Homotopy Theory and Logic

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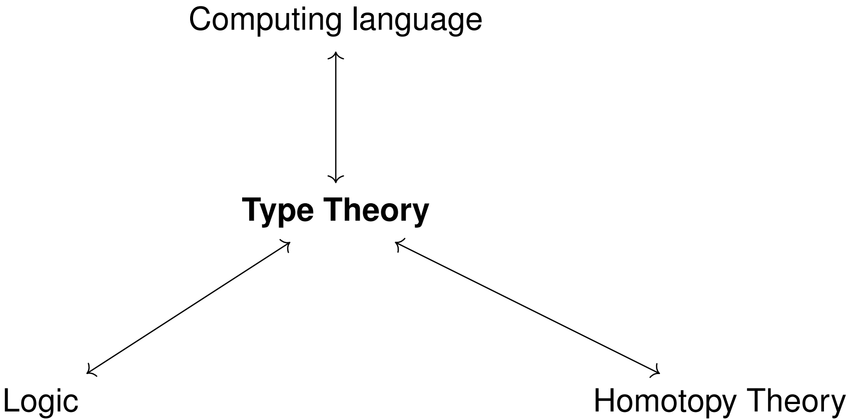
Homotopical Trinitarianism



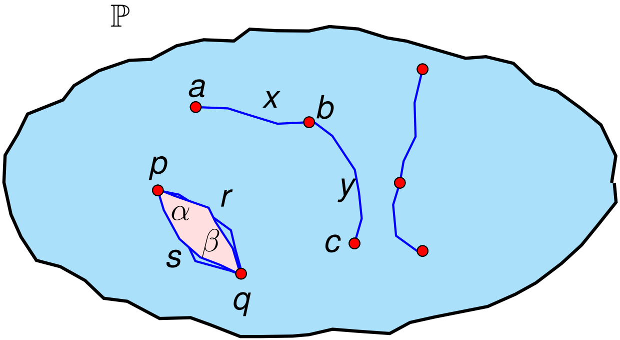
Martin-Löf Type Theory

Martin-Löf dependent type theory (MLTT):

- basis for proof assistants e.g. Lean, Coq, Agda,...
- **Dependent Types:** $a : A \vdash B(a)$. This gives Π-, Σ- and $\mathtt{Id}$- types.
- **Identity Types:** $p, q : P \vdash \mathtt{Id}_p(p, q)$.
- **Iterated identity types**:
  $x, y : \mathtt{Id}_p(p, q) \vdash \mathtt{Id}_{\mathtt{Id}_p(p,q)}(x, y)$...

Identity Types

- $a : A \rightsquigarrow \mathsf{refl}_a : \mathtt{Id}_A(a, a)$.
- $x : \mathtt{Id}_A(a, b) \rightsquigarrow \mathsf{sym}(x) : \mathtt{Id}_A(b, a)$.
- $x : \mathtt{Id}_A(a, b)$ and $y : \mathtt{Id}_A(b, c)$
  $\rightsquigarrow \mathsf{trans}(x, y) : \mathtt{Id}_A(a, c)$.
- $\mathsf{trans}(x, \mathsf{sym}(x)) = \mathsf{trans}(\mathsf{sym}(x), x) = \mathsf{refl}_a$.



Isofibrations

An isofibration in **Gpd** is

$$\begin{array}{ccc} \mathbb{B} & & b \\ \downarrow F & & \downarrow \\ \mathbb{A} & a \xrightarrow{\simeq} & F(b) \end{array}$$

Isofibrations

An isofibration in **Gpd** is

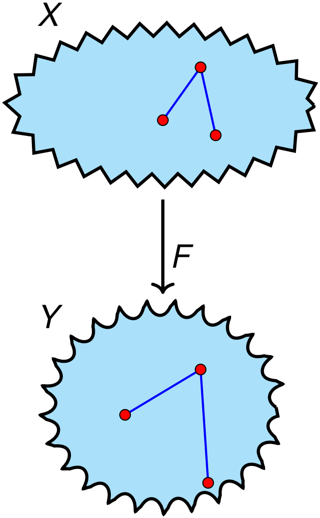
$$
\begin{array}{c} \mathbb{B} \\ \Big\downarrow{\scriptstyle F} \\ \mathbb{A} \end{array}
\qquad\qquad
\begin{array}{ccc} \exists b' & \overset{\simeq}{\dashrightarrow} & b \\ \vdots & & \big\downarrow \\ a & \xrightarrow{\simeq} & F(b) \end{array}
$$

Groupoidal Model

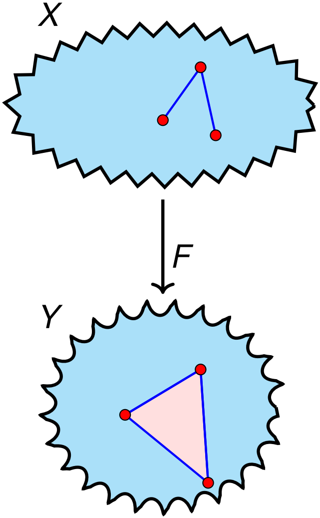
This is due first to Hofmann and Streicher [HS98]:

- Types are modelled by isofibrations.
- $\mathbb{P} \to \mathbf{1}$ is an isofibration, so all groupoids can be thought of as (non-dependent) types.
- There is no higher structure...

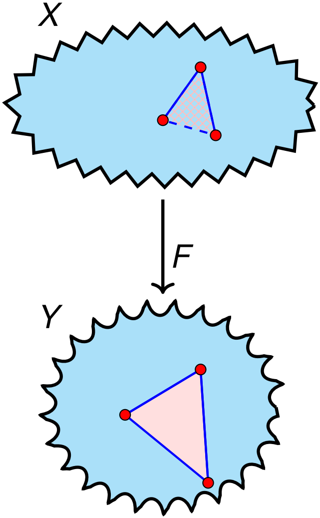
Kan Fibrations



Kan Fibrations



Kan Fibrations



Simplicial Model

This was due first to Voevodksy [KL12]:

- Types are modelled by Kan Fibrations.
- $\mathbb{P} \to \mathbf{1}$ is a Kan fibration if and only if $\mathbb{P}$ is an ∞-groupoid.
- There is higher structure.
- Univalence holds: $(A = B) \simeq (A \simeq B)$.

Model Structures

There is a link between models of MLTT and abstract homotopy theory.

Definition ([Qui67])

Let **M** be a category. A *Quillen model structure* on **M** consists of classes of maps $\mathscr{W}, \mathscr{C}, \mathscr{F}$ satisfying some conditions.

Examples of Model Structures

Example

There is a model structure on **Gpd** :

- $\mathscr{W} = \{$equivalences of categories$\}$
- $\mathscr{C} = \{$injective-on-objects functors$\}$
- $\mathscr{F} = \{$isofibrations$\}$

Examples of Model Structures

Example

There is a model structure on **sSet** :

- $\mathscr{W} = \{$homotopy equivalences$\}$
- $\mathscr{C} = \{$monomorphisms$\}$
- $\mathscr{F} = \{$Kan fibrations$\}$

Examples of Model Structures

Example ([GHSS22])

Let $\mathscr{E}$ be a category with some mild conditions. There is a model structure on $\mathbf{s}\mathscr{E}$:

- $\mathscr{W} = \{$homotopy equivalences$\}$
- $\mathscr{C} = \{$Reedy complemented inclusions$\}$
- $\mathscr{F} = \{$effective Kan fibrations$\}$

This is called the *effective model structure* on $\mathbf{s}\mathscr{E}$.

A model structure on internal groupoids

Theorem (H.)

For a suitable category $\mathcal{E}$, there is a model structure on $\mathbf{Gpd}(\mathcal{E})$:

- $\mathscr{W} = \{$*equivalences of categories*$\}$
- $\mathscr{C} = \{$*complemented inclusion-on-objects functors*$\}$
- $\mathscr{F} = \{$*internal isofibrations*$\}$

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- $\mathscr{F} = \{$*internal isofibrations*$\}$

Examples of $\mathscr{E}$:

- **Set**
- **Cat**
- $\mathbf{Psh}(\mathbb{C})$
- Any Grothendieck topos.
- The effective topos.

An internal groupoidal model of MLTT

Theorem (H.)

Let $\mathscr{E}$ be a category satisfying some conditions. The right adjoint splitting of the comprehension category associated to the

$$(\textbf{TrivCof}, \textbf{Fib})$$

algebraic weak factorisation system on $\mathbf{Gpd}(\mathscr{E})$ *is equipped with strictly stable choices of* Σ, Π *and* $\mathtt{Id}$*-types.*

i.e. internal isofibrations in $\mathbf{Gpd}(\mathscr{E})$ form a model of MLTT.

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Examples for $\mathcal{E}$:

- **Set**
- $\mathbf{Psh}(\mathbb{C})$
- Any Grothendieck topos.
- The effective topos.

A conjecture

Conjecture

Let $\mathscr{E}$ be a category satisfying some conditions. Then the the fibrations of the effective model structure on $\mathbf{s}\mathscr{E}$ give a model of HoTT.

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