An Algebraic Folk Model Structure for Internal Categories

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27th November 2023

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Model Structures

The following is refined from Quillen's original definition [Qui67].

Definition

Let $\mathbf{M}$ be a category. A *model structure* on $\mathbf{M}$ consists of three classes of maps $\mathbf{W}, \mathbf{Cof}, \mathbf{Fib}$ such that

- $\mathbf{W}$ satisfies 3-for-2.
- $(\mathbf{Cof} \cap \mathbf{W}, \mathbf{Fib})$ and $(\mathbf{Cof}, \mathbf{Fib} \cap \mathbf{W})$ form weak factorisation systems.

The Folk Model Structure on **Cat**

Theorem

There is a model structure on **Cat** :

- $\mathbf{W} = \{$*equivalences of categories*$\}$
- $\mathbf{Cof} = \{$*injective-on-objects functors*$\}$
- $\mathbf{Fib} = \{$*isofibrations*$\}$

This is called the folk model structure on **Cat**. *Moreover, it is cofibrantly generated by the sets*

$$I := \{\varnothing \to \mathbf{1}, \mathbf{1} + \mathbf{1} \to \mathbf{2}, \mathbf{P} \to \mathbf{2}\}$$

$$J := \{\mathbf{1} \to \mathbf{I}\}.$$

$\mathbf{Fib} \cap \mathbf{W}$ is the class of functors which are surjective on objects and fully faithful.

The Folk Model Structure on **Cat**

Lemma

Cofibrations lift against trivial fibrations.

(Proof Sketch)

Given $\begin{array}{ccc} \mathbf{A} & \to & \mathbf{X} \\ \scriptstyle\mathbf{Cof}\ni\downarrow & & \downarrow\scriptstyle\in\mathbf{TrivFib} \\ \mathbf{B} & \to & \mathbf{Y} \end{array}$, apply $(-)_0$: **Cat** $\to$ **Set** to obtain

$\begin{array}{ccc} A_0 & \to & X_0 \\ \downarrow & & \downarrow \\ B_0 & \to & Y_0 \end{array}$. Now, **assuming the axiom of choice**, (injective, surjective) is a weak factorisation system of **Set**, meaning we can find the lift. Use fully faithfullness of the trivial fibration to lift this to the required lift.

The Folk Model Structure on **Cat**

Theorem

Assuming the axiom of choice, *there is a model structure on* **Cat** :

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- **Fib** $= \{$*isofibrations*$\}$

This is called the folk model structure on **Cat**.

Internal Categories

Definition

A *small category* is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \underset{p_2}{\overset{\overset{p_1}{\longrightarrow}}{\overset{m}{\longrightarrow}}} C_1 \underset{target}{\overset{\overset{source}{\longrightarrow}}{\overset{i}{\longleftarrow}}} C_0$$

Where $C_0, C_1 \in \mathbf{Set}$.

Internal Categories

Definition

Let $\mathcal{E}$ be a category with pullbacks.
A *category internal to* $\mathcal{E}$ is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \; \overset{p_1}{\underset{p_2}{\overset{m}{\rightrightarrows\!\!\!\rightarrow}}} \; C_1 \; \overset{source}{\underset{target}{\overset{i}{\longleftarrow}}} \; C_0$$

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Where $C_0, C_1 \in \mathcal{E}$.

Examples:

- $\mathcal{E} =$ **Set** recovers the definition of small categories.
- $\mathcal{E} =$ **Cat** recovers the definition of double categories.
- $\mathcal{E} =$ **Man** and restricting to $C_0 = \mathbf{1}$ gives lie groups.
- We are interested in $\mathcal{E} = \mathbf{Set}_{\neg \mathrm{AC}}$.

Cat$(\mathscr{E})$

We can define internal functors so that we obtain a category **Cat**$(\mathscr{E})$.

Definition

An *internal functor* $F : \mathbb{X} \to \mathbb{Y}$ consists of a pair $F_0 : X_0 \to Y_0$, $F_1 : X_1 \to Y_1$ that respects sources, targets, identities and composition, e.g. $F_0 \circ \text{source} = \text{source} \circ F_1$.

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We can also define internal natural transformations and form a 2-category, but for us this will become part of the data of an equivalence of internal categories.

Naïve Statement

Theorem (Not)

Let $\mathcal{E}$ be a catgeory with some structure. There is a model structure on $\mathbf{Cat}(\mathcal{E})$:

- $\mathbf{W} = \{$*equivalences of internal categories*$\}$
- $\mathbf{Cof} = \{$*monomorphic-on-objects functors*$\}$
- $\mathbf{Fib} = \{$*internal isofibrations*$\}$ *(in the sense of Niefield-Pronk [NP19]).*

It is cofibrantly generated by some internal versions of I and J.

Problems

Problem: (mono, epi) is not generally a weak factorisation system on $\mathscr{E}$ unless $\mathscr{E}$ satisfies the external axiom of choice.

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Solution: when $\mathscr{E}$ is lextensive,

$$(\text{complemented inclusion, split epi})$$

is a weak factorisation system that recovers (mono, epi) when choice is true. [GSS22]

Problems

Problem: in the proof that lifting against $\varnothing \to \mathbf{1}$ gives a split epi on objects

$$\begin{array}{ccc} \varnothing & \longrightarrow & \mathbf{X} \\ \downarrow & \nearrow & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{Y} \end{array}$$

By Yoneda, $\mathrm{Hom}(\mathbf{1}, \mathbf{X}) \cong X_0$.
Internally, this cannot work as X_0 is not a set but an object of $\mathscr{E}$.

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By Yoneda, $\mathrm{Hom}(\mathbf{1}, \mathbf{X}) \cong X_0$.
Internally, this cannot work as X_0 is not a set but an object of $\mathscr{E}$.
Solution: Use enriched category theory and the enriched Yoneda lemma to show that $\mathrm{Hom}_{\mathscr{E}}(\mathbf{1}, \mathbb{X}) \cong X_0$.

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Solution: Use the *enriched* small object argument.

The Effective Model Structure

Theorem (The Effective Model Structure [GHSS22])

Let $\mathcal{E}$ be a lextensive category. Then there is a model structure on $s\mathcal{E} := [\Delta^{op}, \mathcal{E}]$ cofibrantly generated by internal versions of horn inclusions and boundary inclusions.

The wfs $(\mathbf{Cof}, \mathbf{TrivFib})$ is the Reedy weak factorisation system lifted from (Comp. inc, Split epi).

Main Theorem

Theorem (Naïve)

Let $\mathcal{E}$ be a Grothendieck topos. There is a model structure on **Cat**$(\mathcal{E})$ *:*

- **W** $=$ {*equivalences of internal categories*}
- **Cof** $=$ {*monomorphic-on-objects functors*}
- **Fib** $=$ {*internal isofibrations*} *(in the sense of Niefield-Pronk [NP19]).*

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- **Fib** $=$ {*internal isofibrations*} *(in the sense of Niefield-Pronk [NP19]).*

It is $\mathcal{E}$-cofibrantly generated by internal versions of I and J.

We call this the effective model structure on internal categories. Taking $\mathcal{E}$ $=$ **Set**, we get a constructive version of the folk model structure on **Cat**.

Internal Generating (Trivial) Cofibrations

For $\mathscr{E}$ a Grothendieck topos, the global sections functor has left adjoint:

$$\mathscr{E} \underset{\text{Hom}(\mathbf{1},-)}{\overset{\underline{(-)}}{\rightleftarrows}} \textbf{Set} \quad (\perp)$$

$$\coprod_{s \in S} \mathbf{1} \longleftarrow\!\shortmid \; S$$

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This preserves limits, and so extends to a functor $\underline{(-)} : \mathbf{Cat} \to \mathbf{Cat}(\mathscr{E})$.

Internal Generating (Trivial) Cofibrations

The generating (trivial) cofibrations of the folk model structure on **Cat** are

$$I := \{\varnothing \to \mathbf{1}, \mathbf{1} + \mathbf{1} \to \mathbf{2}, \mathbf{P} \to \mathbf{2}\}$$

$$J := \{\mathbf{1} \to \mathbf{I}\}.$$

Internal Generating (Trivial) Cofibrations

The "generating" (trivial) cofibrations of the effective model structure on **Cat**$(\mathscr{E})$ are

$$\underline{I} := \{\underline{\varnothing} \to \underline{\mathbf{1}}, \underline{\mathbf{1}} + \underline{\mathbf{1}} \to \underline{\mathbf{2}}, \underline{\mathbf{P}} \to \underline{\mathbf{2}}\}$$

$$\underline{J} := \{\underline{\mathbf{1}} \to \underline{\mathbf{I}}\}.$$

Enrichment

Recall the nerve functor

$$N : \mathbf{Cat}(\mathscr{E}) \to s\mathscr{E}$$

Definition

Let $\mathbb{X}, \mathbb{Y} \in \mathbf{Cat}(\mathscr{E})$. Define

$$\mathrm{Hom}_{\mathscr{E}}(\mathbb{X}, \mathbb{Y}) := \int_{[n]\in\Delta} NY_n^{NX_n}$$

This is precisely the "object of natural transformations" between $N\mathbb{X}, N\mathbb{Y} : \Delta^{op} \to \mathscr{E}$.

Enrichment

Definition

Fix $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$. We define an evaluation map

$$\mathbb{X}(-) : \quad \mathbf{Cat} \longrightarrow \mathcal{E}$$

$$K \longmapsto \mathrm{Hom}_{\mathcal{E}}(\underline{K}, \mathbb{X})$$

Enrichment

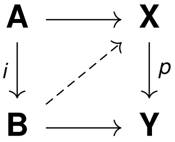
Lemma

Let X be an internal category. We can calculate the following:

1. $Hom_{\mathcal{E}}(\underline{\emptyset}, X) \cong \underline{\mathbf{1}}$.
2. $Hom_{\mathcal{E}}(\underline{\mathbf{1}}, X) \cong X_0$.
3. $Hom_{\mathcal{E}}(\underline{\mathbf{2}}, X) \cong X_1$.
4. $Hom_{\mathcal{E}}(\underline{\mathbf{1}} + \underline{\mathbf{1}}, X) \cong X_1 \times_{X_0} X_1$.
5. $Hom_{\mathcal{E}}(X, \underline{\mathbf{1}}) = \underline{\mathbf{1}}$.

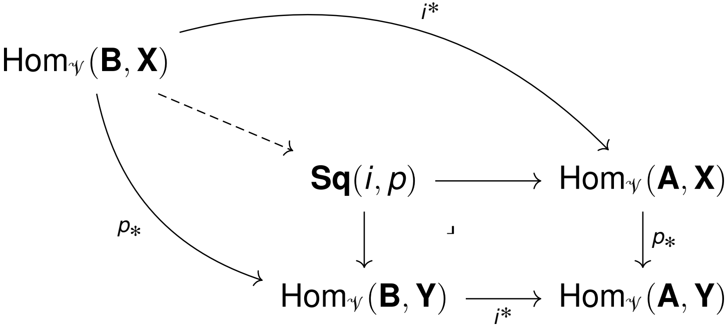
Enriched Lifting

Let $i : \mathbf{A} \to \mathbf{B}, p : \mathbf{X} \to \mathbf{Y}$.
Ordinary lifting: we say $i \boxslash p$ to mean that for any commutative square:



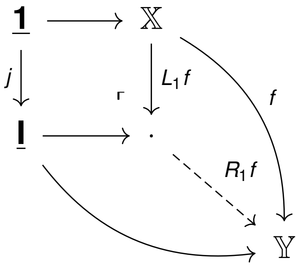
Enriched Lifting

Let $i : \mathbf{A} \to \mathbf{B}$, $p : \mathbf{X} \to \mathbf{Y}$.
Enriched lifting: we say $i \boxslash p$ to mean that the dotted arrow is a split epimorphism:



Enriched Small Object Argument

The algebraic small object argument applied to $\underline{J} = \{j : \underline{\mathbf{1}} \to \underline{\mathbf{I}}\}$ involves:



Enriched Small Object Argument

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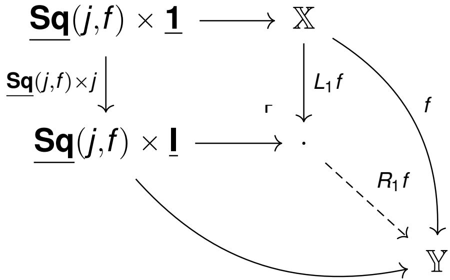

Internal Isofibrations

Definition

Internal isofibrations are maps which have the enriched right lifting property against $\underline{\mathbf{1}} \to \underline{\mathbf{I}}$.

Unraveling this, an internal isofibration is an internal functor $f : \mathbb{X} \to \mathbb{Y}$ such that

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$$\mathsf{Hom}_{\mathscr{E}}(\underline{\mathbf{I}}, \mathbb{X}) \to \mathsf{Hom}_{\mathscr{E}}(\underline{\mathbf{I}}, \mathbb{Y}) \times_{\mathsf{Hom}_{\mathscr{E}}(\underline{\mathbf{1}}, \mathbb{Y})} \mathsf{Hom}_{\mathscr{E}}(\underline{\mathbf{1}}, \mathbb{X})$$

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is a split epi.

$$\mathbb{X}(\mathbf{I}) \to \mathbb{Y}(\mathbf{I}) \times_{Y_0} X_0$$

is a split epi. This recovers the notion of internal isofibration between groupoids given in [NP19].

Lifting weak factorisation systems

In [GHSS22],

(comp. inc, split epi) on $\mathscr{E} \rightsquigarrow (\textbf{Cof}, \textbf{TrivFib})$ on $s\mathscr{E}$.

as the induced Reedy weak factorisation system.

Lifting weak factorisation systems

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as the induced Reedy weak factorisation system.
For this model structure:

$$\text{(comp. inc, split epi) on } \mathcal{E} \rightsquigarrow (\textbf{Cof}, \textbf{TrivFib}) \text{ on } \textbf{Cat}(\mathcal{E}).$$

via a morally similar lifting of the base weak factorisation system.

The Folk Model Structure on $\mathbf{Cat}(\mathcal{E})$

The following is from [EKVdL05]

Theorem

If $\mathcal{E}$ is a finitely complete category such that $\mathbf{Cat}(\mathcal{E})$ is finitely cocomplete then a model category structure is defined on $\mathbf{Cat}(\mathcal{E})$ by choosing **W** *the class of homotopy equivalences,* **Cof** *the class of functors having the homotopy extension property and* **Fib** *the class of functors having the homotopy lifting property.*

Algebraic Model Structures

The following is due to [Rie11].

Definition

Let **M** be a category, and let **W** be a class of maps satisfying 3-for-2. An *algebraic model structure* on $(\mathbf{M}, \mathbf{W})$ consists of two algebraic weak factorisation systems $(\mathbb{C}_t, \mathbb{F}), (\mathbb{C}, \mathbb{F}_t)$ together with a morphism of awfs $\xi : (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ such that the underlying ordinary wfs forms a model structure on **M** with **W** being the class of weak equivalences.

Algebraic Folk model Structure

Theorem

Let $\mathscr{E}$ be a Grothendieck topos. There is an ***algebraic*** *model structure on* $\mathbf{Cat}(\mathscr{E})$ *whose underlying ordinary model structure is the effective model structure on internal categories.*

Gpd$(\mathscr{E})$ and Models of Type Theory

Restrict attention to **Gpd**$(\mathscr{E})$. We have an awfs $(\mathbb{C}_t, \mathbb{F})$ such that algebras for $\mathbb{F}$ are isofibrations and coalgebras for $\mathbb{C}_t$ are trivial cofibrations.

Theorem ([GL19])

Let $(\mathbb{L}, \mathbb{R})$ be an awfs on a category **M** *with the structure of a type theoretic awfs. Then the right adjoint splitting of the comprehension category associated to $(\mathbb{L}, \mathbb{R})$ is equipped with strictly stable choices of* Σ-, Π- *and* $\mathtt{Id}$*-types.*

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Theorem

Let $\mathscr{E}$ be a Grothendieck topos. Then the right adjoint splitting of the comprehension category associated to $(\mathbb{C}_t, \mathbb{F})$ of **Gpd**$(\mathscr{E})$ *is equipped with strictly stable choices of* Σ*-,* Π*- and* $\mathtt{Id}$*-types.*

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Taking $\mathcal{E}$ = **Set**, we get a constructive model of MLTT.

Further Work

- Let $\mathscr{E} = \mathbf{Cat}$ and compare to existing notions of model structures on double categories.
- Investigate the properties when $\mathscr{E} = \mathrm{Asm}^{K_1}$.
- Let $\mathscr{E} = \mathbf{sSet}$ and compare it to Horel's.
- Compare with the Joyal-Tierney model structure on $\mathbf{Cat}(\mathscr{E})$.

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