Colimits of internal categories

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Outline

1. Background
2. Coequalisers in $\mathbf{Cat}(\mathscr{E})$
   - Parallel functors that agree on objects
   - Coequalisers out of a discrete category
   - putting it together
3. Future work

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Internal Categories

Definition

A *small category* is:

$$\begin{tikzcd} \cdots \arrow[r] & C_1 \times_{C_0} C_1 \arrow[r, shift left=3, "p_1"] \arrow[r, "m"] \arrow[r, shift right=3, "p_2"'] & C_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & C_0 \arrow[l, "i"'] \end{tikzcd}$$

Where $C_0, C_1 \in$ **Set**. These are the objects of a 2-category **Cat**.

Internal Categories

Definition

Let $\mathscr{E}$ be a category with pullbacks.
A category internal to $\mathscr{E}$ is:

$$
\begin{tikzcd}
\cdots \arrow[r] & C_1 \times_{C_0} C_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & C_0 \arrow[l, "i" description]
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Where $C_0, C_1 \in \mathscr{E}$.
These are the objects of a 2-category $\mathbf{Cat}(\mathscr{E})$.

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$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \;\; \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow[p_2]{} \end{matrix} \;\; C_1 \;\; \begin{matrix} \xrightarrow{d_1} \\ \xleftarrow{i} \\ \xrightarrow[d_0]{} \end{matrix} \;\; C_0$$

Where $C_0, C_1 \in \mathcal{E}$.
These are the objects of a 2-category $\mathbf{Cat}(\mathcal{E})$.

GOAL: We wish to find conditions on $\mathcal{E}$ which ensure that $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits.

Internal Categories

Theorem

Let $\mathcal{E}$ be locally finitely presentable. Then $\mathbf{Cat}(\mathcal{E})$ has 2-colimits.

Proof: Using limit sketches and [AR94].

2-colimits

Recall the following result:

Theorem ([Kel82])

A 2-category $\mathscr{K}$ has (finite) 2-colimits if and only if it has (finite) coproducts, copowers by the free living arrow in **Cat** *(which we denote* **2***) and coequalisers.*

2-colimits

Recall the following result:

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$$\mathscr{K}(\mathbf{2} \odot A, B) \cong \mathbf{Cat}(\mathbf{2}, \mathscr{K}(A, B)).$$

Coproducts

Coproducts in **Cat**: easy!

$$(\mathscr{C} + \mathscr{D})_0 := \mathscr{C}_0 + \mathscr{D}_0$$

$$(\mathscr{C} + \mathscr{D})_1 := \mathscr{C}_1 + \mathscr{D}_1$$

$$(\mathscr{C} + \mathscr{D})_1 \times_{(\mathscr{C}+\mathscr{D})_0} (\mathscr{C} + \mathscr{D})_1 \cong (\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1) + (\mathscr{D}_1 \times_{\mathscr{D}_0} \mathscr{D}_1)$$

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For $\mathscr{E}$ an extensive category, the coproduct in **Cat**$(\mathscr{E})$ is calculated in exactly the same way.

Copowers by **2**

$$\mathscr{K}(\mathbf{2} \odot A, B) \cong \mathbf{Cat}(\mathbf{2}, \mathscr{K}(A, B)).$$

Copowers by **2** in **Cat**: given by the cartesian product.

Copowers by $\mathbf{2}$

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Copowers by $\mathbf{2}$ in $\mathbf{Cat}$: given by the cartesian product.

For a lextensive category $\mathscr{E}$, copowers by $\mathbf{2}$ are given by cartesian product with the internal category $\mathbf{2}_{\mathscr{E}}$:

$$\begin{tikzcd} \cdots \arrow[r] & \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1} \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & \mathbf{1}+\mathbf{1}+\mathbf{1} \arrow[r, shift left=2, "d_1"] \arrow[r, leftarrow, "i"] \arrow[r, shift right=2, "d_0"'] & \mathbf{1}+\mathbf{1} \end{tikzcd}$$

Coequalisers: Naive attempt

Coequalisers in **Cat**: much more subtle...

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\textit{source}"] \arrow[r, shift right, "\textit{target}"'] & \bullet \to \bullet \arrow[r, "\textit{coequalise}"] & \bullet \arrow[loop right] \end{tikzcd}$$

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This tells us three things:

- Coequalisers are not calculated levelwise in **Cat**.
- **Cat**(**FinSet** does not have coequalisers.
- Coequalisers in **Cat** utilise free categories on graphs.

Coequalisers in **Cat**

A method for calculating coequalisers in **Cat** is described by Bednarczyk, Borzyszkowski, and Pawlowski in [BBP99].

Outline

Recipe for calculating coequalisers in **Cat**$(\mathscr{E})$

Consider the following diagram in **Cat**$(\mathscr{E})$:

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \end{tikzcd}$$

Recipe for calculating coequalisers in $\mathbf{Cat}(\mathcal{E})$

STEP 1: Calculate the coequaliser of

$$\begin{tikzcd} \mathbf{disc}(A_0) \arrow[r] & \mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, dashed, "K"] & \mathbb{D} \end{tikzcd}$$

Recipe for calculating coequalisers in **Cat**$(\mathcal{E})$

STEP 1: Calculate the coequaliser of

$$\begin{tikzcd} \mathbf{disc}(A_0) \arrow[r] & \mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, dashed, "K"] & \mathbb{D} \end{tikzcd}$$

STEP 2: form the coequaliser $P : \mathbb{D} \to \mathbb{C}$ of the parallel pair of internal functors KF and KG.

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "K \cdot F"] \arrow[r, bend right, "K \cdot G"'] & \mathbb{D} \arrow[r, dashed, "P"] & \mathbb{C}. \end{tikzcd}$$

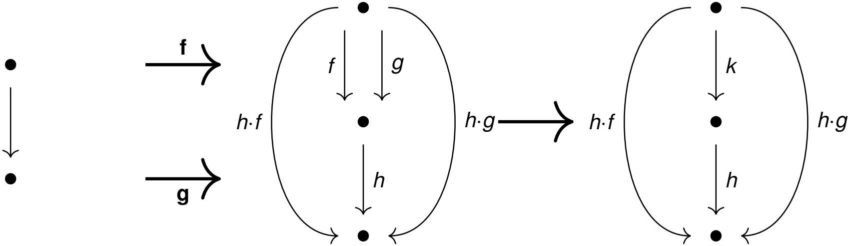
Outline

Parallel functors that agree on objects

Consider $\mathbb{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B}$ such that $F_0 = G_0$.

Naive attempt: coequalise $A_1 \underset{G_1}{\overset{F_1}{\rightrightarrows}} B_1$

Counterexample to naive attempt



Parallel functors that agree on objects

$$
\begin{tikzcd}
 & & L \arrow[dll, dashed] \arrow[d, dashed] \arrow[drr, dashed] & & \\
B_1 \arrow[dr, "d_0"'] & & A_1 \arrow[dl, "F_0 d_1"'] \arrow[dr, "F_0 d_0"] & & B_1 \arrow[dl, "d_1"] \\
 & B_0 & & B_0 &
\end{tikzcd}
$$

Parallel functors that agree on objects

This induces a diagram in $\mathscr{E}$

$$L \underset{1\times G_1\times 1}{\overset{1\times F_1\times 1}{\rightrightarrows}} B_3 \xrightarrow{m^2} B_1 \overset{Q_1}{\dashrightarrow} C_1$$

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Lemma

Let $\mathcal{E}$ be a category with pullbacks and pullback stable coequalisers. Then (B_0, C_1) can be given the structure of an internal category.

Parallel functors that agree on objects

identity assigner:

$$B_0 \xrightarrow{i} B_1 \xrightarrow{Q_1} C_1$$

Parallel functors that agree on objects

Sources and target:

$$
\begin{tikzcd}
L \arrow[r, shift left, "1\times F_1\times 1"] \arrow[r, shift right, "1\times G_1\times 1"'] & B_3 \arrow[r, "m^2"] & B_1 \arrow[r, "Q_1"] \arrow[d, "d_i"] & C_1 \arrow[d, dashed, ":= d_i"] \\
 & & B_0 \arrow[r, equal] & B_0
\end{tikzcd}
$$

Parallel functors that agree on objects

Composition: assuming coequalisers are stable under pullback in $\mathscr{E}$

$$\begin{tikzcd}
L \times_{B_0} L \arrow[r, shift left, "m^2 \cdot \widetilde{F} \times_{B_0} m^2 \cdot \widetilde{F}"] \arrow[r, shift right, "m^2 \cdot \widetilde{G} \times m^2 \cdot \widetilde{G}"'] & B_1 \times_{B_0} B_1 \arrow[r, "Q_1 \times_{Q_0} Q_1"] \arrow[d, "m"] & C_1 \times_{C_0} C_1 \arrow[d, dashed, "m"] \\
& B_1 \arrow[r, "Q_1"] & C_1
\end{tikzcd}$$

Parallel functors that agree on objects

Proposition

Let $\mathscr{E}$ be a category with pullbacks and pullback stable coequalisers. Then $\mathbf{Cat}(\mathscr{E})$ has coequalisers of parallel pairs of functors that agree on objects.

Coequifiers

Corollary

Let $\mathscr{E}$ be a lextensive category with pullbacks and pullback stable coequalisers. The 2-category **Cat**$(\mathscr{E})$ has coequifiers.

$$\begin{tikzcd} \mathbb{A} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\overline{\alpha}\Downarrow \quad \Downarrow\overline{\beta}"] & & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{2}_{\mathscr{E}} \times \mathbb{A} \arrow[rr, bend left, "\hat{\alpha}"] \arrow[rr, bend right, "\hat{\beta}"'] & & \mathbb{B} \end{tikzcd}$$

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Coequalisers out of a discrete category

Consider $\quad \mathbf{disc}(A) \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B}$ such that $A \in \mathscr{E}$.

Naive attempt: coequalise $\quad A \underset{G_0}{\overset{F_0}{\rightrightarrows}} B_0$

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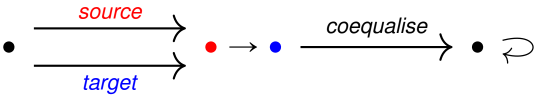

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- Assuming that we have free internal categories on internal graphs in $\mathcal{E}$: take the free category $\mathbb{F}(\mathcal{G})$.

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- We get an arrow $\mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$.
- We force this to be an internal functors with coequifiers, to get an internal functor $\mathbb{B} \to \mathbb{C}$.

Coequalisers out of a discrete category

Proposition

Let $\mathscr{E}$ be an extensive category with pullbacks and pullback stable coequalisers in which the forgetful functor $\mathscr{U} : \mathbf{Cat}(\mathscr{E})_1 \to \mathbf{Gph}(\mathscr{E})$ has a left adjoint. Then $\mathbf{Cat}(\mathscr{E})$ has coequalisers of parallel pairs of functors out of a discrete category.

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Examples

Extensive categories with pullbacks and pullback stable coequalisers in which the forgetful functor $\mathscr{U}$: $\mathbf{Cat}(\mathscr{E})_1 \to \mathbf{Gph}(\mathscr{E})$ has a left adjoint:

- List-arithmetic pretoposes with pullback stable coequalisers [Mai10].

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Non-examples: $\mathscr{E} = \mathbf{Cat}$, $\mathbf{Cat}(\mathscr{E})$, ...

Pullback stable coequalisers in **Cat**?

Coequalisers are not stable under pullback in **Cat**:
Consider the pushout:

$$\begin{tikzcd}
(1) \arrow[r] \arrow[d] & (1 \to 2) \arrow[d] \\
(0 \to 1) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (0 \to 1 \to 2)
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Pull this back along $(0 \to 2) \to (0 \to 1 \to 2)$:

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\varnothing \arrow[r] \arrow[d] & (2) \arrow[d] \\
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- We have a converse result for our assumptions.
- We can refine this result when $\mathscr{E}$ is an arithmetic Π-pretopos.
- possibly: $\mathbf{Cat}(\mathscr{E})$ has lax-pullback stable coequalisers?

ArXiv link

Colimits of internal categories, written with Adrian Miranda, 2024.

https://arxiv.org/abs/2403.03647

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Are any coequalisers stable under pullback?

Definition

In $\mathbf{Cat}(\mathscr{E})$, an internal functor $F : \mathbb{X} \to \mathbb{Y}$ is called a *discrete Conduché fibration* if the following is a pullback square:

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{F_1 \times_{F_0} F_1} & Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ X_1 & \xrightarrow[F_1]{} & Y_1 . \end{array}$$

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Proposition

Let $\mathcal{E}$ be a category with pullbacks. Then $\mathcal{E}$ has pullback stable coequalisers if and only if coequalisers of parallel pairs of internal functors which agree on objects are stable under pullback along discrete Conduché fibrations.

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$$\begin{tikzcd} L \arrow[r, shift left, "1 \times F_1 \times 1"] \arrow[r, shift right, "1 \times G_1 \times 1"'] & B_3 \arrow[r, "m^2"] & B_1 \arrow[r, dashed, "Q_1"] & C_1 \end{tikzcd}$$

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Let $\mathcal{E}$ be a category with pullbacks. Then $\mathcal{E}$ has pullback stable coequalisers if and only if coequalisers of parallel pairs of internal functors which agree on objects are stable under pullback along discrete Conduché fibrations.

Corollary: in $\mathbf{DblCat} = \mathbf{Cat}(\mathbf{Cat})$, Conduché fibrations are not all exponentiable (cf. [Nie20]).

Free internal categories from 2-colimits

Lemma ([Bou10])

Let $\mathscr{E}$ be a category with pullbacks, and suppose that $\mathbf{Cat}(\mathscr{E})$ *has finite 2-colimits. Then the forgetful functor* $\mathscr{U} : \mathbf{Cat}(\mathscr{E})_1 \to \mathbf{Gph}(\mathscr{E})$ *has left adjoint.*

Let $\mathscr{G} = (G_0, G_1, s, t)$ be an internal graph in $\mathscr{E}$. Consider the coinserter in $\mathbf{Cat}(\mathscr{E})$

$$\begin{tikzcd} \mathbf{disc}(G_1) \arrow[rr, bend left, "\mathbf{disc}(s)"] \arrow[rr, bend right, "\mathbf{disc}(t)"'] & & \mathbf{disc}(G_0) \arrow[r, dashed, "Q"] & \mathbb{F}(\mathscr{G}). \end{tikzcd}$$

Converse result

Theorem

Let $\mathcal{E}$ be a category with pullbacks. Then $\mathcal{E}$ is extensive, has pullback stable coequalisers, and the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint if and only if the 2*-category $\mathbf{Cat}(\mathcal{E})$ is extensive, has* 2*-colimits and has pullbacks and coequalisers of parallel pairs of functors that agree on objects are stable under pullback along discrete Conduché fibrations.*