Internal categories, algebraic model structures and type theory

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An example of some kind of (small) $(2,1)$-pretopos should be satisfied by **Gpd**

... and **Gpd**$(\mathscr{E})$ for suitable $\mathscr{E}$

Outline

1. 2-categorical axioms
2. Algebraic homotopy theory
3. Type theory
4. Future work

Outline

Internal categories

Let $\mathscr{E}$ be a category with pullbacks.

Definition

A category internal to $\mathscr{E}$ is:

$$C_1 \times_{C_0} C_1 \overset{p_1}{\underset{p_2}{\overset{m}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}}}} C_1 \overset{d_1}{\underset{d_0}{\overset{i}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}}} C_0$$

These are the objects of a 2-category $\mathbf{Cat}(\mathscr{E})$.

We denote the $(2,1)$-category of internal groupoids by $\mathbf{Gpd}(\mathscr{E})$.

Cateads

Note that for $\mathbb{C} \in \mathbf{Cat}(\mathscr{E})$, the 2-colimit of

$$C_1 \times_{C_0} C_1 \;\underset{p_2}{\overset{\overset{p_1}{\longrightarrow}}{\overset{m}{\longrightarrow}}}\; C_1 \;\underset{d_0}{\overset{\overset{d_1}{\longrightarrow}}{\overset{i}{\longleftarrow}}}\; C_0$$

is $\mathbb{C}$.

Cateads

Definition (Bourne-Penon, Bourke)

For a 2-category $\mathscr{K}$, a *catead* is

$$C_1 \times_{C_0} C_1 \;\underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}\!\!\to}}\; C_1 \;\underset{d_0}{\overset{d_1}{\underset{i}{\leftrightarrows}\!\!\to}}\; C_0$$

such that (d_1, d_0) forms a 2-sided discrete fibration.
We call its 2-colimit a *codescent object*.

So for $\mathscr{K} = \mathbf{Cat}(\mathscr{E})$, every object is a codescent object of a *discrete* catead.

Bourke's exactness axioms

Let $\mathscr{K}$ be a 2-category.

Bourke's axioms:

1. $\mathscr{K}$ has pullbacks and powers by $\mathbf{2}$.
2. $\mathscr{K}$ has codescent objects of cateads and they are effective.
3. Codescent morphisms are effective in $\mathscr{K}$.
4. Discrete objects in $\mathscr{K}$ are projective.
5. $\mathscr{K}$ has enough projectives.

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$$f \downarrow f \downarrow f \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows\!\!\to}}} f \downarrow f \underset{d_0}{\overset{d_1}{\underset{i}{\rightleftarrows\!\!\to}}} X \overset{f}{\twoheadrightarrow} Y$$

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$Y \in \mathscr{K}$ is discrete if

$$\mathbb{X} \underset{g}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \phi}}}} Y \implies f = g \text{ and } \phi = 1_f$$

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$$P \twoheadrightarrow \mathbb{Y}$$

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5. $\mathscr{K}$ has enough projectives.

Theorem (Carboni-Vitale)

An exact 1*-category is an exact completion if and only if it has enough projectives. In this case, it is the exact completion of its projective objects.*

Bourke's characterisation of $\mathbf{Cat}(\mathscr{E})$

Theorem (Bourke)

$\mathscr{E}$ *has pullbacks* $\implies$ $\mathbf{Cat}(\mathscr{E})$ *satisfies Bourke's axioms.*

Conversely, if $\mathscr{K}$ *satisfies Bourke's axioms*
$\implies$ $\mathscr{K} \simeq \mathbf{Cat}(\mathscr{E})$ *where* $\mathscr{E} := \mathbf{Disc}(\mathscr{K})$.

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Theorem (Bourke-Garner)

$\mathscr{E} \mapsto$ **Cat**$(\mathscr{E})$ *is a kind of 2-exact completion.*

Refining this

Theorem (H.)

Let $\mathscr{E}$ be a locally cartesian closed, lextensive category with coequalisers and a NNO
$\implies$ $\mathbf{Cat}(\mathscr{E})$ *satisfies* $(1)-(4)$.

Conversely $\mathscr{K}$ *satisfies* $(1)-(4) \implies \mathscr{K} \simeq \mathbf{Cat}(\mathscr{E})$ *for* $\mathscr{E}$ *a locally cartesian closed lextensive category with coequalisers and NNO.*

1. Bourke's axioms.
2. 2-lextensivity.
3. Discrete opfibrations are exponentiable.
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Follows from (Street-Verity).

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$\mathcal{K}$ satisfies $(1)-(2) \rightsquigarrow$ a "small $\mathcal{F}_{\mathrm{BO}}$-pretopos".

Examples

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- **Set**

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- Categories of assemblies **Asm** (cf. the effective topos (Hyland))

Homotopical structure

Theorem (Everaert-Kieboom-Van der Linden)

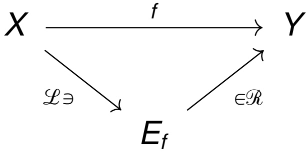
There is a model structure on **Cat**$(\mathcal{E})$*:*

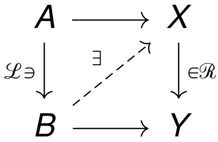
- *the weak equivalences are the representable weak equivalences.*
- *the fibrations are the representable isofibrations.*
- *the cofibrations are the complemented inclusion on objects functors.*

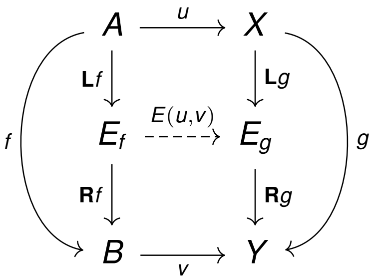
$f : \mathbb{X} \to \mathbb{Y}$ is a complemented inclusion on objects if $f_0 \cong \iota_{X_0} : X_0 \hookrightarrow X_0 + C$.

Outline

Functorial wfs







giving endofunctors $\mathbf{L}, \mathbf{R} : \mathbf{C}^\mathbf{2} \to \mathbf{C}^\mathbf{2}$

Relation to algebraicity

$\mathbf{L}, \mathbf{R} : \mathbf{C^2} \to \mathbf{C^2}$ are moreover (co)pointed!

$$
\begin{array}{ccc}
X & = & X \\
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An $(\mathbf{R}, \eta)$-algebra structure for f is given by a square

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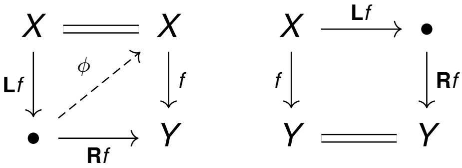

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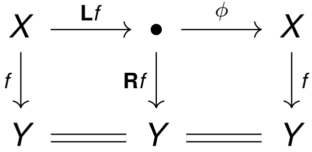

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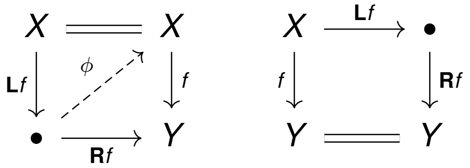

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Lemma

There exists a $(\mathbf{R}, \eta)$-algebra structure on $f \iff f \in \mathscr{R}$.

Algebraic weak factorisation systems

Definition (Grandis and Tholen)

An *algebraic weak factorisation system* on a category **C** is a pair $(\mathbb{L}, \mathbb{R})$ of a comonad and a monad on $\mathbf{C}^\mathbf{2}$ such that $(\overline{\mathbb{L}\text{-Coalg}}, \overline{\mathbb{R}\text{-Alg}})$ is a wfs.

Algebraic model structures

Definition (Riehl)

An *algebraic model structure* on a homotopical category $(\mathbf{C}, \mathscr{W})$ is a pair of algebraic weak factorisation systems $(\mathbb{TC}, \mathbb{F})$ and $(\mathbb{C}, \mathbb{TF})$ satisfying some conditions.

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Example

Any cofibrantly generated model structure!

Algebraic model structure

Theorem (H.)

1. *There is an algebraic model structure on* $\mathbf{Cat}(\mathcal{E})$.
2. *The (co)monads on this are described explicitly.*
3. *It has underlying model structure of (Everaert, Kieboom, Van der Linden).*
4. *It is cartesian monoidal.*
5. *It is cofibrantly generated.*
6. *The algebraic fibrations are the cloven isofibrations.*

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6. *The algebraic fibrations are the cloven isofibrations.*

Corollary: we can lift Everaert, Kieboom and Van der Linden's model structure to the category of $\mathbb{M}$-modules for an internal monoidal category $\mathbb{M}$.

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Groupoids

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The algebraic model structure on $\mathbf{Cat}(\mathscr{E})$ restricts to $\mathbf{Gpd}(\mathscr{E})$.

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There is a $(2,1)$-version of Bourke's theorem.
The algebraic model structure on $\mathbf{Cat}(\mathscr{E})$ restricts to $\mathbf{Gpd}(\mathscr{E})$.
Such a $(2,1)$-category models MLTT.

Type theoretic awfs

Definition (Gambino-Larrea)

A *type-theoretic algebraic weak factorisation system* on a category **C** is an awfs $(\mathbb{TC}, \mathbb{F})$ with some extra structure and satisfying certain conditions.

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Theorem (Gambino-Larrea)

Type theoretic awfs model MLTT with Σ, Π *and* $\mathtt{Id}$*-types.*

The $\mathbb{F}$-algebras model the dependent types.

Internal groupoidal model of MLTT

Theorem (H.)

The awfs $(\mathbb{TC}, \mathbb{F})$ *on the category* **Gpd**$(\mathcal{E})$ *is equipped with the structure of a type theoretic awfs.*

So cloven isofibrations form a model of MLTT.

Examples

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- Any presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. Note that $\mathbf{Gpd}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]) \cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Gpd}]$.
- Any Grothendieck topos.
- Any elementary topos with a natural numbers object.
- Arithmetic Π-pretoposes.
- Palmgren's CETCS.
- Categories of assemblies **Asm**

Outline

Realisability 2-topos

In joint work with Sam Speight:

- $\mathbf{Asm}^{\text{ex/reg}} = \mathscr{E}\mathbf{ff}$.
- There is a modest discrete opfib. classifier in $\mathbf{Cat}(\mathbf{Asm})$ (cf. Weber's elementary 2-toposes).
- It is not a Grothendieck 2-topos.
- It is a 2-category with a class of small discrete opfibs.
- $\mathbf{Gpd}(\mathbf{Asm})$ models **MLTT**. In this case the classifier becomes a univalent universe of small 0-types.

Related work by Awodey-Emmenegger and Agwu and HoTTLean (Hua, Awodey, Carneiro, Hazratpour, Nawrocki, Woolfson, Xu)

Directed type theory

Together with Fernando Chu:
Cat$(\mathscr{E})$ models directed type theory.

Summary

- For $\mathscr{K}$ a 2-category satisfying some axioms, $\mathscr{K}$ is of the form $\mathscr{K} \simeq \mathbf{Cat}(\mathscr{E})$ for richly structured $\mathscr{E}$.
- It therefore has an algebraic homotopy theory.
- Its $(2, 1)$-core models higher dimensional logic i.e MLTT.
- Such a thing should be an example of a small $(2, 1)$-pretopos.

Arxiv: The algebraic internal groupoid model of Martin-Löf type theory, 2025.

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