The Elementary Theory of the 2-Category of Small Categories

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Outline

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1. Motivation
2. Review of ETCS
3. ET2CSC
4. 2-Categories of Categories

ZFC and ETCS

ZFC

- First order theory.
- Well-founded trees.
- Axiomatises "$x \in X$".

ZFC and ETCS

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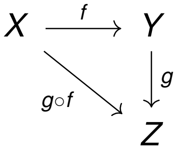
- Axiomatises "$x \in X$".

<u>ETCS</u> (Lawvere, 1964)

- Assume the existence of a category $\mathscr{E}$ satisfying some properties.
- Axiomatises



- We can do "naïve" set theory here.

ETCS and ET2CSC

ET2CSC

- Assume the existence of a 2-category $\mathscr{K}$ satisfying some properties.
- Axiomatises

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{B} \\ & {\scriptstyle G\circ F}\searrow & \downarrow{\scriptstyle G} \\ & & \mathscr{C} \end{array}$$

$$\mathscr{A} \underset{G}{\overset{F}{\underset{\alpha\Downarrow}{\rightrightarrows}}} \mathscr{B}$$

ETCS (Lawvere, 1964)

- Assume the existence of a category $\mathscr{E}$ satisfying some properties.
- Axiomatises

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g\circ f}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array} .$$

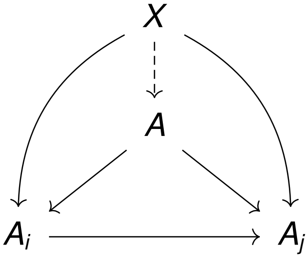
- We can do “naïve” set theory here.

Outline

ETCS

$\mathscr{E} \models$ ETCS if:

- It has finite limits.



ETCS

$\mathscr{E} \models$ ETCS if:

- It has finite limits.
- It is cartesian closed.

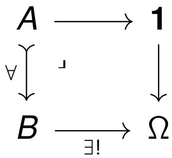
$$X, Y \in \mathscr{E} \rightsquigarrow X^Y \in \mathscr{E}$$

$$[Y \times Z, X] \cong \left[Z, X^Y\right]$$

ETCS

$\mathcal{E} \models$ ETCS if:

- It has finite limits.
- It is cartesian closed.
- It has a subobject classifier.



ETCS

$\mathcal{E} \models$ ETCS if:

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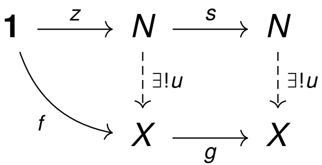
- It has a natural numbers object.



ETCS

$\mathscr{E} \models$ ETCS if:

- It has finite limits.
- It is cartesian closed.
- It has a subobject classifier.
- It has a natural numbers object.
- It is well-pointed.

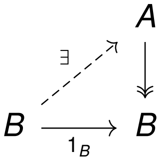
$$\mathbf{1} \xrightarrow{\forall x} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

$$fx = gx \implies f = g$$

ETCS

$\mathcal{E} \models$ ETCS if:

- It has finite limits.
- It is cartesian closed.
- It has a subobject classifier.
- It has a natural numbers object.
- It is well-pointed.
- It satisfies the external axiom of choice.



Examples

Assuming AOC:

- **Set** $\models$ ETCS.

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- Let λ be an uncountable, strong limit cardinal. Then **Set**$_\lambda$ $\models$ ETCS.

Examples

Assuming AOC:

- **Set** $\models$ ETCS.
- Let λ be an uncountable, strong limit cardinal. Then $\mathbf{Set}_\lambda \models$ ETCS.

A trivial example:

- **1** $\models$ ETCS.

Non-examples

- For $1 < \lambda \leqslant \aleph_0$,
  $\mathbf{Set}_\lambda$ has no NNO.

Non-examples

- For $1 < \lambda \leqslant \aleph_0$, **Set**$_\lambda$ has no NNO.
- **Cat**$_1$ is not well-pointed.

$$\mathbf{1} \xrightarrow{\forall x} \mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$$

$$Fx = Gx \implies F_0 = G_0$$

Non-examples

- For $1 < \lambda \leqslant \aleph_0$, $\mathbf{Set}_\lambda$ has no NNO.
- $\mathbf{Cat}_1$ is not well-pointed.

$$\mathbf{2} \xrightarrow{\forall f} \mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$$

$$Ff = Gf \implies F = G$$

Non-examples

- For $1 < \lambda \leqslant \aleph_0$, **Set**$_\lambda$ has no NNO.
- **Cat**$_1$ is not well-pointed.
- **Grp** is not well pointed.

$$\mathbf{1} \xrightarrow{\exists ! e} G \underset{g}{\overset{f}{\rightrightarrows}} H$$

$$fe = ge \nRightarrow f = g$$

ZFC in ETCS

Let $\mathscr{E} \models$ ETCS.

- $X \in \mathscr{E} \rightsquigarrow$ sets.

ZFC in ETCS

Let $\mathscr{E} \models$ ETCS.

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- $\mathbf{1} \to X \rightsquigarrow$ elements of X.

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Let $\mathscr{E} \models$ ETCS.

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- $\Omega^X \rightsquigarrow$ power set of X.

ZFC in ETCS

Let $\mathscr{E} \models$ ETCS.

- $X \in \mathscr{E} \rightsquigarrow$ sets.
- $\mathbf{1} \to X \rightsquigarrow$ elements of X.
- $\Omega^X \rightsquigarrow$ power set of X.
- + more...

Large sets

ETCS corresponds to a fragment of ZFC.

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Cannot form the set

$$\bigsqcup_{n \in \mathbb{N}} f_n(\mathbb{N})$$

where $f_{n+1}(X) := \mathbb{N}^{f_n(X)}$.

Large sets

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Cannot form the set

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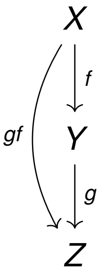
where $f_{n+1}(X) := \mathbb{N}^{f_n(X)}$.

In ZFC $\rightsquigarrow$ replacement.

Replacement

$$
\begin{array}{ccl}
X & & \supseteq f^{-1}(y) \\
\downarrow f & \searrow gf & \\
Y & & \supseteq g^{-1}(z) \ni y \\
\downarrow g & & \\
Z & & \ni z
\end{array}
$$

Replacement



$\supseteq f^{-1}(y)$

$\supseteq g^{-1}(z) \ni y$

$\ni z$

$$gf^{-1}(z) = \bigsqcup_{y \in g^{-1}(z)} f^{-1}(y)$$

Replacement

$$
\begin{array}{ccc}
X & & \supseteq f^{-1}(y) \\
\downarrow f & & \\
Y & & \supseteq g^{-1}(z) \ni y \\
\downarrow g & & \\
Z & & \ni z
\end{array}
\qquad (gf : X \to Z)
$$

$$
gf^{-1}(z) = \coprod_{y \in g^{-1}(z)} f^{-1}(y)
$$

To axiomatise classes and sets $\rightsquigarrow$ Joyal and Moerdijk's algebraic set theory.

Outline

Internal Categories

Definition

A *small category* is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \underset{p_2}{\overset{\overset{p_1}{\longrightarrow}}{\xrightarrow{m}}} C_1 \underset{d_0}{\overset{\overset{d_1}{\longrightarrow}}{\xleftarrow{i}}} C_0$$

Where $C_0, C_1 \in \mathbf{Set}$.

These are the objects of a 2-category **Cat**.

Internal Categories

Definition

Let $\mathscr{E}$ be a category with pullbacks.
A category internal to $\mathscr{E}$ is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}}} C_1 \underset{d_0}{\overset{d_1}{\underset{i}{\leftrightarrows}}} C_0$$

Where $C_0, C_1 \in \mathscr{E}$.
These are the objects of a 2-category $\mathbf{Cat}(\mathscr{E})$.

Idea!

$\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat} \rightsquigarrow \mathscr{E} \models$ ETCS, then $\mathbf{Cat}(\mathscr{E}) \approx \mathbf{Cat}$.

Limits and Cartesian Closure

Theorem

1. *$\mathscr{E}$ has finite limits iff* $\mathbf{Cat}(\mathscr{E})$ *has finite* 2*-limits.*
2. *$\mathscr{E}$ is cartesian closed if and only if* $\mathbf{Cat}(\mathscr{E})$ *is* 2*-cartesian closed. ([BE69] [Mir18])*

Limits and Cartesian Closure

Theorem

1. *$\mathscr{E}$ has finite limits iff $\mathbf{Cat}(\mathscr{E})$ has finite 2-limits.*
2. *$\mathscr{E}$ is cartesian closed if and only if $\mathbf{Cat}(\mathscr{E})$ is 2-cartesian closed. ([BE69] [Mir18])*

The proofs use:

- The theory of limit sketches.
- The nerve $N : \mathbf{Cat}(\mathscr{E}) \to \mathbf{s}\mathscr{E}$
- $\mathbf{disc} \dashv (-)_0 \dashv \mathbf{indisc} : \mathscr{E} \to \mathbf{Cat}(\mathscr{E})$

Full Subobject Classifiers

Full monos are defined representably in $\mathscr{K}$.

Definition

Let $\mathscr{K}$ be a 2-category. A *full subobject classifier* is:

- a full monomorphism $\underline{\top} : \underline{\mathbf{1}} \to \underline{\Omega}$

such that:

- $\forall$ full monos $i : A \to B$, $\exists! \chi_i : B \to \underline{\Omega}$ making the following square a pullback.

$$\begin{array}{ccc} A & \xrightarrow{!} & \underline{\mathbf{1}} \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow{\scriptstyle \underline{\top}} \\ B & \xrightarrow[\chi_i]{} & \underline{\Omega}. \end{array}$$

FSOC example

For $\mathscr{K} = \mathbf{Cat}$, the full subobject classifier is given by $\mathbf{1} \to \mathbb{I} := \{0 \cong 1\}$.

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Note that $\mathbb{I} = \textbf{indisc}(\{0, 1\})$.

Full Subobject Classifiers

Theorem

Let $\mathcal{E}$ be a category with finite limits. TFAE

- *$\mathcal{E}$ has a subobject classifier.*
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Proof. (Sketch)
$\mathbf{1} \to \Omega$ SOC for $\mathscr{E} \rightsquigarrow \mathbf{1} \to \mathbf{indisc}(\Omega)$ is FSOC for $\mathbf{Cat}(\mathscr{E})$.
$\mathbf{1} \to \underline{\Omega}$ FSOC for $\mathbf{Cat}(\mathscr{E}) \rightsquigarrow \mathbf{1} \to \underline{\Omega}_0$ SOC for $\mathscr{E}$.

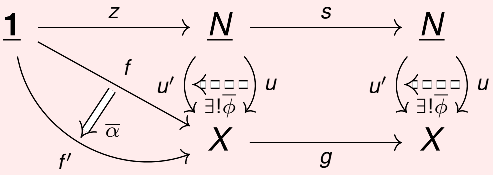
Natural Numbers Object

Definition

Let $\mathscr{K}$ be a 2-category with a terminal object $\underline{\mathbf{1}}$.

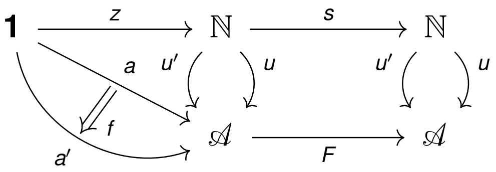
$$\underline{\mathbf{1}} \xrightarrow{z} \underline{N} \xrightarrow{s} \underline{N}$$

is called a *natural numbers object* in $\mathscr{K}$ if it is a natural numbers object for the underlying 1-category of $\mathscr{K}$ and if:



NNO example

For $\mathscr{K} = \mathbf{Cat}$, the natural numbers object is given by $\mathbf{disc}(\mathbb{N})$.



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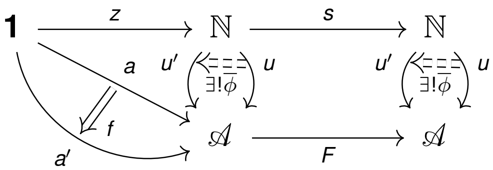

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Proof. (Sketch)
(N, z, s) NNO for $\mathcal{E} \rightsquigarrow (\mathbf{disc}(N), \mathbf{disc}(z), \mathbf{disc}(s))$ is a NNO for $\mathbf{Cat}(\mathcal{E})$.
$(\underline{N}, \underline{z}, \underline{s})$ NNO in $\mathbf{Cat}(\mathcal{E}) \rightsquigarrow (\underline{N}_0, \underline{z}_0, \underline{s}_0)$ is an NNO for $\mathcal{E}$.

Well-pointedness

Let $\mathbf{2}$ denote the walking arrow in $\mathbf{Cat}$.

Definition

A 2-category $\mathscr{K}$ is called 2-*well-pointed* if the following conditions hold.

1. $\mathscr{K}$ has a terminal object $\underline{\mathbf{1}}$.
2. The copower $\mathbf{2} \odot \underline{\mathbf{1}}$ exists in $\mathscr{K}$.
3. The family containing just $\mathbf{2} \odot \underline{\mathbf{1}}$ is a generator for the 2-category $\mathscr{K}$.

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For $\mathscr{K} = \mathbf{Cat}$, $\mathbf{2} \odot \mathscr{A} := \mathbf{2} \times \mathscr{A}$.

$$\mathbf{2} \xrightarrow{\forall f} \mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$$

$$Ff = Gf \implies F = G$$

Well-pointedness

Theorem

Let $\mathscr{E}$ be a lextensive, cartesian closed category. TFAE

- *$\mathscr{E}$ is well-pointed.*
- *$\mathbf{Cat}(\mathscr{E})$ is 2-well-pointed.*

Well-pointedness

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We construct an internal free living arrow $\mathbf{2}_{\mathcal{E}}$.

Well-pointedness

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- **Cat**$(\mathcal{E})$ *is* 2*-well-pointed.*

We construct an internal free living arrow $\mathbf{2}_{\mathcal{E}}$.
As a stepping stone in our proof, we also show that $\mathcal{E}$ is lextensive iff **Cat**$(\mathcal{E})$ is lextensive in the 2 dimensional sense.

Axiom of Choice

Proposition

Let $\mathscr{E}$ be a category with pullbacks. TFAE:

1. The external axiom of choice holds in $\mathscr{E}$.
2. Any fully faithful and epi-on-objects functor internal to $\mathscr{E}$ has a section in the 2-category **Cat**$(\mathscr{E})$.

Axiom of Choice

Lemma

There is an (epi, mono) orthogonal factorisation system on $\mathcal{E}$ if and only if there is an

$$(\textit{epi-on-objects, full monomorphism})$$

orthogonal factorisation system on $\mathbf{Cat}(\mathcal{E})$.

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$(\mathscr{L}, \mathscr{R})$ on $\mathscr{E} \rightsquigarrow$
$(\mathscr{L}$ on objects, $\mathscr{R}$ on objects and fully faithful$)$ on **Cat**$(\mathscr{E})$.

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$(\mathscr{L}, \mathscr{R})$ on $\mathscr{E} \rightsquigarrow$
$(\mathscr{L}$ on objects, $\mathscr{R}$ on objects and fully faithful$)$ on **Cat**$(\mathscr{E})$.

Definition ([Str82])

A morphism in a 2-category $\mathscr{K}$ which is left orthogonal to all fully faithful monomorphisms in $\mathscr{K}$ will be called *acute*.

Axiom of Choice

Definition

Say that a 2-category $\mathscr{K}$ satisfies the *categorified axiom of choice* if any acute fully faithful morphism has a section.

Axiom of Choice

Theorem

Let $\mathcal{E}$ be a category with pullbacks, products and an (epi, mono)-orthogonal factorisation system. TFAE:

1. *The category $\mathcal{E}$ satisfies the external axiom of choice.*
2. *The 2-category $\mathbf{Cat}(\mathcal{E})$ satisfies the categorified axiom of choice.*

Summary

For the 1-category $\mathscr{E}$:

- finite limits
- cartesian closure
- subobject classifier
- natural numbers object
- well-pointed
- axiom of choice

For the 2-category $\mathbf{Cat}(\mathscr{E})$:

- finite 2-limits
- cartesian closure
- full subobject classifier
- natural numbers object
- 2-well-pointed
- categorified axiom of choice

Summary

For the 1-category $\mathscr{E}$:

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Can we replace $\mathbf{Cat}(\mathscr{E})$ with $\mathscr{K}$?

Bourke's characterisation of $\mathbf{Cat}(\mathcal{E})$

Theorem ([Bou10])

If $\mathcal{E}$ is a category with pullbacks then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ satisfies the conditions listed below, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$.

1. *$\mathcal{K}$ has pullbacks and powers by **2**.*
2. *$\mathcal{K}$ has codescent objects of categories internal to $\mathcal{K}$ whose source and target maps form a two-sided discrete fibration.*
3. *Codescent morphisms are effective in $\mathcal{K}$.*
4. *Discrete objects in $\mathcal{K}$ are projective.*
5. *For every object $A \in \mathcal{K}$, there is a projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*

Bourke's characterisation of **Cat**$(\mathscr{E})$

Theorem ([Bou10])

If $\mathscr{E}$ is a category with pullbacks then the 2-category $\mathscr{K} := \mathbf{Cat}(\mathscr{E})$ satisfies Bourke's axioms. Conversely, if $\mathscr{K}$ satisfies Bourke's axioms, then there is a 2-equivalence $K \simeq \mathbf{Cat}(\mathscr{E})$ where $\mathscr{E} := \mathbf{Disc}(\mathscr{K})$.

ET2CSC

Definition

We say that the 2-category $\mathscr{K}$ models the *elementary theory of the* 2*-category of small categories* (ET2CSC) if the following properties hold:

1. It satisfies Bourke's axioms.
2. It has a terminal object.
3. It is cartesian closed.
4. It is 2-well-pointed.
5. It has a natural numbers object.
6. It has a full subobject classifier.
7. It satisfies the categorified axiom of choice.

We write $\mathscr{K} \models$ ET2CSC

Examples

Assuming the axiom of choice:

- **Cat** $\models$ ET2CSC.

Examples

Assuming the axiom of choice:

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Examples

Assuming the axiom of choice:

- **Cat** $\models$ ET2CSC.
- Let λ be an uncountable, strong limit cardinal. Then **Cat**(**Set**$_\lambda$) $\models$ ET2CSC.

Trivial example:

- **1** $=$ **Cat**(**1**) $\models$ ET2CSC.

Main Result

Theorem

1. *Let $\mathscr{E}$ be a category. Then $\mathscr{E} \models$ ETCS if and only if $\mathbf{Cat}(\mathscr{E}) \models$ ET2CSC, and in this case $\mathscr{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathscr{E}))$.*
2. *Let $\mathscr{K}$ be a 2-category. Then $\mathscr{K} \models$ ET2CSC if and only if $\mathbf{Disc}\,(\mathscr{K}) \models$ ETCS, and in this case $\mathscr{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathscr{K}))$.*
3. *This extends to a biequivalence*

$$\mathbf{ETCS} \underset{\mathbf{Cat}(-)}{\overset{\mathbf{Disc}(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\sim}}}} \mathbf{ET2CSC}$$

Outline

Elementary 2 toposes

For the 1-category $\mathscr{E}$:

- finite limits
- cartesian closure
- subobject classifier

is an elementary 1-topos!

Elementary 2 toposes

For the 1-category $\mathscr{E}$:

- finite limits
- cartesian closure
- subobject classifier

is an elementary 1-topos!

For the 2-category $\mathbf{Cat}(\mathscr{E})$

- finite 2-limits
- cartesian closure
- full subobject classifier

... is not an elementary 2-topos in the sense of [Web07].

Discrete Opfibrations

A discrete opfibration in **Cat** is

$$\begin{array}{ccccc} \mathscr{A} & & & & a \\ \big\downarrow{\scriptstyle F} & & & & \big\downarrow \\ \mathscr{B} & & b & \longrightarrow & F(a) \end{array}$$

Discrete Opfibrations

A discrete opfibration in **Cat** is

$$\begin{array}{ccccc} \mathscr{A} & & \exists a' & \dashrightarrow & a \\ \Big\downarrow{\scriptstyle F} & & \vdots & & \Big\downarrow \\ \mathscr{B} & & b & \longrightarrow & F(a) \end{array}$$

Discrete Opfibrations

A discrete opfibration in **Cat** is

$$
\begin{array}{ccccc}
\mathscr{A} & \qquad & \exists a' & \dashrightarrow & a \\
\big\downarrow F & & \big\downarrow & & \big\downarrow \\
\mathscr{B} & & b & \longrightarrow & F(a)
\end{array}
$$

In a 2-category $\mathscr{K}$, we define discrete opfibrations representably.

Discrete Opfibration Classifiers

Recall the Grothendieck construction:

$$\begin{array}{ccc} \mathrm{el}_{\mathscr{A}}(F) & \longrightarrow & \mathbf{Set}_* \\ \big\downarrow & \lrcorner & \big\downarrow p \\ \mathscr{A} & \xrightarrow[F]{} & \mathbf{Set} \end{array}$$

$\mathrm{el}_{\mathscr{A}} : \mathbf{CAT}(\mathscr{A}, \mathbf{Set}) \to \mathbf{DopFib}/\mathscr{A}$.

Discrete Opfibration Classifiers

Definition ([Web07])

A *discrete opfibration classifier* for $\mathscr{K}$ consists of a discrete opfibration $p : S_* \to S$ such that for any $X \in \mathscr{K}$, the functor $\mathrm{el}_X : \mathscr{K}(X, S) \to \mathbf{Dopfib}/X$ which sends $f \in \mathscr{K}(X, S)$ to the pullback of p along f is fully faithful.

2-toposes

Definition ([Web07])

An *elementary* 2*-topos* is a 2-category $\mathscr{K}$ such that

- It has finite 2-limits.
- It is cartesian closed.
- It has a duality involution.
- It has a discrete opfibration classifier.

2-toposes

Definition ([Web07])

An *elementary* 2*-topos* is a 2-category $\mathscr{K}$ such that

- It has finite 2-limits.
- It is cartesian closed.
- It has a duality involution.
- It has a discrete opfibration classifier.

So **Cat** is not a 2-topos, but rather **CAT** is.

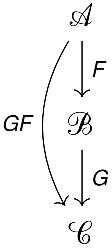
Replacement

In **SET** :

$$
\begin{array}{ccc}
X & & \supseteq f^{-1}(y) \\
\downarrow f & & \\
Y & & \supseteq g^{-1}(z) \ni y \\
\downarrow g & & \\
Z & & \ni z
\end{array}
\qquad gf : X \to Z
$$

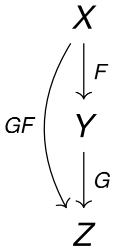
2D replacement

In **CAT** :



2D replacement

Shulman: if $\mathscr{K}$ has a discrete opfibration classifier
$p : S_* \to S$.



This is a 2-categorical axiom of replacement!

2-categories of categories

Definition

Let $\mathscr{K}$ be a 2-category with a discrete opfibration classifier $p : S_* \to S$, and let $\mathscr{K}_\sigma \hookrightarrow \mathscr{K}$ denote the full-sub-2-category of small objects. Then $(\mathscr{K}, p : S_* \to S)$ is said to be a 2-*category of categories* if the following conditions hold.

1. $\mathscr{K}_\sigma, \mathscr{K} \models$ ET2CSC .
2. $\mathscr{K}_\sigma \hookrightarrow \mathscr{K}$ is a morphism of models of ET2CSC.
3. Small discrete opfibrations are closed under composition.

Example

$(\mathbf{CAT}, p : \mathbf{Set}_* \to \mathbf{Set})$ is a 2-category of categories.

$$\mathbf{Cat} \hookrightarrow \mathbf{CAT}$$

Example

Let μ be an inaccessible cardinal and $\lambda > \mu$ be a strong limit cardinal.

Write $\mathbf{Set} := \mathbf{Set}_\mu$, $\mathbf{Cat} := \mathbf{Cat}(\mathbf{Set})$ and $\mathbf{CAT} := \mathbf{Cat}(\mathbf{Set}_\lambda)$. Then:

$(\mathbf{CAT}, p : \mathbf{Set}_* \to \mathbf{Set})$ is a 2-category of categories.

$$\mathbf{Cat} \hookrightarrow \mathbf{CAT}$$

Work In Progress

- 2CoC satisfy Shulman's 2D AST.

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- Free cocompletions in 2CoC.

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- A more genuinely 2D NNO.

Work In Progress

- 2CoC satisfy Shulman’s 2D AST.
- Free cocompletions in 2CoC.
- A more genuinely 2D NNO.
- Coequalisers in ET2CSC.

ArXiv link

The Elementary Theory of the 2-Category of Small Categories, written with Adrian Miranda, 2024. To appear in TAC: special volume for Bill Lawvere.

https://arxiv.org/abs/2403.03647

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