Models of Martin-Löf Type Theory and Algebraic Model Structures

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Propositions as Sets (BHK)

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- Given a set I and a family of sets $\{A_i\}_{i \in I}$, the set $\Pi_{i \in I} A_i$ corresponds to universal quantification $\forall i \in I, A_i$.
- Similarly, the set $\Sigma_{i \in I} A_i$ corresponds to existential quantification $\exists i \in I, A_i$.

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- $p \in P$ is a proof that P is true, but in ZFC is also a proposition as it is a set.
- The law of the excluded middle holds in this model, so it is non-constructive.
- Given $p, q \in P$, we can ask externally if $p = q$ but cannot form a set that is the proposition of this this (We could form the set representing $p \cong q$ though, but this is different.)

Propositions as Types

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- **Identity Types:** given $p, q : P$, we can form the type $\mathtt{Id}_p(p, q)$.
- **Iterated identity types**: but given $x, y : \mathtt{Id}_p(p, q)$, we can form $\mathtt{Id}_{\mathtt{Id}_p(p,q)}(x, y)$...

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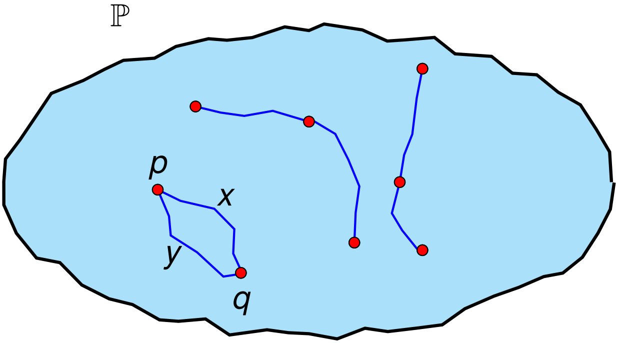
- $\mathrm{trans}(x, \mathrm{sym}(x)) = \mathrm{trans}(\mathrm{sym}(x), x) = \mathrm{refl}_a$.

Groupoidal Model



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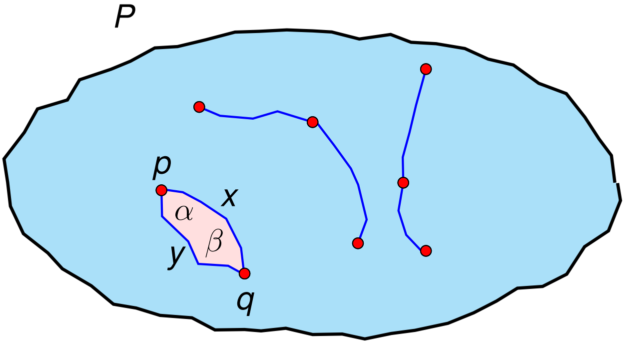
- For $p, q \in \mathbb{P}$, we model the identity type by $\mathrm{Hom}_{\mathbf{Gpd}}(p, q)$.
- There is no higher structure...

Simplicial Model



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- For $p, q \in \mathbb{P}$, we model the identity type by $\mathrm{Hom}_{\mathbf{sSet}}(p, q)$.
- There is higher structure.
- In this model, univalence holds.

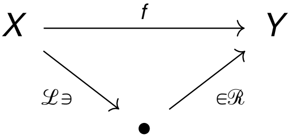
Weak Factorisation Systems

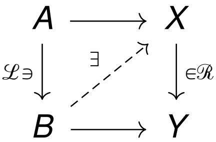
Both isofibrations and Kan fibrations are the right class of a weak factorisation system:

Definition

A *weak factorisation system* (wfs) on a category **M** is a pair $(\mathscr{L}, \mathscr{R})$ of classes of morphisms in **M** such that:

1. Every map $f : X \to Y$ can be factorised as a map in $\mathscr{L}$ followed by a map in $\mathscr{R}$.
2. $\mathscr{L} = {}^{\pitchfork}\mathscr{R}$ and $\mathscr{R} = \mathscr{L}^{\pitchfork}$.





Examples of WFSs

Example

(injective, surjective)

forms a wfs for **Set**.

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Example

Assuming the axiom of choice, (injective, surjective) forms a wfs for **Set**.

Examples of WFSs

Example ([GSS22])

~~**Assuming the axiom of choice**~~ (complemented inclusions, split epimorphism) forms a wfs on **Set**.

Examples of WFSs

Example

(inj-on-objects, isofibrations + equiv) forms a weak factorisation system on **Cat**.

Examples of WFSs

Example

Assuming the axiom of choice (inj-on-objects, isofibrations + equiv) forms a weak factorisation system on **Cat**.

Model Structures

The following is refined from Quillen's original definition [Qui67].

Definition

Let **M** be a category. A *model structure* on **M** consists of three classes of maps $\mathscr{W}, \mathscr{C}, \mathscr{F}$ such that:

1. $\mathscr{W}$ satisfies 3-for-2.
2. $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ form weak factorisation systems.

Examples of Model Structures

Example

Assuming the axiom of choice, there is a model structure on **Cat** :

- $\mathscr{W} = \{\text{equivalences of categories}\}$
- $\mathscr{C} = \{\text{injective-on-objects functors}\}$
- $\mathscr{F} = \{\text{isofibrations}\}$

Examples of Model Structures

Example

Assuming the axiom of choice, there is a model structure on **sSet** :

- $\mathscr{W} = \{$homotopy equivalences$\}$
- $\mathscr{C} = \{$monomorphisms$\}$
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~~**Assuming the axiom of choice**~~, there is a model structure on **sSet** :

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Examples of Model Structures

Example ([GHSS22])

Let $\mathscr{E}$ be a category with some mild conditions. There is a model structure on $\mathbf{s}\mathscr{E}$:

- $\mathscr{W} = \{$homotopy equivalences$\}$
- $\mathscr{C} = \{$Reedy complemented inclusions$\}$
- $\mathscr{F} = \{$Kan fibrations$\}$

This is called the *effective model structure* on $\mathbf{s}\mathscr{E}$.

A Constructive model structure on **Cat**

Theorem

Assuming the axiom of choice, *there is a model structure on* **Cat** :

- $\mathscr{W} = \{$*equivalences of categories*$\}$
- $\mathscr{C} = \{$*injective-on-objects functors*$\}$
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We call this the classical natural model structure on **Cat**. *This is cofibrantly generated and monoidal with respect to* $\times$.

A Constructive model structure on **Cat**

Theorem (H.)

~~***Assuming the axiom of choice***~~*, there is a model structure on* **Cat** *:*

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Internalising

Theorem (H.)

Let $\mathscr{E}$ be a category satisfying some conditions (for example, a Grothendieck topos).
There is a model structure on $\mathbf{Cat}(\mathscr{E})$:

- $\mathscr{W} = \{$*equivalences of categories*$\}$
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we call this the natural model structure on $\mathbf{Cat}(\mathscr{E})$.
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Algebraic WFSs

It is an open problem whether Voevodsky's simplicial model of HoTT is constructive. One suggestion on how to solve this is using more algebraic formulations.

Definition

An *algebraic weak factorisation system* (awfs) on a category **M** is a pair $(\mathbb{L}, \mathbb{R})$ of a comonad and a monad on $\mathbf{M}^{\rightarrow}$ such that:

1. $(\overline{\mathbb{L}\text{-Coalg}}, \overline{\mathbb{R}\text{-Alg}})$ is a wfs on **M**.
2. A certain canonical map forms a distributive law.

Think of $(f, \alpha) \in \overline{\mathbb{R}\text{-Alg}}$ as a fibration with the extra data α of a lifting function.

Type Theoretic AWFS

Definition ([GL23])

An awfs $(\mathbb{L}, \mathbb{R})$ is said to have the structure of a *type theoretic algebraic weak factorisation system* (ttawfs) if the following hold:

1. Maps in $\overline{\mathbb{R}\text{-Alg}}$ are exponentiable.
2. It has a frobenius structure: $\overline{\mathbb{R}\text{-Alg}}$ is stable under pullback along maps in $\overline{\mathbb{L}\text{-Colg}}$.
3. It has the structure of a stable, functorial choice of path objects.

Type Theoretic AWFS

Theorem ([GL23], Theorem 4.12)

Let $(\mathbb{L}, \mathbb{R})$ *be an awfs with the structure of a ttawfs. Then the right adjoint splitting of the comprehension category associated to the awfs is equipped with strictly stable choices of* Σ, Π *and* $\mathtt{Id}$*-types i.e. it forms a model of MLTT.*

Type Theoretic AWFS

Theorem ([GL23], Theorem 5.5)

In **Gpd***, (strong deformation retracts, normal isofibrations) underlies an awfs which has the structure of a ttawfs.*

Algebraic Model Structures

The following was defined by Emily Riehl [Rie11].

Definition

Let **M** be a category, and $\mathscr{W}$ be a class of maps which satisfies 3-for-2. An *algebraic model structure* on $(\mathbf{M}, \mathscr{W})$ is a pair of awfss $(\mathbb{C}, \mathbb{TF})$ and $(\mathbb{TC}, \mathbb{F})$ such that:

- The underlying wfss of these form a model structure on **M** with $\mathscr{W}$ the class of weak equivalences.
- There is a morphism of awfss $\xi : (\mathbb{TC}, \mathbb{F}) \to (\mathbb{C}, \mathbb{TF})$.

Constructive Algebraic Model Structures

Theorem (H.)

Let $\mathcal{E}$ be a category satisfying some conditions. There is an algebraic *model structure on* **Cat**$(\mathcal{E})$ *which has the natural model structure on* **Cat**$(\mathcal{E})$ *as its underlying model structure.*

In particular, there exists an awfs $(\mathbb{TC}, \mathbb{F})$ *such that* $\mathbb{TC}$*-coalgebras are complemented inclusion on objects functors which are equivalences and* $\mathbb{F}$*-algebras are isofibrations.*

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In particular, there exists an awfs $(\mathbb{TC}, \mathbb{F})$ such that $\mathbb{TC}$-coalgebras are complemented inclusion on objects functors which are equivalences and $\mathbb{F}$-algebras are isofibrations.

Theorem (H.)

Let $\mathcal{E}$ be a category satisfying some conditions. There is an algebraic model structure on **s**$\mathcal{E}$ *which has the effective model structure on* **s**$\mathcal{E}$ *as its underlying model structure.*

An internal model of MLTT

Theorem (H.)

Let $\mathcal{E}$ be a category satisfying some conditions. In **Gpd**$(\mathcal{E})$, $(\mathbb{TC}, \mathbb{F})$ *has the structure of a ttawfs.*

Summary

	Category	QMS?	Alg?	MLTT?	agree?
Class.	**Gpd**	✓	✓	✓	✓
	sSet	✓	✓	✓	✓
Const.	**Gpd**			[GL23]	
	sSet	[GSS22]			
Int.	**Gpd**$(\mathscr{E})$				
	s$\mathscr{E}$	[GHSS22]			

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	sSet	[GSS22]	✓		
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	s$\mathcal{E}$	[GHSS22]	✓		

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	sSet	[GSS22]	✓	?	
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	s$\mathscr{E}$	[GHSS22]	✓	?	

Questions and Further Work

- Can we prove that the awfs for **sSet** / **s**$\mathcal{E}$ form a ttawfs? This would give a constructive simplicial model of MLTT.

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- Can we prove that the awfs for **sSet** / **s**$\mathcal{E}$ form a ttawfs? This would give a constructive simplicial model of MLTT.
- Should there be a link between models of MLTT and (algebraic) model structures?
- Explore the properties of the $\mathbf{Gpd}(\mathcal{E})$ model. Do different choices of $\mathcal{E}$ give different type theories?
- In particular, look at $\mathbf{Gpd}(\mathbf{Eff})$. Does this give a model of “computable” type theory? (With Sam Speight) Or a realisability 2-topos?

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Assumptions needed on $\mathcal{E}$

- Lextensivity.
- Cartesian closed.
- Finite limits.
- Some conditions ensuring **Cat**$(\mathcal{E})$ has colimits (locally finitely presentable will do, but probably so will elementary topos + NNO).
- Local Cartesian Closure.

A Grothendieck topos satisfies all of these. So does the effective topos.