A Visual Introduction to Homology and Homotopy

Calum Hughes

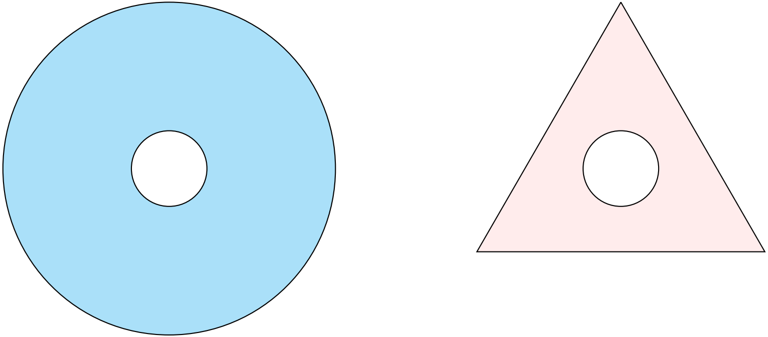
School of Mathematics and Statistics, University of Sheffield

28th April 2022

The University Of Sheffield.

Topological Structures

Are these two shapes the *same*?



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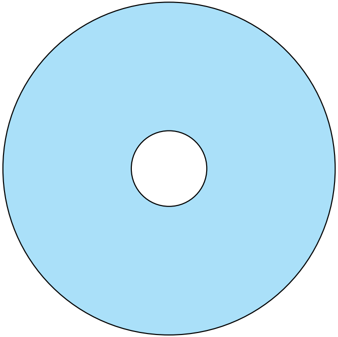

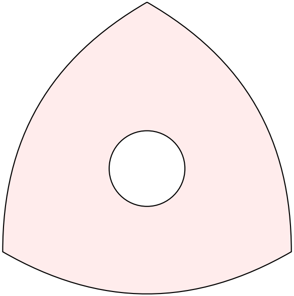

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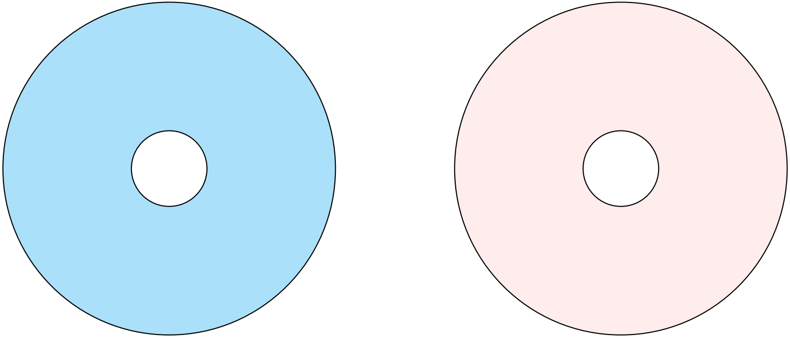
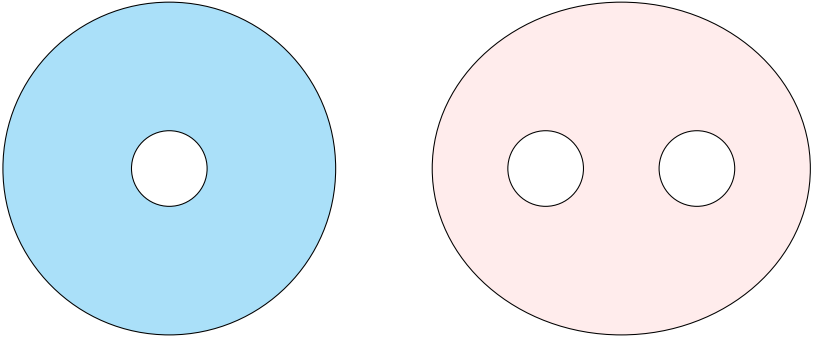


Topological Structures

How about these two?



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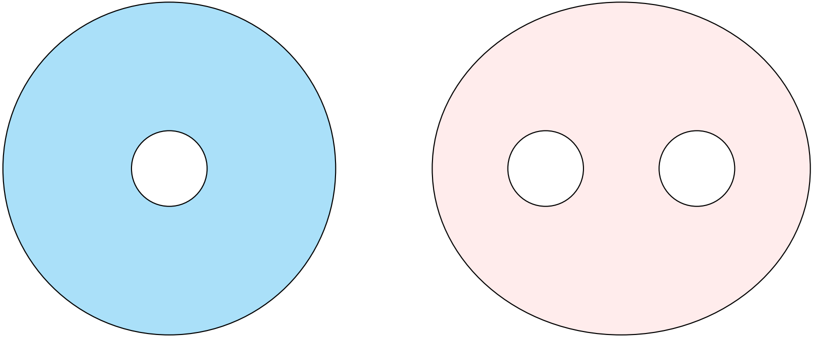
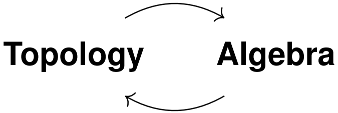


How would we go about *proving* that they are not?

Algebraic Topology

The aim of algebraic topology is to translate topological questions into algebraic ones.



Building Blocks

We use simplices as 'building blocks' . These can be thought of as n-dimensional triangles.

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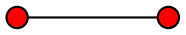
- The 1-simplex is a line. We call this Δ^1. 

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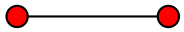
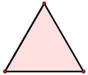
- The 1-simplex is a line. We call this Δ^1. 
- The 2-simplex is a triangle. We call this Δ^2.



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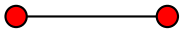
- The 1-simplex is a line. We call this Δ^1. 
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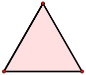
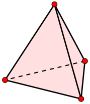


- The 3-simplex is a tetrahedron. We call this Δ^3.



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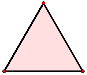

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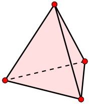
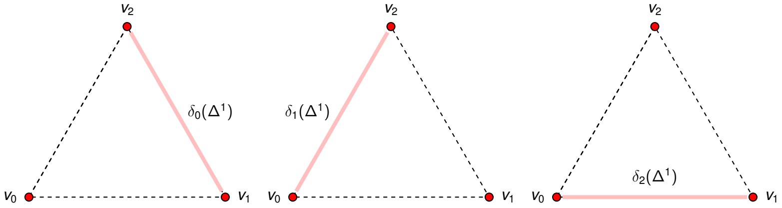
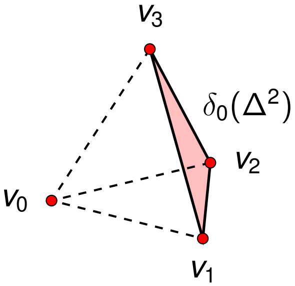


- and so on up to higher dimensions...

Face maps

We can consider maps onto the faces of simplices. These maps help us "glue" the shape back together.

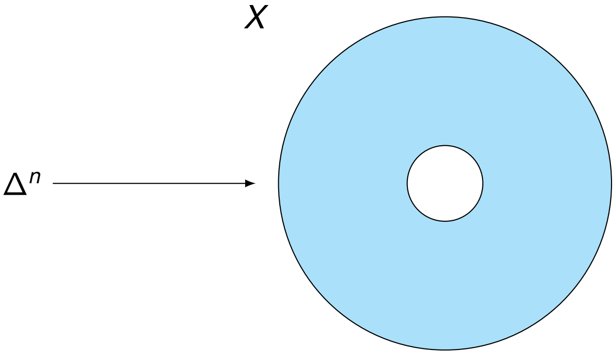




Singular Complex

Let X be a topological space.
Consider the set:

$$S_n X = \{u : \Delta^n \to X : \ u \text{ is continuous}\}$$



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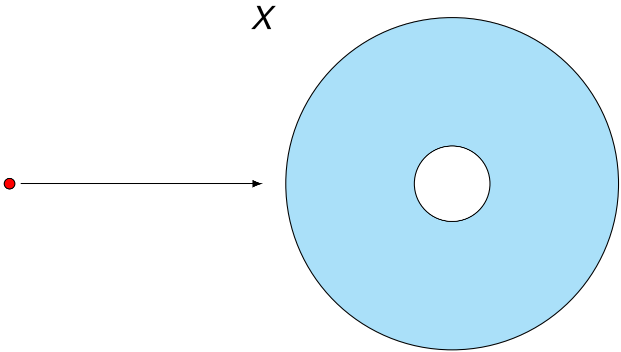

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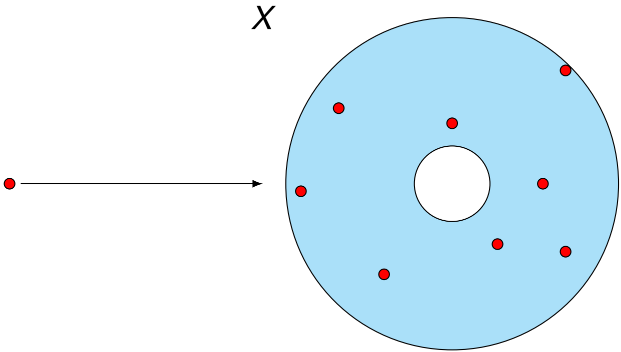

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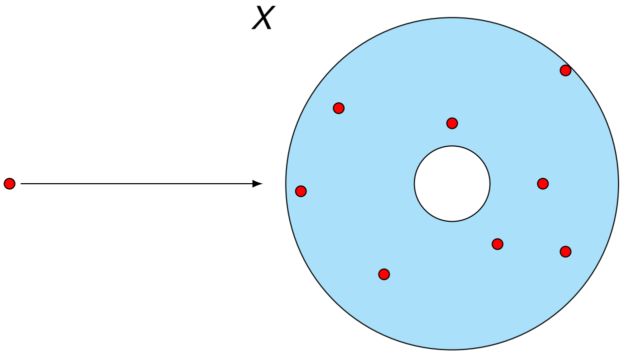

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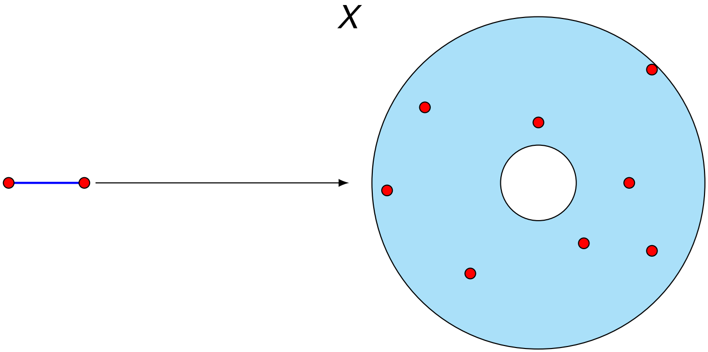

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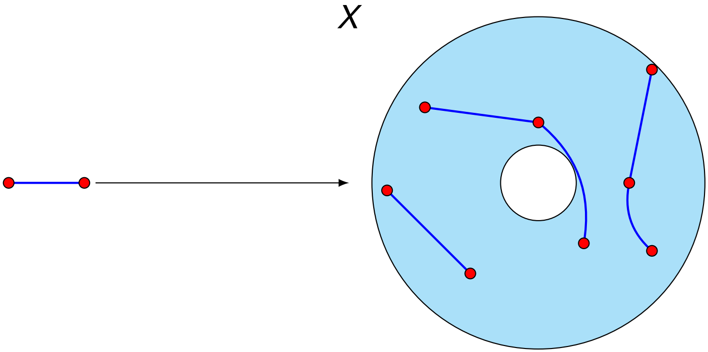

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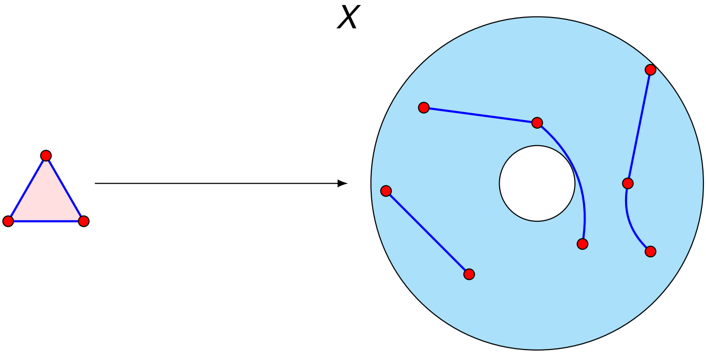

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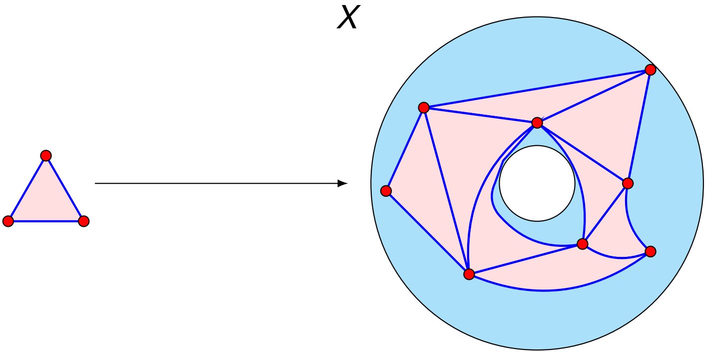

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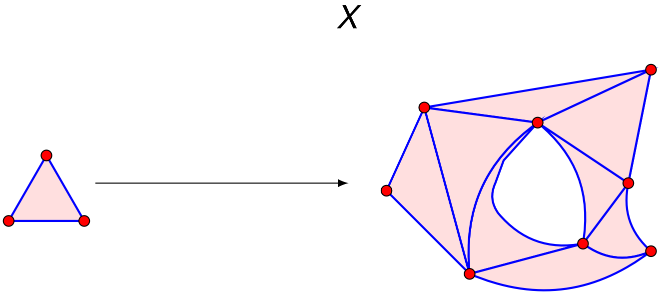

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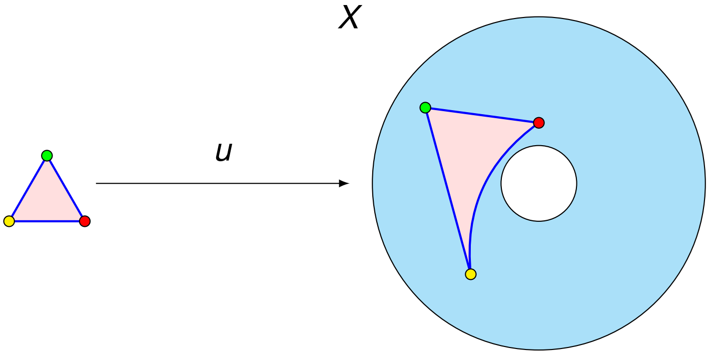

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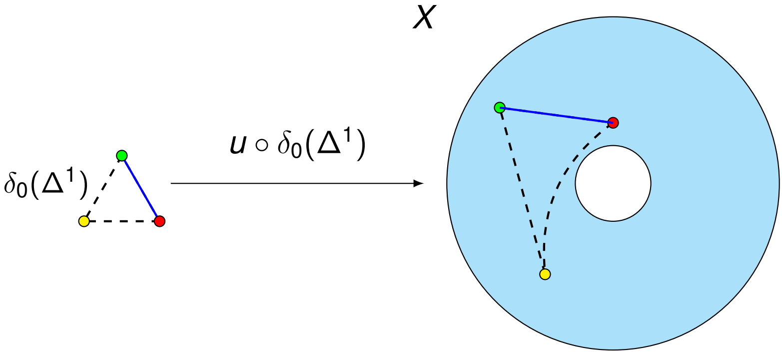

The translation

$$\left\{ \begin{array}{c} \text{Topological} \\ \text{spaces} \end{array} \right\} \underset{|-|}{\overset{S_*}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Simplicial} \\ \text{Sets} \end{array} \right\}$$

$$X \xmapsto{\quad S_* \quad} S_*X$$

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$$X \xmapsto{\quad S_* \quad} S_*X \xrightarrow{\quad \mathbb{Z} \quad} \mathbb{Z}S_*X$$

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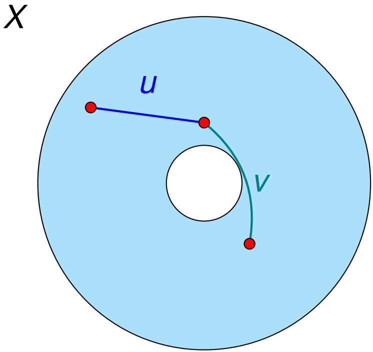
$$3u_1 - 4u_2 + 15u_3 \in \mathbb{Z}S_2X$$

Why?



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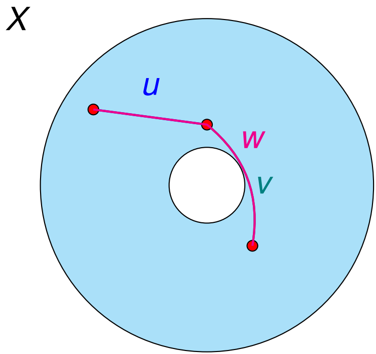

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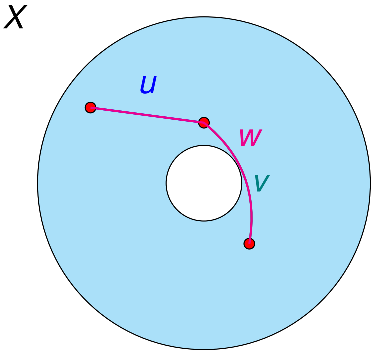
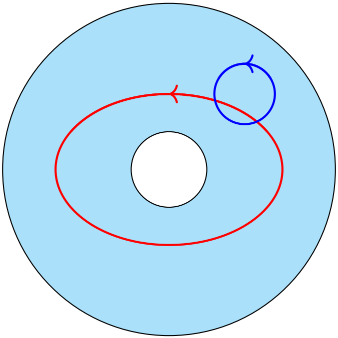


We would like to be able to say something like $u + v = w$.

Homology



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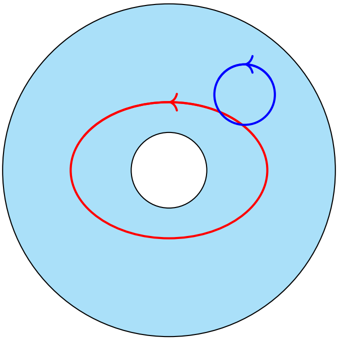

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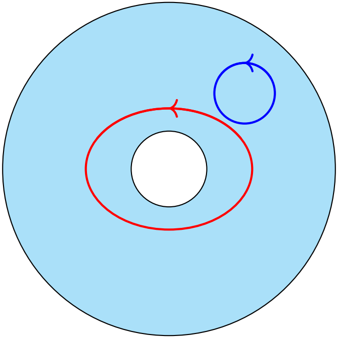

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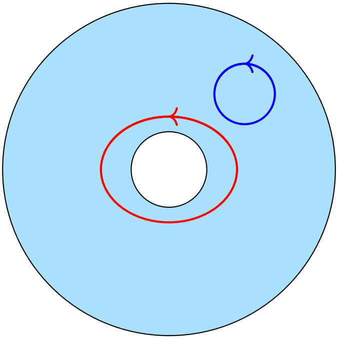

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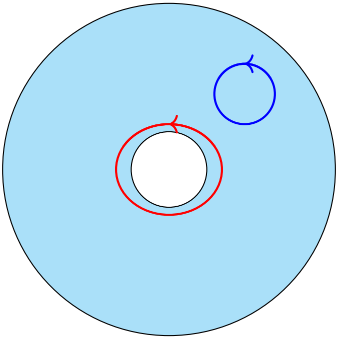

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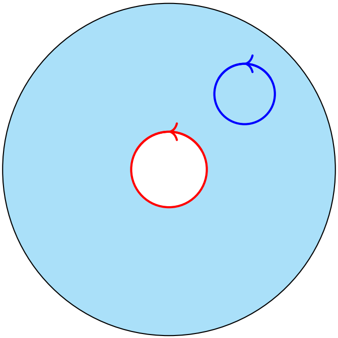

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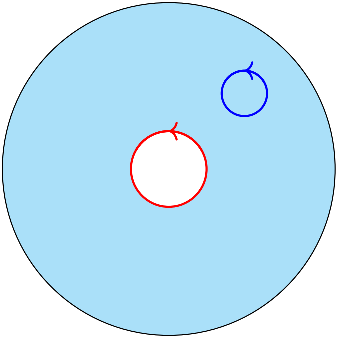

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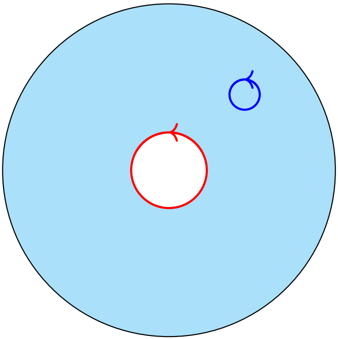

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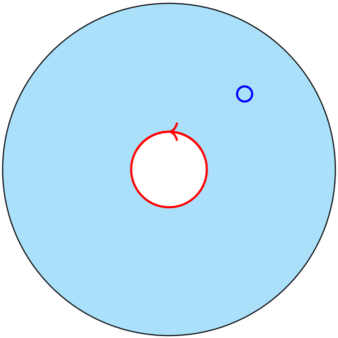

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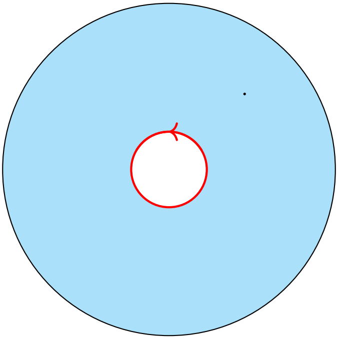

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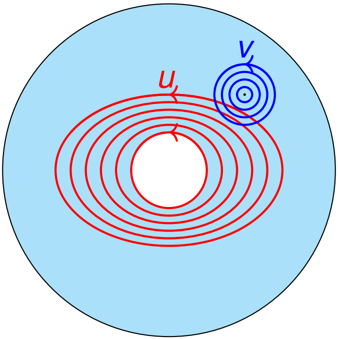
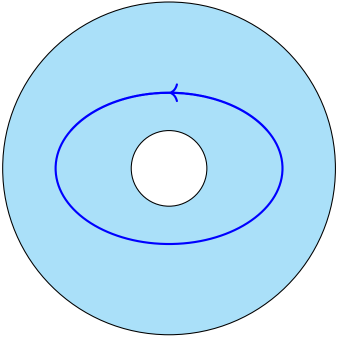


Simplicial Homology



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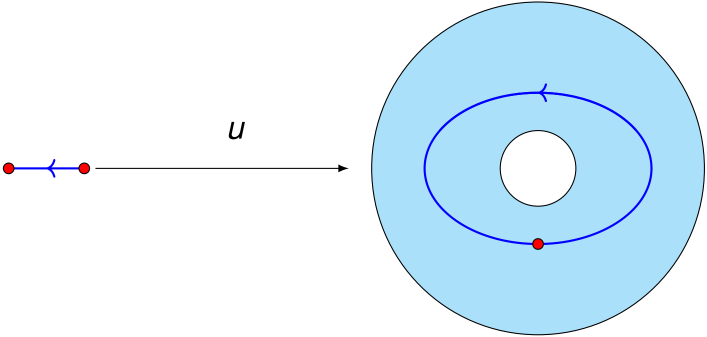

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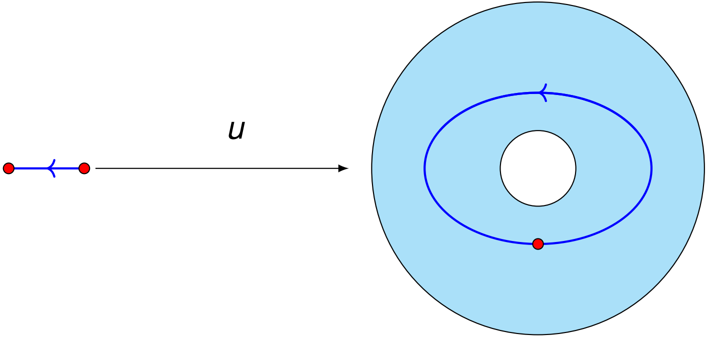

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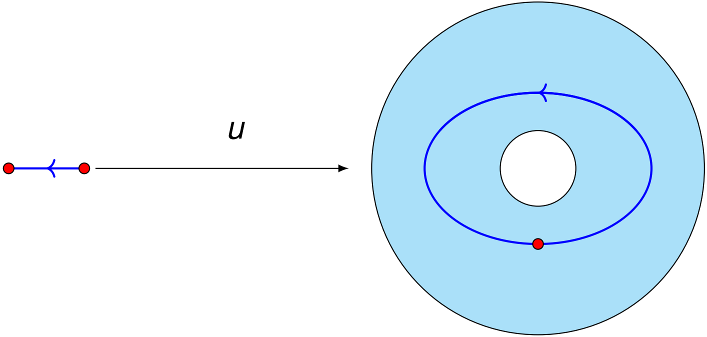

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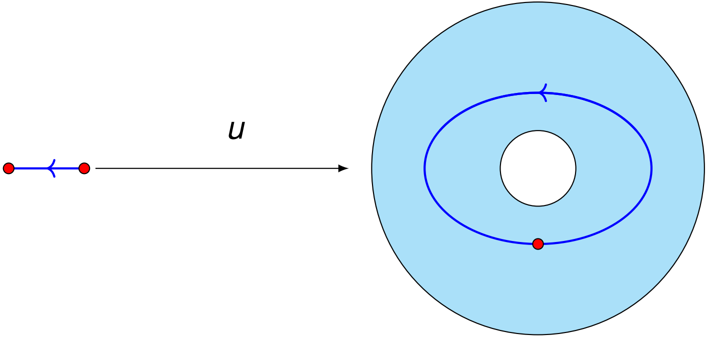

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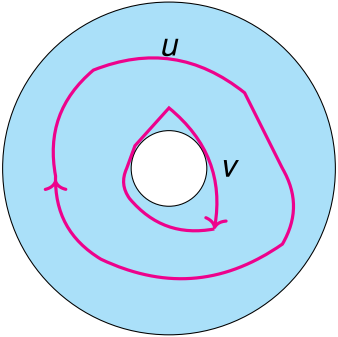

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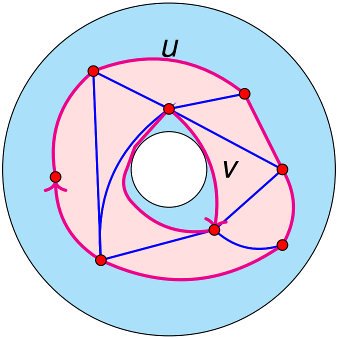

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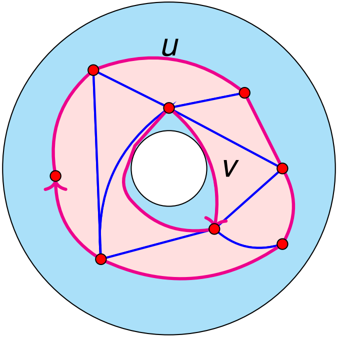

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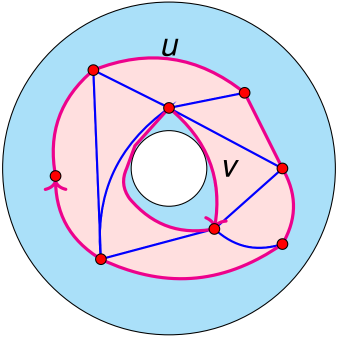

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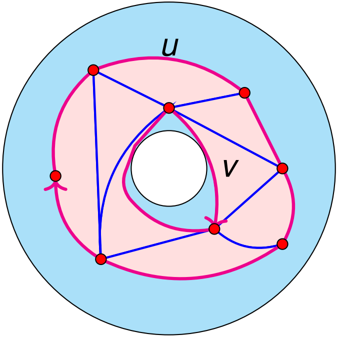

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Homology Groups

We describe this situation algebraically:

- Both $\mathrm{Im}(d_{n+1})$ and $\ker(d_n)$ are subgroups of $\mathbb{Z}S_nX$.

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- This is an invariant under homeomorphism of topological spaces: if X is homeomorphic to Y, then $H_n(\mathbb{Z}S_nX) = H_n(\mathbb{Z}S_nY)$ for all n.

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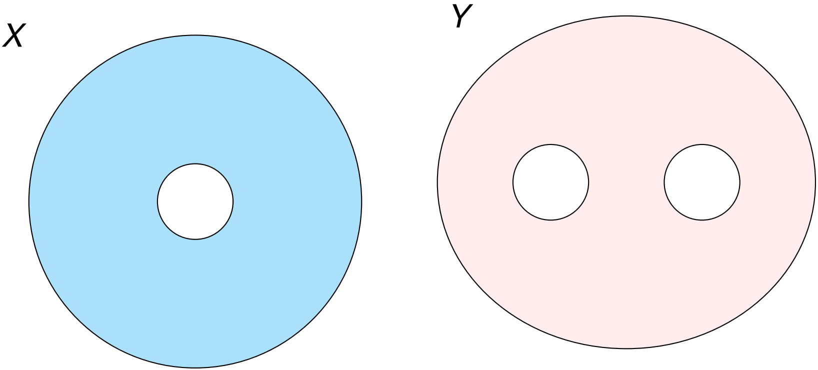
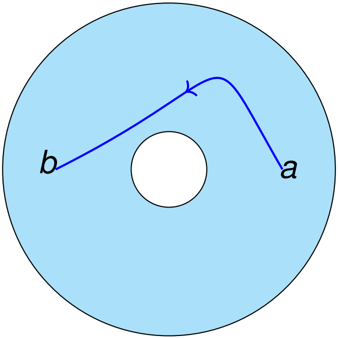
- Both $\operatorname{Im}(d_{n+1})$ and $\ker(d_n)$ are subgroups of $\mathbb{Z}S_nX$.
- $d_n \circ d_{n+1} = 0$, so $\operatorname{Im}(d_{n+1})$ is a subgroup of $\ker(d_n)$.
- Therefore, we can form the quotient group $H_n(\mathbb{Z}S_nX) = \ker(d_n)/\operatorname{Im}(d_{n+1})$.
- This identifies cycles that can be morphed into one another!
- This is an invariant under homeomorphism of topological spaces: if X is homeomorphic to Y, then $H_n(\mathbb{Z}S_nX) = H_n(\mathbb{Z}S_nY)$ for all n.
- The converse is often useful: if $H_n(\mathbb{Z}S_nX) \neq H_n(\mathbb{Z}S_nY)$, then X is not homeomorphic to Y.

Back to our example...

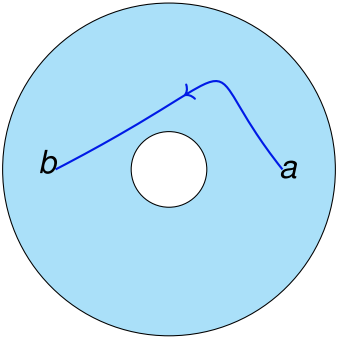


We can caluclate that $H_1(\mathbb{Z}S_1X) \cong \mathbb{Z}$, whereas $H_1(\mathbb{Z}S_1Y) \cong \mathbb{Z}^2$ and so X is ***not*** homeomorphic to Y!

Homotopy



Homotopy



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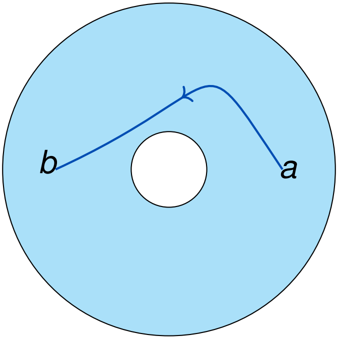

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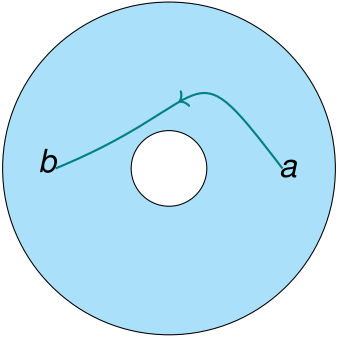

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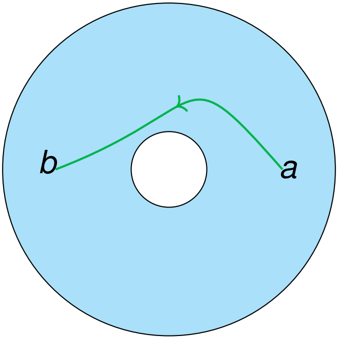

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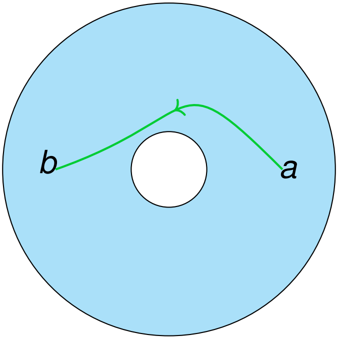

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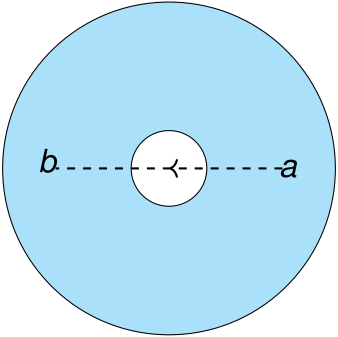

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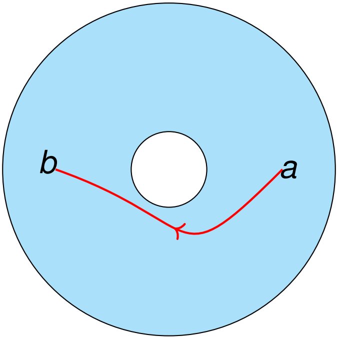

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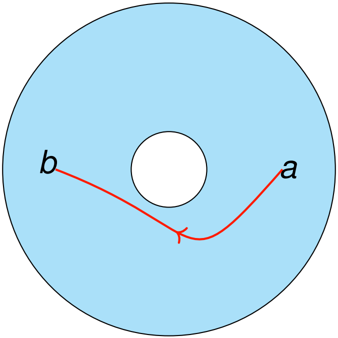

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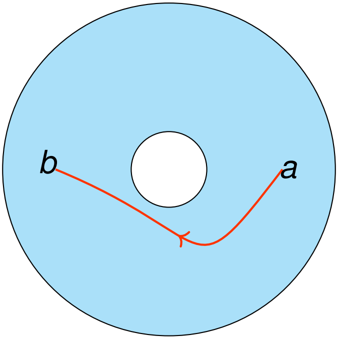

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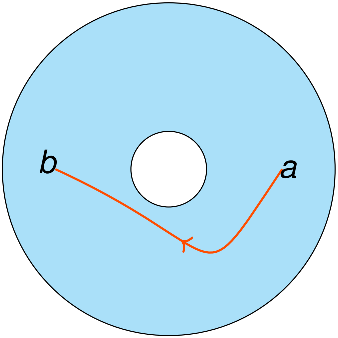

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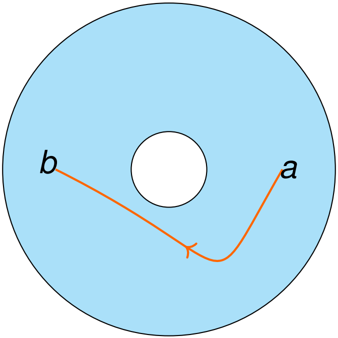

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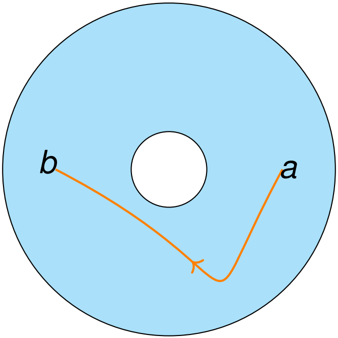

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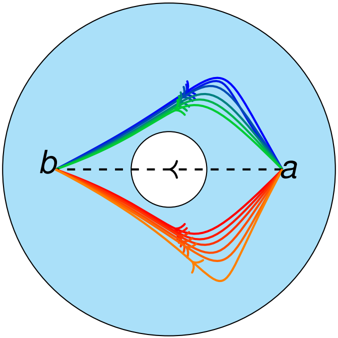
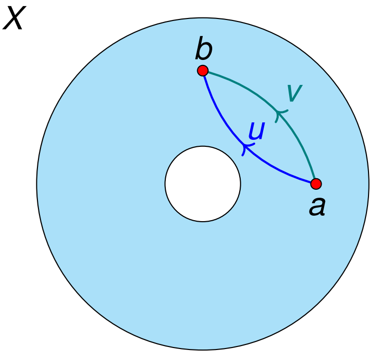


Simplicial Homotopy



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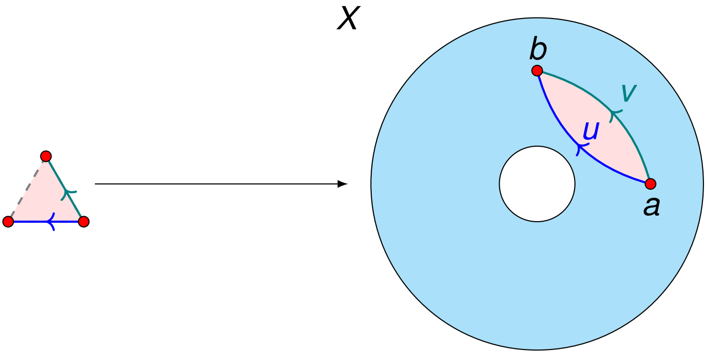

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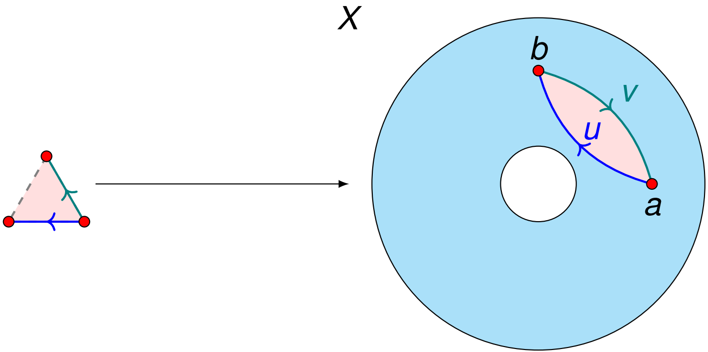

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Theorem

Let X be a topological space. Then

$$\pi_n(\mathbb{Z}SX) = H_n\left(\bigcap_{i=0}^{n-1} \ker(\delta_i : \mathbb{Z}S_nX \to \mathbb{Z}S_{n-1}X)\right)$$

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  This allows us to do homotopy theory in a more general setting.
- Abstract homotopy is useful in many other areas, such as computer science and logic with the invention of Homotopy Type Theory.

References

For further reading, I recommend:

- To learn more about homological algebra: Charles. A Weibel *An Introduction to Homological Algebra*, 1995.
- To learn more about simplicial sets: Greg Friedman *An Elementary Illustrated Introduction to Simplicial Sets* , 2011.
- For a very readable introduction category theory, a great souce is: Emily Riehl, *Category Theory in Context*, 2014.