Class $(2,1)$-categories

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HoTT MURI 2025
work in progress

MANCHESTER 1824
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Outline

Elementary topos theory

	1-cats	$(2,1)$-cats
Object	elementary topos	
internal logic	0 dimensional MLTT	
Key example	**Set**	

Definition (Lawvere-Tierney)

An elementary topos is a cartesian closed category with finite limits and a subobject classifier.

Elementary topos theory

	1-cats	$(2,1)$-cats
Object	elementary topos	Weber $(2,1)$-topos
internal logic	0 dimensional MLTT	?
Key example	**Set**	...?

Definition (Weber)

An elementary $(2,1)$-topos is a cartesian closed $(2,1)$-category with finite limits and a discrete opfibration classifier and a duality involution.

Example of an Weber $(2,1)$-topos

Gpd the $(2,1)$-category of small groupoids:

- finite limits and duality involution ✓
- cartesian closed ✓
- discrete opfib classifier ✓✗ (i.e. $\top \to \{\bot, \top\}$)

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GPD$_\mu$ the $(2,1)$-category of μ-small groupoids for some $\mu > \lambda$

- finite limits and duality involution ✓
- cartesian closed ✓
- discrete opfib classifier ✓(i.e **Set**$_{\lambda *}$ $\to$ **Set**$_\lambda$)

The category of classes

Von Neumann-Bernays-Gödel class theory:
Class has:

- objects: $\{x \text{ a set} : \phi(x) \text{ is true for } \phi \text{ a formula in FOL}\}$
- morphims: class functions

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We define **GPD** $:=$ **Gpd**(**Class**).

Class categories

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internal logic	small obs: 0D MLTT	
Key example	**Class**	

Class categories: (Joyal-Moerdijk), but see also (Awodey-Butz-Simpson-Streicher, van den Berg-Moerdijk, Simpson,...)

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Outline

1. Motivation
2. Axioms
3. Properties
4. Extra axioms
5. Future work

Cateads

Definition (Bourne-Penon, Bourke)

For a $(2,1)$-category $\mathscr{K}$, a *catead* is

$$C_1 \times_{C_0} C_1 \underset{p_2}{\overset{p_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{m}}}}} C_1 \underset{d_0}{\overset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{i}}}}} C_0$$

such that (d_1, d_0) forms a 2-sided discrete fibration.
We call its 2-colimit a *codescent object*.

Codescent morphisms are a $(2,1)$-dimensional analogue of a regular epimorphism in a 1-category.

Exactness

Given $f : X \to Y$

$$f \downarrow f \downarrow f \;\begin{array}{c}\xrightarrow{p_1}\\ \xrightarrow{m}\\ \xrightarrow[p_2]{}\end{array}\; f \downarrow f \;\begin{array}{c}\xrightarrow{d_1}\\ \xleftarrow{i}\\ \xrightarrow[d_0]{}\end{array}\; X \overset{q}{\twoheadrightarrow} C$$

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$$

Definition (Bourke-Garner)

A $(2,1)$-category $\mathscr{K}$ is called $\mathscr{F}_{BO}$*-regular* if it has finite $(2,1)$-limits and codescent objects of higher kernels exist and are closed under $(2,1)$-pullback.
It is called $\mathscr{F}_{BO}$*-exact* if codescent objects and morphisms are effective.

Pre-class structure

Let $\mathscr{K}$ be an $\mathscr{F}_{\mathsf{BO}}$-regular and extensive $(2,1)$-category, $^\circ : \mathscr{K}^{\mathsf{co}} \to \mathscr{K}$ a duality involution and $\mathscr{S}$ a class of discrete opfibrations. We call $(\mathscr{K}, ^\circ, \mathscr{S})$ a *pre-class* $(2,1)$*-category.*

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We call a discrete object $X \in \mathbf{Disc}(\mathscr{K})$ *small* if $! : X \to \mathbf{1}$ is in $\mathscr{S}$.
We call a general object $\mathbb{X} \in \mathscr{K}$ *small* if there exists a small discrete object and a codescent morphism $q : X \twoheadrightarrow \mathbb{X}$, such that $(s,t) : q \downarrow q \to X^\circ \times X$ is in $\mathscr{S}$.

$$\begin{array}{ccc} q \downarrow q & \xrightarrow{\;s\;} & X \\ {\scriptstyle t}\downarrow & \cong & \downarrow{\scriptstyle q} \\ X & \underset{q}{\twoheadrightarrow} & \mathbb{X} \end{array}$$

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Define the full sub-$(2,1)$-category of small objects by $\mathscr{K}_{\mathrm{small}}$.

Axioms

Let $(\mathscr{K}, ^\circ, \mathscr{S})$ be a pre-class $(2,1)$-category. Consider:

1. Replacement.
2. Stability.
3. $0 \to \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \to \mathbf{1}$ belong to $\mathscr{S}$.
4. Sums.
5. Quotients.
6. Exponentiality.
7. Representability.
8. Cancellability.
9. Small NNO.
10. Small projectivity.
11. Small exactness

Any isomorphism is in $\mathscr{S}$ and $\mathscr{S}$ is closed under composition.

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In any $(2,1)$-pullback square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle G}\downarrow & \cong & \downarrow{\scriptstyle F} \\ B & \longrightarrow & Y \end{array}$$

If $F \in \mathscr{S}$ then $G \in \mathscr{S}$.

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If $X \to Y$ and $X' \to Y'$ belong to $\mathscr{S}$ then so does $X + X' \to Y + Y'$.

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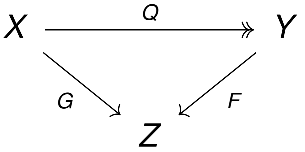
In any commutative diagram



where F and G are discrete opfibrations, if Q is codescent and G belongs to $\mathscr{S}$ then so does F.

Axioms

Let $(\mathscr{K}, ^\circ, \mathscr{S})$ be a pre-class $(2, 1)$-category. Consider:

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Every map in $\mathscr{S}$ is exponentiable

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There exists a classifier for maps in $\mathscr{S}$, which is iteself in $\mathscr{S}$.

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Let $F : X \to Y$ and $G : Y \to Z$ be discrete opfibrations. If $GF \in \mathscr{S}$ then $F \in \mathscr{S}$.

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Every small discrete object in $\mathscr{K}$ is projective, i.e.

$$\mathrm{Hom}_{\mathbf{Gpd}}(X, -) : \mathscr{K} \to \mathbf{Gpd}$$

preserves codescent morphisms.

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Effectivity of small codescent objects/ morphisms.

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Definition

If $(\mathscr{K}, \circ, \mathscr{S})$ satisfies 1-11, we call it a *class* $(2,1)$*-category.*

Examples

$\mathscr{K}$	$\mathscr{K}_{\text{small}}$	$\mathscr{S}$
GPD	**Gpd**	**Set**-sized fibers

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- There is a Grothendieck construction.

Properties

Let $(\mathscr{K}, \circ, \mathscr{S})$ be a class $(2,1)$-category.

Theorem

The $(2,1)$-category $\mathscr{K}_{small} \simeq \mathbf{Gpd}(\mathscr{E})$ for $\mathscr{E} := \mathbf{Disc}(\mathscr{K})$. Moreover, $\mathscr{E}$ is a locally cartesian closed, extensive category with a natural numbers object.

The hard part of this follows from John Bourke's PhD thesis.

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Theorem

$\mathscr{K}_{small}$ is a model of MLTT. Therefore $\mathscr{K}$ models MLTT with a univalent universe of small 0-types.

(See HoTTLEAN)

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Extra axioms

Let $(\mathscr{K}, \circ, \mathscr{S})$ be a class $(2,1)$-category. Consider:
Inspired by Lurie's ∞-toposes...

- $\mathscr{K}$ has a small full subobject classifier.

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Let $(\mathscr{K}, \circ, \mathscr{S})$ be a class $(2,1)$-category. Consider:
Inspired by Lawvere...

- $\mathscr{K}$ has a small full subobject classifier.
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- $\mathscr{K}$ satisfies the categorified axiom of choice.

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Therefore, $\mathscr{K}_{\text{small}}$ has the same logical power as ETCS, which has the same logical power as ZFC without replacement.

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Therefore, $\mathscr{K}$ has the same logical power as ZFC.

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Future work

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- Stability under (op)fibrational slicing.

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- More algebraic set theory.
- Removing the duality involution.
- Considering $\mathbf{Gpd}(\mathbf{Asm}_A)$...
- Optimising the axioms.
- Stability under (op)fibrational slicing.
- Comparison to Joseph Helfer's 2-toposes $\rightsquigarrow$ the classifier is an "internal 1-topos".

Summary

	1-cats	$(2,1)$-cats
Object	class categories	class $(2,1)$-categories
internal logic	small obs: 0D MLTT	small obs: 1D MLTT
Key example	**Class**	**GPD**

Adding axioms to a class $(2,1)$-category, we can give an $(2,1)$-categorical description of a logic which is as powerful as ZFC.

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