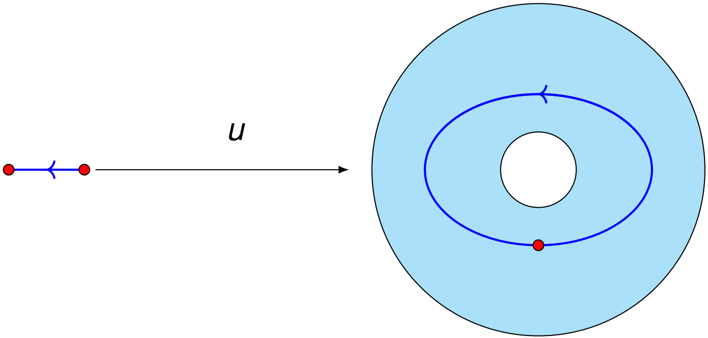

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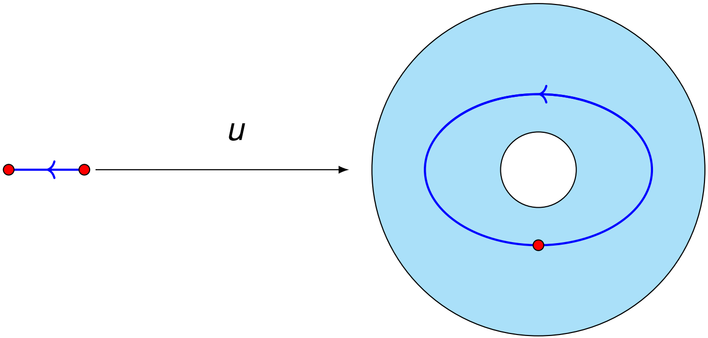
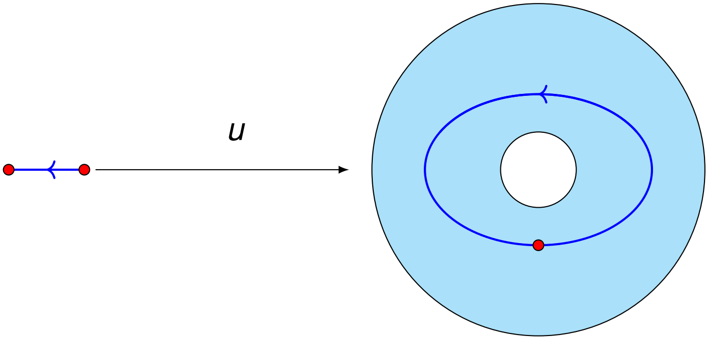


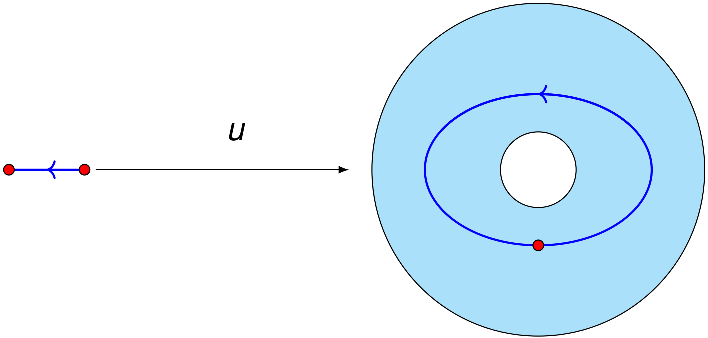
So cycles correspond to maps $u \in S_1 X$ with
$u \circ \delta_0(\Delta_1) - u \circ \delta_1(\Delta_1) = 0$.

Simplicial Homology



So cycles correspond to maps $u \in S_1X$ with $u \circ \delta_0(\Delta_1) - u \circ \delta_1(\Delta_1) = 0$. For $u \in S_nX$ we define the differential $d_n : S_nX \to S_{n-1}X$ by $d_n(u) = \sum_{i=0}^{n}(-1)^i u \circ \delta_i$. We extend this linearly for $u \in \mathbb{Z}S_nX$.

Simplicial Homology



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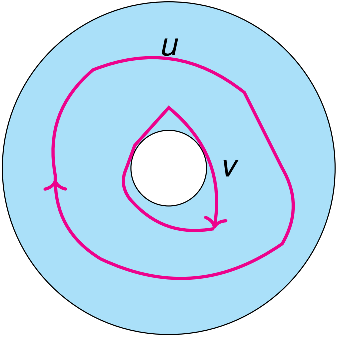
Simplicial Homology



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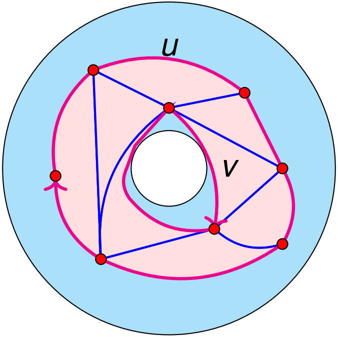

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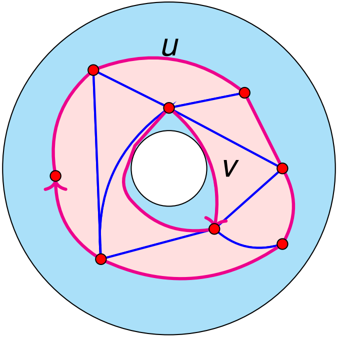

So, two cycles are to be thought of as the same if they can be “filled in” by a collection of 2-simplices.

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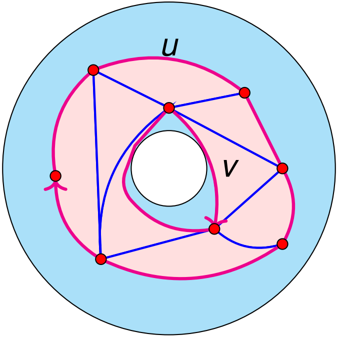

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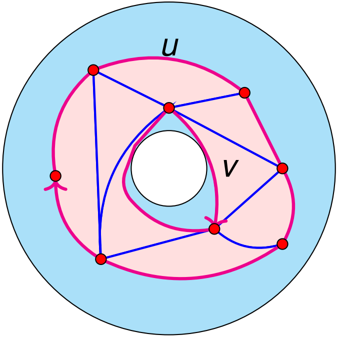

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Homology Groups

We describe this situation algebraically:

- Both $\mathrm{Im}(d_{n+1})$ and $\ker(d_n)$ are subgroups of $\mathbb{Z}S_nX$.

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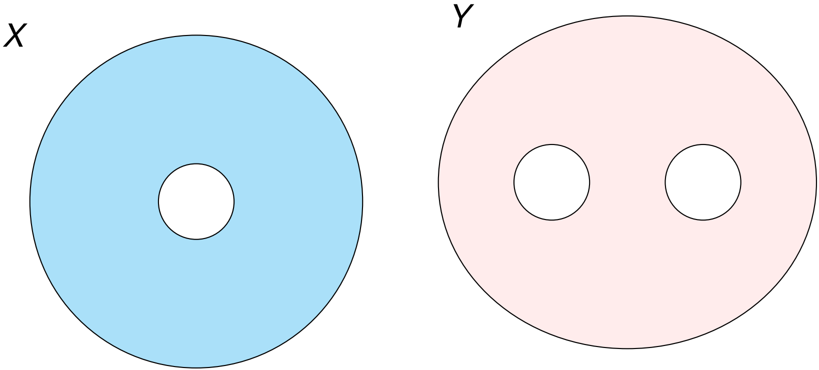
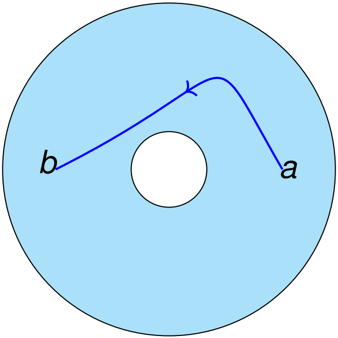
- The converse is often useful: if $H_n(\mathbb{Z}SX) \neq H_n(\mathbb{Z}SY)$, then X is not homeomorphic to Y.

Back to our example...



We can caluclate that $H_1(\mathbb{Z}SX) \cong \mathbb{Z}$, whereas $H_1(\mathbb{Z}SY) \cong \mathbb{Z}^2$ and so X is ***not*** homeomorphic to Y!

Homotopy



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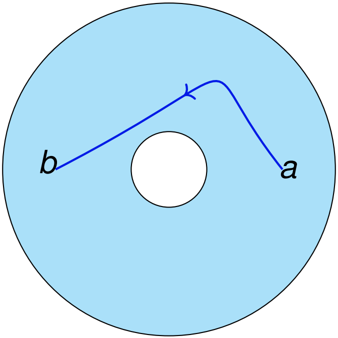

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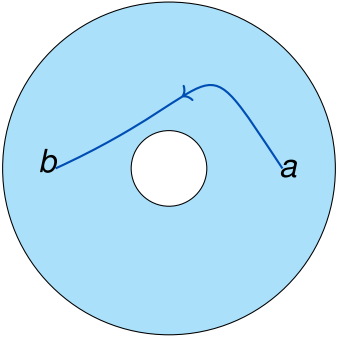

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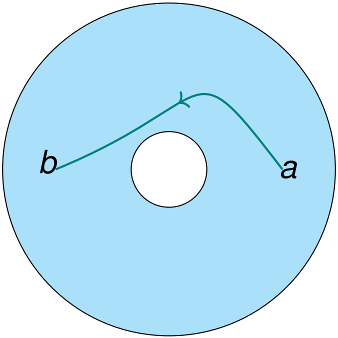

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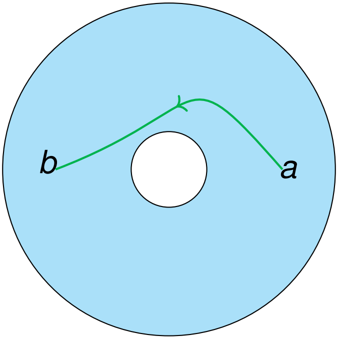

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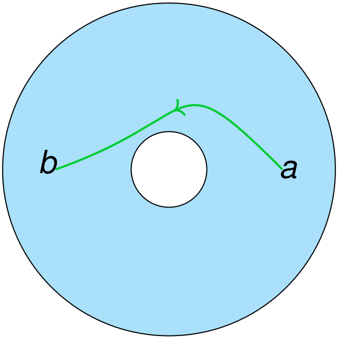

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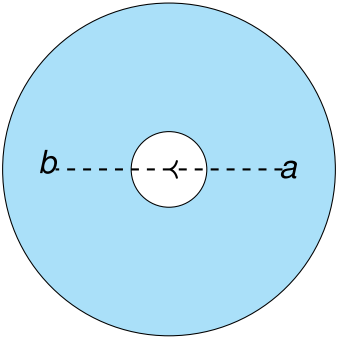

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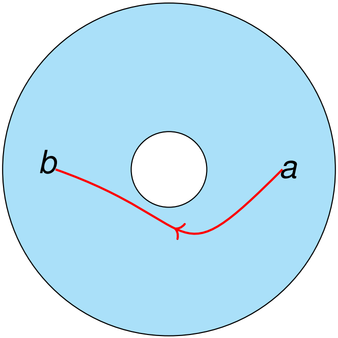

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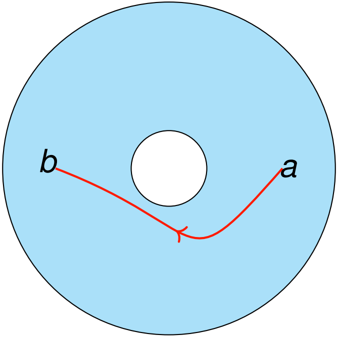

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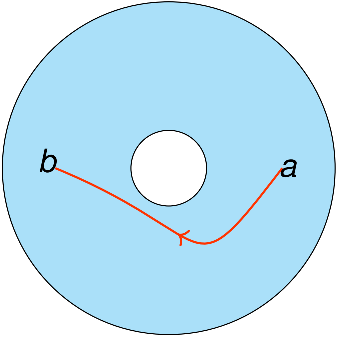

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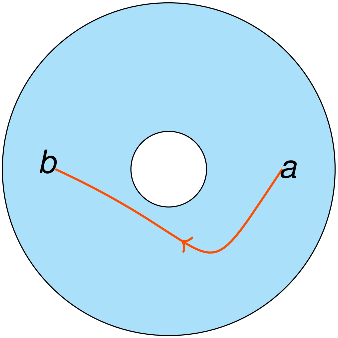

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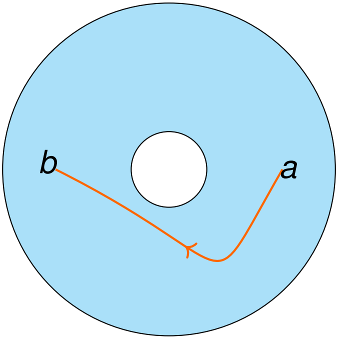

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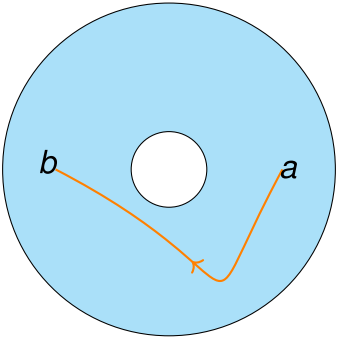

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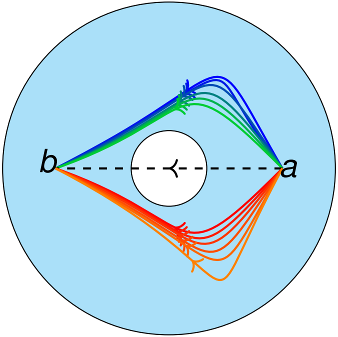
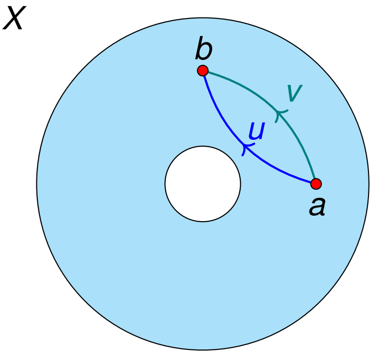


Simplicial Homotopy



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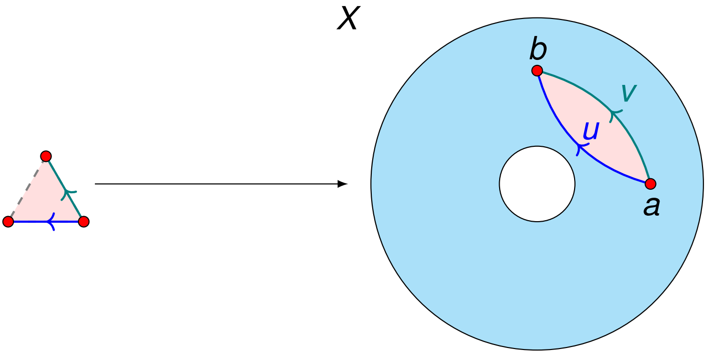

We say u is *homotopic* to v, or $u \sim v$.

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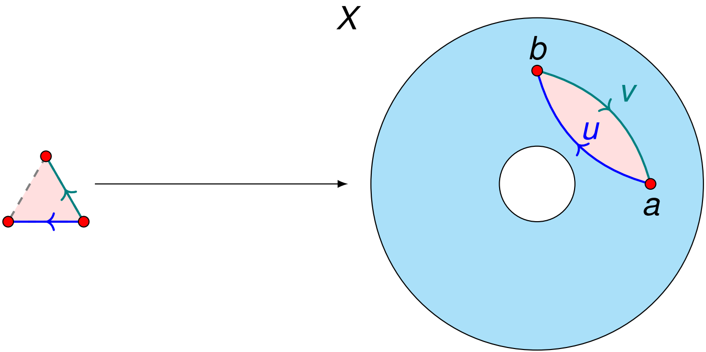

We say u is *homotopic* to v , or $u \sim v$. Also $u - v$ is a cycle, i.e. $u - v \in \ker(d_n)$.

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- This identifies homotopic elements!

The Dold-Kan Correspondence

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Theorem

Let X be a topological space. Then

$$\pi_n(\mathbb{Z}SX) = H_n\left(\bigcap_{i=0}^{n-1} \ker(\delta_i : \mathbb{Z}S_nX \to \mathbb{Z}S_{n-1}X)\right)$$

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  This allows us to do homotopy theory in a more general setting.
- Abstract homotopy is useful in many other areas, such as computer science and logic with the invention of Homotopy Type Theory.

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- It helps to keep this intuitive concept in mind when thinking more abstractly.
- Incredibly abstract mathematics can be applied to many other areas, for example in homotopy type theory.

References

For further reading, I recommend:

- To learn more about homological algebra: Charles. A Weibel *An Introduction to Homological Algebra*, 1995.
- To learn more about simplicial sets: Greg Friedman *An Elementary Illustrated Introduction to Simplicial Sets* , 2011.
- For a very readable introduction category theory, a great souce is: Emily Riehl, *Category Theory in Context*, 2014.

I welcome any questions!