Categorified Choice Principles

Calum Hughes

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MANCHESTER 1824
The University of Manchester

Outline

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1. Background
2. Categorifying
3. Type Theory (WIP)

The Axiom of Choice

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Axiom of choice

Every surjection $f : X \twoheadrightarrow Y$ has a splitting.

i.e. there exists $g : Y \to X$ such that

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

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External axiom of choice in $\mathcal{E}$ (AC)

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Example

- Assuming the axiom of choice, **Set** satisfies (AC).
- **FinSet** satisfies (AC).
- **Top** does not satisfy (AC).

The elementary theory of the category of sets

Definition (Lawvere)

A category $\mathcal{E}$ is said to satisfy the *elementary theory of the category of sets* (ETCS) if

- It is an elementary topos.
- It has a natural numbers object.
- It is well-pointed.
- It satisfies (AC).

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Morally, ETCS characterises the category of ZFC sets.

Weaker forms of choice

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For all $X \in \mathcal{E}$ there exists a choice object $P \in \mathcal{E}$ and an epimorphism $P \twoheadrightarrow X$.
Alternatively, we say $\mathcal{E}$ has *enough choice objects*.

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Clearly, (AC) $\implies$ (P) but not in general the converse.

The constructive elementary theory of the category of sets

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Morally, CETCS characterises the category of CZF sets/ setoids (cf. Aczel).

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	1D	2D
Object	$\mathscr{E}$ a 1-category (e.g. topos)	$\mathscr{K}$ a 2-category
internal logic	set theory (ZFC, CZF,...)	category theory
Key example	**Set**	**Cat**

Internal categories

Let $\mathscr{E}$ be a category with pullbacks.

Definition

A category internal to $\mathscr{E}$ is

$$\mathrm{Mor}(C) \times_{\mathrm{ob}(C)} \mathrm{Mor}(C) \xrightarrow{\text{comp}} \mathrm{Mor}(C) \;\underset{\text{target}}{\overset{\text{source}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\text{id}}}}}}\; \mathrm{Ob}(C)$$

These are the objects of a 2-category $\mathbf{Cat}(\mathscr{E})$.

Note: when $\mathscr{E} = \mathbf{Set}$, $\mathbf{Cat}(\mathscr{E}) = \mathbf{Cat}$.

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Note: when $\mathcal{E} = \mathbf{Set}$, $\mathbf{Cat}(\mathcal{E}) = \mathbf{Cat}$.
Idea: to check "correctness" of our 2D definitions $\rightsquigarrow$
$\mathcal{E}$ satisfies the 1D property iff $\mathbf{Cat}(\mathcal{E})$ satisfies 2D property.

Axiom of choice in **Cat**

Axiom of choice

Every surjection $f : X \twoheadrightarrow Y$ has a splitting.

is equivalent to the following axiom:

Axiom of choice III

Every essentially surjective on objects and fully faithful functor is an equivalence of categories.

Axiom of choice in **Cat**

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Axiom of choice III

Every surjective on objects and fully faithful functor is a split equivalence of categories.

i.e. given $F : \mathbb{C} \to \mathbb{D}$ that is surjective on objects and fully faithful there exists $G : \mathbb{D} \to \mathbb{C}$ such that $FG = \mathrm{id}_{\mathbb{D}}$ and there exists $\alpha : GF \cong \mathrm{id}_{\mathbb{C}}$.

Axiom of choice for internal categories

External axiom of choice in $\mathcal{E}$ (AC)

Every epimorphism $f : X \twoheadrightarrow Y$ in $\mathcal{E}$ has a splitting.

is equivalent to the following axiom:

External axiom of choice for $\mathbf{Cat}(\mathcal{E})$ (2AC)

Every epimorphic on objects and fully faithful internal functor is a split internal equivalence of internal categories.

i.e. given $F : \mathbb{C} \to \mathbb{D}$ that is epimorphic on objects and fully faithful there exists $G : \mathbb{D} \to \mathbb{C}$ such that $FG = \mathrm{id}_{\mathbb{D}}$ and there exists $\alpha : GF \cong \mathrm{id}_{\mathbb{C}}$.

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Problem: (2AC) cannot be phrased in an arbitrary 2-category.

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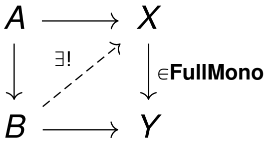

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$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \overset{\exists !}{\nearrow} & \big\downarrow {\scriptstyle \in \textbf{FullMono}} \\ B & \longrightarrow & Y \end{array}$$

External axiom of choice for $\mathscr{K}$ (2AC II)

Every acute and fully faithful 1-cell is a split equivalence.

i.e. given $F : X \to Y$ that is acute and fully faithful there exists $G : Y \to X$ such that $FG = \mathrm{id}_Y$ and there exists a 2-cell $\alpha : GF \cong \mathrm{id}_X$.

ET2CSC

Definition (H. and Miranda)

A 2-category $\mathscr{K}$ is said to satisfy the *elementary theory of the* 2*-category of small categories* (ET2CSC) if

- ...
- It satisfies (2AC II).

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Morally, ET2CSC characterises the 2-categorical properties of the 2-category of small categories in ZFC. What about the 2-category of small categories in CZF?

Categorified choice objects

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Example

In **Cat**$(\mathscr{E})$, the choice objects are precisely those $\mathbb{X} \in$ **Cat**$(\mathscr{E})$ such that $\mathrm{Ob}(X)$ is a choice object in $\mathscr{E}$.

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Such things have been studied:

- Anthony Agwu's PhD thesis.
- Steve Awodey and Jacapo Emmenegger's coherent groupoids.
- Cofibrant objects of a model structure on $\mathbf{Cat}(\mathscr{E})$ (Everaert-Kieboom-Van der Linden).
- They are related to $\mathscr{F}_{\mathrm{SO}}$-exactness (Bourke-Garner).

The categorified presentation axiom

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A 2-category $\mathscr{K}$ is said to satisfy the *constructive elementary theory of the 2-category of small categories* (ET2CSC) if

- It is $\mathscr{F}_{\mathrm{SO}}$-exact.
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- It satisfies (2P).

Theorem (H.)

There is a biequivalence

$$\mathbf{CETCS} \simeq \mathbf{ET2CSC}$$

Morally, CET2CSC characterises the properties of the 2-category of small categories in CZF/ setoids.

Outline

Models of MLTT

Restricting from **Cat**$(\mathcal{E})$ to **Gpd**$(\mathcal{E})$...

Theorem (H.)

For nice enough $\mathcal{E}$ *(e.g. satisfies ETCS or CETCS), the* $(2,1)$*-category* **Gpd**$(\mathcal{E})$ *models Martin-Löf Type Theory (MLTT) with* Σ*-,* Π*- and* $\mathtt{Id}$*-types.*

Groupoids model non-dependent types.
Trivial fibrations are exactly the epimorphic on objects split equivalences.

Projective types

Observation: split-epi on object and fully faithful functors are exactly the dependent types with contractible fibres.

Definition (WIP)

A type A is a choice type if for all $a : A \vdash B(a)$ type

$$\left(\exists_{b_0 : B(a)} \prod_{b : B(a)} \mathtt{Id}(b_0, b) \right) \to \left(\sum_{b_0 : B(a)} \prod_{b : B(a)} \mathtt{Id}(b_0, b) \right)$$

Modulo the correctness of the above definition, the choice groupoids $\mathbb{X} \in \mathbf{Gpd}(\mathcal{E})$ model the choice types.

Type Theoretic choice

Axiom of choice for MLTT (AC_{MLTT})

Every non-dependent type is projective.

Question: how does this compare to the axiom of choice already considered? (e.g. AC_{-1} in the HoTT book).

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Theorem

- $\mathcal{E}$ *satisfies (AC)* $\implies$ $\mathbf{Gpd}(\mathcal{E})$ *models MLTT* $+$ AC_{MLTT}
- $\mathcal{E}$ *satisfies (P)* $\implies$ $\mathbf{Gpd}(\mathcal{E})$ *models MLTT* $+$ P_{MLTT}

Summary

- Using internal categories, we came up with 2-dimensional abstract versions of the axiom of choice and the presentation axiom.
- These are interesting for 2-categorical foundations of mathematics.
- To do this, we had to come up with a 2-dimensional definition of projective object. This is related to various concepts in the literature.
- This has applications to formulating the axiom of choice and presentation axioms in higher dimensional logic such as MLTT.

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