COLIMITS OF INTERNAL CATEGORIES

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Abstract. We show that for a list-arithmetic pretopos $\mathcal{E}$ with pullback stable coequalisers, the 2-category $\mathbf{Cat}(\mathcal{E})$ of internal categories, functors and natural transformations has finite 2-colimits.

Contents

1. Introduction

1.1. **Context and Motivation.** It is well known that **Cat** has finite colimits, with coproducts computed at the level of underlying simplicial sets. On the other hand, the coequaliser of a parallel pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ has a more complicated description involving not just equivalence classes of objects and morphisms of $\mathcal{D}$ but also equivalence classes of paths, as described in [BBP99].

The goal of this work is to provide conditions on a category $\mathcal{E}$ such that the 2-category $\mathbf{Cat}(\mathcal{E})$ of internal categories, internal functors, and internal natural transformations has finite 2-colimits. It is well-known that to show that a 2-category has finite 2-colimits it suffices to show that it has coproducts, copowers by the free-living arrow in **Cat** (which we denote **2**) and coequalisers (See ([Kel89], §3) for example). Lextensivity of $\mathcal{E}$ suffices for coproducts and copowers by **2** to exist in $\mathbf{Cat}(\mathcal{E})$, as shown in Lemma 5.2 and Theorem 5.5 of [HM24b] and reviewed in Section 3. In contrast, exactness properties between coequalisers and pullbacks in $\mathcal{E}$ only give rise to very special coequalisers in $\mathbf{Cat}(\mathcal{E})$, as treated in Section 4. The following example illustrates that exactness properties in $\mathcal{E}$ are insufficient for $\mathbf{Cat}(\mathcal{E})$ to have coequalisers.

Example 1.1. Consider the following diagram in $\mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{FinSet}$, the category of finite sets. The two functors in this diagram pick out the source and target of the free-living arrow.

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$$\mathbf{1} \underset{d^0}{\overset{d^1}{\rightrightarrows}} \mathbf{2}$$

The coequaliser of this diagram in $\mathbf{Cat}(\mathbf{Set})$ is given by the monoid of natural numbers, which is not a finite category and hence does not live in $\mathbf{FinCat} := \mathbf{Cat}(\mathbf{FinSet})$.

Internalising this construction to some category $\mathcal{E}$, one obtains the free monoid on the terminal object in $\mathcal{E}$ if this free monoid exists. Remark D5.3.4 of [Joh02a] shows that in an elementary topos $\mathcal{E}$, the existence of such a free monoid is equivalent to $\mathcal{E}$ having a natural numbers object. In the absence of cartesian closure and a subobject classifier, having a *parametrised list object* on A implies the existence of the free monoid on A. This follows from ([Mai10], Proposition 7.3) by restricting the construction of the free internal category on a free internal graph to one object categories and graphs. As such, we will assume that parametrised list objects exist in $\mathcal{E}$ on top of exactness properties between pullbacks and finite colimits.

On the other hand, if we assume that $\mathcal{E}$ is locally finitely presentable, then the existence of 2-colimits in $\mathbf{Cat}(\mathcal{E})$ is relatively easy to prove.

Proposition 1.2. *Let $\mathcal{E}$ be accessible. Then $\mathbf{Cat}(\mathcal{E})$ is accessible as a 1-category. Furthermore, if $\mathcal{E}$ also has finite colimits (so is locally finitely presentable), then $\mathbf{Cat}(\mathcal{E})$ has 2-colimits.*

Proof. Recall that $\mathbf{Cat}(\mathcal{E})$ is of the form $\mathbf{Mod}(\mathcal{S}, \mathcal{E})$, the category of models for a finite limit sketch $\mathcal{S}$ in $\mathcal{E}$. As $\mathcal{E}$ is accessible, we can apply ([LT23], Proposition 5.13) and deduce that $\mathbf{Mod}(\mathcal{S}, \mathcal{E})$ is accessible. For $\mathcal{E}$ locally finitely presentable, we instead apply Proposition 1.53 of [AR94], and conclude that $\mathbf{Cat}(\mathcal{E})_1$ is locally finitely presentable, so has finite colimits, in particular coequalisers. Therefore, $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits. □

We restrict ourselves to the elementary setting of a list-arithmetic pretopos with finite pullback stable coequalisers— that is: an exact, extensive category with finite pullback stable coequalisers and parameterised list objects. Our main result is Theorem 7.2, which says that $\mathbf{Cat}(\mathcal{E})$ has coequalisers under these assumptions. Finite 2-colimits follow as a consequence ([Kel89], §3; [HM24b], §5).

Examples of list-arithmetic pretoposes with pullback stable coequalisers are given in Section 2, and include univalent universes of dependent type theory that satisfy axiom K and are closed under the empty type, unit type, sum types, dependent sum types, propositional truncations, quotient sets, and parameterised natural numbers type— that is: models of extensional Martin-Löf type theory [Str93]. These examples are of interest in logic; Maietti [Mai10] proposes list-arithmetic pretoposes as an appropriate setting to capture Joyal's notion of an arithmetic universe [Joy05].

The study of 2-categories of internal categories has been of increasing interest in recent years. [Bou10] shows that assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ is a kind of 2-exact completion of the 1-category $\mathcal{E}$. 2-categories of internal categories are also of interest for matters relating to 2-dimensional foundations of mathematics. In previous work [HM24b] we described the elementary theory of the 2-category of small categories, which extends Lawvere's elementary theory of the category of sets to the higher dimensional setting. This will be extended in future work [HM24a] where we will describe 2-categories of categories, which should be examples of elementary 2-toposes. Although many possible definitions of elementary 2-toposes have been given [Web07, Str80, Hel24], it is generally agreed that 2-toposes should have 2-colimits. Hence, it is important to understand 2-categories which have 2-colimits, and our present work establishes this for 2-categories of internal categories under appropriate assumptions on $\mathcal{E}$. Relatedly, our result allows for a proof that the model structure on internal categories described in [EKVdL05] is cofibrantly generated and algebraic, in upcoming work [Hug24].

It was claimed as folklore in Section 2.2 of [EKVdL05] that for $\mathcal{E}$ an elementary topos with natural numbers object, $\mathbf{Cat}(\mathcal{E})$ has coequalisers which build upon this construction. We give a detailed verification of this claim and generalise it from elementary toposes with natural numbers objects to list-arithmetic pretoposes with finite pullback-stable coequalisers, which need not be cartesian closed or have a subobject classifier.

1.2. **Structure of the paper.** After giving some preliminary definitions in Section 2, this work is divided into five further sections. Section 3 recalls the construction of coproducts and copowers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$, and gives a more detailed outline of our strategy in constructing coequalisers. Section 4 constructs coequalisers of parallel pairs of internal functors that agree on objects. This simple case allows us to construct coequifiers in $\mathbf{Cat}(\mathcal{E})$. Section 5 recalls the construction of the free internal category on an internal graph (Theorem 5.2) which is due to ([Mai10], Proposition 7.3). This construction uses the internal type theory of a list-arithmetic pretopos, internally mimicking ([ML13], §II.7, Theorem 1). Section 6 uses free internal categories on internal graphs to construct coequalisers of pairs of arrows out of a discrete category. Finally, Section 7 brings together all these parts to prove that $\mathbf{Cat}(\mathcal{E})$ has coequalisers for an arbitrary pair of parallel morphisms.

1.3. **Notational conventions.** We adopt the notation for internal categories that was established in ([HM24b], §2).

2. Preliminaries and Setting

In this section, we define the setting within which we work and give examples of such settings.

Definition 2.1 ([Mai10], Definition 2.4)**.** Let $\mathcal{E}$ be a category with finite limits. We say that $\mathcal{E}$ has *parametrised list objects* if for any $X \in \mathcal{E}$, there exists an object $L(X) \in \mathcal{E}$ together with morphisms $r_0^X : \mathbf{1} \to L(X)$ and $r_1^X : L(X) \times X \to L(X)$ such that for any $b : B \to Y$ and $g : Y \times X \to Y$, there exists a unique $u : B \times L(X) \to Y$ making the following diagram commute:

$$\begin{array}{ccccc} B & \xrightarrow{(1_B, r_0^X \cdot !_B)} & B \times L(X) & \xleftarrow{1_B \times r_1^X} & B \times (L(X) \times X) \\ & \searrow^{b} & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle (u \times 1_X) \cdot \sigma} \\ & & Y & \xleftarrow[g]{} & Y \times X \end{array}$$

in which $\sigma : B \times (L(X) \times X) \to (B \times L(X)) \times X$ is the associative isomorphism of the cartesian product.

Remark 2.2. We note that for any category $\mathcal{E}$ with parametrised list objects, the assignment $X \mapsto L(X)$ extends to a functor $L : \mathcal{E} \to \mathcal{E}$; on morphisms $f : X \to Y$, we define $L(f) : L(X) \to L(Y)$ by the universal property of the parametrised list objects, taking $B = \mathbf{1}, Y = L(Y), b = r_0^Y$ and $g = \pi_{L(Y)} : L(Y) \times X \to L(Y)$ in the above definition. Moreover, there is a multiplication action $\mu_X : L(X) \times L(X) \to L(X)$ defined by the universal property by taking $B = L(X), Y = L(X), b = 1_{L(X)}$ and $g = r_1^X$. We also have a unit $\nu_X : X \to L(X)$ given by the the composite:

$$X \xrightarrow{(r_0^X \cdot !, 1_X)} L(X) \times X \xrightarrow{r_1^X} L(X).$$

The maps μ_X, ν_X furnish $L(X)$ with the structure of a monoid in $(\mathcal{E}, \times, \mathbf{1})$.

Example 2.3. Useful intuition is provided by the case $\mathcal{E} = \mathbf{Set}$. For any set X, $L(X)$ is defined to be the set of words with alphabet X, otherwise known as the free monoid generated by X. The morphism $r_0^X : \mathbf{1} \to L(X)$ is given by the empty list. The morphism $r_1^X : L(X) \times X \to L(X)$ takes a word $(x_1...x_n)$ and an element $y \in X$ and outputs the word $(x_1...x_n y)$. The morphism $\mu_X : L(X) \times L(X) \to L(X)$ concatenates two words $((x_1...x_n), (y_1...y_m)) \mapsto (x_1, ...x_n y_1...y_m)$. The morphism $\nu_X : X \to L(X)$ takes an element $x \in X$ and forms the singleton word $(x) \in L(X)$.

Remark 2.4. Any category with parametrised list objects has a parametrised natural numbers object by taking $X = \mathbf{1}$. We also remark that if $\mathcal{E}$ is cartesian closed, then the existence of parametrised lists objects (resp. a parametrised natural numbers objects) is equivalent to the existence of list objects (resp. a natural numbers objects) [Joh02b].

Definition 2.5. A *pretopos* is an exact and extensive category.
If a pretopos has parameterised list objects, we call it a *list-arithmetic pretopos*.

In particular, a list-arithmetic pretopos satisfies the following useful properties.

- It is extensive and has finite products, so it is distributive [CLW93].
- It is exact, so it is regular, so it has finite limits by definition.
- It is extensive, so it has finite coproducts.
- It has coequalisers ([Mai10], §3.9)

Definition 2.6. Let $\mathcal{E}$ be a category with pullbacks. We say that $\mathcal{E}$ has *pullback stable coequalisers* if for any morphism $f : X \to Y$ in $\mathcal{E}$ the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ preserves coequalisers.

Our main result, Theorem 7.2 assumes that $\mathcal{E}$ is a list-arithmetic pretopos with finite pullback stable coequalisers. Below, we record some examples of suitable categories in decreasing generality.

Definition 2.7. If a list-arithmetic pretopos is also locally cartesian closed, we call it an *arithmetic* Π*-pretopos*.

This is a suitable setting for our work; indeed: coequalisers are pullback stable as we prove in Corollary 2.9 using the following lemma.

Lemma 2.8. *Let $\mathcal{E}$ be a cartesian closed exact category, and consider coequaliser diagrams*

$$A \underset{G}{\overset{F}{\rightrightarrows}} B \xrightarrow{Q} C \qquad X \underset{J}{\overset{H}{\rightrightarrows}} Y \xrightarrow{K} Z$$

Then the following diagram is also a coequaliser in $\mathcal{E}$:

$$A \times X \underset{G\times J}{\overset{F\times H}{\rightrightarrows}} B \times Y \xrightarrow{Q\times K} C \times Z.$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
A \times X & \overset{F\times 1_X}{\underset{G\times 1_X}{\rightrightarrows}} & B \times X & \xrightarrow{Q\times 1_X} & C \times X \\
{\scriptstyle 1_A\times H}\downarrow\downarrow{\scriptstyle 1_A\times J} & & {\scriptstyle 1_B\times H}\downarrow\downarrow{\scriptstyle 1_B\times J} & & {\scriptstyle 1_C\times H}\downarrow\downarrow{\scriptstyle 1_C\times J} \\
A \times Y & \overset{F\times 1_Y}{\underset{G\times 1_Y}{\rightrightarrows}} & B \times Y & \xrightarrow{Q\times 1_Y} & C \times Y \\
{\scriptstyle 1_A\times K}\downarrow & & \downarrow{\scriptstyle 1_B\times K} & & \downarrow{\scriptstyle 1_C\times K} \\
A \times Z & \overset{F\times 1_Z}{\underset{G\times 1_Z}{\rightrightarrows}} & B \times Z & \xrightarrow{Q\times 1_Z} & C \times Z.
\end{array}$$

We want to show that the diagonal composite of this diagram is a coequaliser diagram.

Note that, for any $E \in \mathcal{E}$, $E \times -$ and hence $- \times E$ is left adjoint to $(-)^E$, and so preserves all colimits, in particular coequaliser. Hence, in the above diagram, all rows and columns are coequalisers. In an exact category, coequalisers are effective and effective epimorphisms are closed under composition, so it follows that $Q \times K : B \times Y \to C \times Z$ is an effective epimorphism. Hence, it is the coequaliser of its kernel pair, so it remains to show that the following square is a pullback:

$$\begin{array}{ccc}
A \times X & \xrightarrow{G\times J} & B \times Y \\
{\scriptstyle F\times H}\downarrow & & \downarrow{\scriptstyle Q\times K} \\
B \times Y & \xrightarrow[Q\times K]{} & C \times Z
\end{array}$$

Again, since coequalisers are effective in an exact category, K and Q are effective epimorphisms, so there are pullback squares

$$\begin{array}{ccc}
A & \xrightarrow{F} & B \\
{\scriptstyle G}\downarrow & \lrcorner & \downarrow{\scriptstyle Q} \\
B & \xrightarrow[Q]{} & C
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{H} & Y \\
{\scriptstyle J}\downarrow & \lrcorner & \downarrow{\scriptstyle K} \\
Y & \xrightarrow[K]{} & Z
\end{array}$$

Now, pullback squares are closed under product, this follows representably from the easy-to-verify result in **Set**. □

Corollary 2.9. *Let $\mathcal{E}$ be an arithmetic Π-pretopos. Then coequalisers are stable under pullback.*

Proof. Let $\mathcal{E}$ be an arithmetic Π-pretopos and let $f : A \to B$ in $\mathcal{E}$. We apply Lemma 2.8 to the category $\mathcal{E}/B$ which is exact since exactness is stable under slicing ([BB04], Appendix A), cartesian closed because $\mathcal{E}$ was locally cartesian closed and has finite colimits because colimits in the slice are calculated as in $\mathcal{E}$. Here, products are pullbacks over B, and coequalisers are computed as in $\mathcal{E}$. □

A class of examples of categories $\mathcal{E}$ satisfying the assumptions of Corollary 2.9 are given by univalent universes of dependent type theory that satisfy axiom K and are closed under the empty type, unit type, sum types, dependent sum types, product types, dependent product types, propositional truncations, quotient sets, and parameterised natural numbers type. Such things are models of extensional Martin-Löf type theory [Str93].

As a consequence of Theorem 2.5.17 of [Joh02a], any locally cartesian closed positive coherent category with natural numbers object has list objects. To give intuition for why this is true, we give this proof in **Set** and argue that all the constructions can be interpreted in the internal logic of any exact, locally cartesian closed category with natural numbers object.

Recall that the pushforward of a pair of composable functions (f, g) in **Set** is given by

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{z \in Z} \prod_{y \in g^{-1}(z)} f^{-1}(y) \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Let $A \in \mathbf{Set}$. By choosing the correct $f : X \to Y, g : Y \to Z$, we can write the free monoid on A as the pushforward of two maps in **Set**. Pushforwards of maps exist in any locally cartesian closed category.

First, take $Z = \mathbb{N}$. Then, take $Y \rightarrowtail \mathbb{N} \times \mathbb{N}$ to be the subset

$$\{(x, y), x \in \mathbb{N}, y \in \mathbb{N} : \exists k \in \mathbb{N} : x + s(k) = y\},$$

which can be formed using the internal language of an exact category using equalisers and regular epimorphisms. Take $g : Y \to \mathbb{N}$ to be the following composite:

$$Y \rightarrowtail \mathbb{N} \times \mathbb{N} \xrightarrow{\pi_2} \mathbb{N}.$$

Finally, take $X = A \times Y$ and $f = \pi_Y : A \times Y \to Y$. Then the pushforward of these maps is precisely $\coprod_{n \in \mathbb{N}} \prod_{i \in \mathbb{N}} A$, which is the list object on A.

Conversely, any arithmetic Π-pretopos has a natural numbers object. Hence, any Π-pretopos (a locally cartesian closed pretopos) with a natural numbers object is equivalent to an arithmetic Π-pretopos.

An arithmetic Π-pretopos is cartesian closed as it is locally cartesian closed and has a terminal object. However, it need not have a subobject classifier.

Any elementary topos with natural numbers object is an arithmetic Π-pretopos; indeed: it is locally cartesian closed. Hence, any model of the elementary theory of the category of sets [LM05] is a suitable setting for this work too. This is of interest in relation to [HM24b].

3. Constructing finite 2-colimits of internal categories via simpler colimits

Recall (for example from ([Kel89], §3) that finite 2-colimits can be constructed using finite coproducts, coequalisers of parallel pairs, and copowers by **2**. We briefly review the construction of finite coproducts and copowers by **2** in 2-category $\mathbf{Cat}(\mathcal{E})$ under the assumption that $\mathcal{E}$ is lextensive. We then outline the construction of coequalisers of parallel pairs in $\mathbf{Cat}(\mathcal{E})$ which we will develop over the subsequent Sections.

First, we describe an internal free-living arrow in $\mathbf{Cat}(\mathcal{E})$, which we denote $\mathbf{2}_{\mathcal{E}}$. For any object $A \in \mathcal{K}$, the cartesian product $\mathbf{2}_{\mathcal{E}} \times A$ will have the universal property of the copower of A by **2**. The internal category $\mathbf{2}_{\mathcal{E}}$ can be concretely described as a truncated simplicial object, with n-simplices given by the $(n+2)$-fold coproduct of the terminal object $\mathbf{1} \in \mathcal{E}$; see Example 2.3.2 of [Mir18] for further details. Abstractly, it is the image of **2** under $\mathbf{Cat}(F) : \mathbf{Cat}(\mathbf{FinSet}) \to \mathbf{Cat}(\mathcal{E})$, where $F : \mathbf{FinSet} \to \mathbf{Cat}(\mathcal{E})$ is the unique coproduct and terminal object preserving functor, which is described in Definition 5.4 of [HM24b]. We note that with the additional assumption of cartesian closure, Proposition 3.1 (2) is Theorem 5.5 (2) of [HM24b], but this proof is more general as we only assume lextensivity.

Proposition 3.1. *Let $\mathcal{E}$ be lextensive. Then $\mathbf{Cat}(\mathcal{E})$ has*

(1) extensive coproducts which are created by $N : \mathbf{Cat}(\mathcal{E})_1 \to [\Delta^{op}_{\leq 3}, \mathcal{E}]$.

*(2) copowers by **2**, which for an internal category $\mathbb{A}$ are given by $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$.*

Proof. For part (1), the coproduct of a pair of internal categories $\mathbb{A}$ and $\mathbb{B}$ is given levelwise by $n \mapsto \mathbb{A}_n + \mathbb{B}_n$. We refer the reader to (Lemma 5.2 ,[HM24b]) for a full proof and details. For part (2), the internal functor $\mathbf{2}_{\mathcal{E}} \times \mathbb{A} \to \mathbb{B}$ corresponding to an internal natural transformation $\alpha : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ is given via the description of $\mathbf{2}_{\mathcal{E}}$ by two morphisms $(f_0, g_0) : A_0 + A_0 \to B_0$ and $(f_1, m.\underline{\alpha}, g_1) : A_1 + A_1 + A_1 \to B_1$ in $\mathcal{E}$. Further details can be found in [Mir18].

□

In light of Proposition 3.1, to show that $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits it suffices to show that the 2-category $\mathbf{Cat}(\mathcal{E})$ has coequalisers of parallel pairs. Moreover, since $\mathbf{Cat}(\mathcal{E})$ has powers by $\mathbf{2}$, it suffices to show that the underlying category $\mathbf{Cat}(\mathcal{E})_1$ has coequalisers of parallel pairs.

A naive attempt at constructing a coequaliser of a pair of internal functors would be to do this levelwise. We have already seen in Example 1.1 that this does not work even internal to **Set** since pairs of morphisms may become newly composable once a coequaliser is also taken at the level of objects. In Example 1.1, the single non-identity morphism of the free living arrow becomes composable with itself after gluing together its source and target; this new composite is not created by coequalising on morphisms, and so one must take the free category on the graph obtained by coequalising on objects and then morphisms.

Our construction of coequalisers of arbitrary parallel pairs of internal functors $F,\ G :\ \mathbb{A} \to \mathbb{B}$ decomposes into the following two steps.

(1) First restrict F and G along $\varepsilon_{\mathbb{A}} : \mathbf{disc}(\mathbb{A}) \to \mathbb{A}$ and form the coequaliser $K : \mathbb{B} \to \mathbb{D}$ of the parallel pair $F \cdot \varepsilon_{\mathbb{A}}$ and $G \cdot \varepsilon_{\mathbb{A}}$.

$$\mathbf{disc}(A_0) \underset{G\cdot\varepsilon_{\mathbb{A}}}{\overset{F\cdot\varepsilon_{\mathbb{A}}}{\rightrightarrows}} \mathbb{B} \xrightarrow{K} \mathbb{D}.$$

In Proposition 6.6 we show that if $\mathcal{E}$ is a list arithmetic pretopos with pullback stable coequalisers then coequalisers of parallel pairs of internal functors out of discrete categories exist in $\mathbf{Cat}(\mathcal{E})$.

(2) Next, form the coequaliser $P : \mathbb{D} \to \mathbb{C}$ of the parallel pair of internal functors KF and KG.

$$\mathbb{A} \underset{K\cdot G}{\overset{K\cdot F}{\rightrightarrows}} \mathbb{D} \xrightarrow{P} \mathbb{C}.$$

Note that since K coequalises $F.\varepsilon_{\mathbb{A}}$ and $G.\varepsilon_{\mathbb{A}}$, the functors KF and KG agree on objects. In Proposition 4.1 we show that if $\mathcal{E}$ has pullback stable coequalisers then $\mathbf{Cat}(\mathcal{E})$ has coequalisers of parallel pairs of internal functors that agree on objects.

Finally, in Section 7 we show that for abstract reasons these steps combine in such a way that $Q := PK : \mathbb{B} \to \mathbb{C}$ is the coequaliser of the original parallel pair $F, G : \mathbb{A} \to \mathbb{B}$. We prove Proposition 6.6, as required for step (1) above, using the following two auxiliary constructions.

i The construction of free categories on graphs. We use their universal property, which is established for list arithmetic pretoposes in [Mai10] and reviewed in Section 5.

ii The construction of coequifiers of parallel pairs of internal natural transformations. We show in Corollary 4.3 that when $\mathcal{E}$ has pullback stable coequalisers then $\mathbf{Cat}(\mathcal{E})$ has coequifiers of arbitrary pairs of internal natural transformations.

In step (1) above, we first forget about any morphisms in $\mathbb{A}$ and instead generate the coequaliser on objects and consider the graph $\mathcal{G}$ which has equivalence classes of objects in $\mathbb{B}$ as objects and morphisms in $\mathbb{B}$ as edges. The free category on this graph gives us a category whose morphisms are strings of morphisms in $\mathbb{B}$ that become composable once the we coequalise on objects. We require an internal functor $\mathbb{B} \to \mathbf{F}(\mathcal{G})$, but the construction so far only guarantees us a morphism of their underlying graphs. The final two coequifiers extend this to a morphism of graphs which respects identities and composition.

Step (2) then considers the morphisms of $\mathbb{A}$, and takes the coequaliser just on morphisms. This requires only exactness properties in $\mathcal{E}$.

Remark 3.2. It is interesting to compare this construction with the method used in §4 of [BBP99] in the context of **Cat**. Let $F, G : \mathcal{A} \to \mathcal{B}$. The construction of a coequaliser in [BBP99] first constructs a relation ${}_F{=}_G$ on $\mathcal{B}$ generated by F and G defined on objects by $a_F =_G a \in \mathcal{A}_0$ iff $F(a) = G(a)$ and on morphisms by $f_F =_G f$ iff $F(f) = G(f)$. It then constructs the *generalised congruence* ${}_F{\simeq}_G$ generated by this relation, which closes this relation on morphisms under some axioms. It then quotients $\mathcal{B}$ by this generalised congruence, and the result is the coequaliser. In contrast, Step (1) of our construction constructs a category in which the *generalised* congruence on $\mathcal{B}$ is simply an ordinary congruence (in the standard sense of [ML13], for example) on this new category. In other words, the category constructed by Step (1) is the setting in which the generalised congruence is defined. In internal category theory, one must be very careful to state precisely where things are defined. Step (2) takes the usual quotient of a category by a congruence.

We do not, however, attempt to define the notion of a generalised congruence on an internal category.

4. Coequalisers of arrows that agree on objects

Throughout this section, $\mathcal{E}$ will be assumed to be a category with pullbacks and pullback stable coequalisers. The goal of this section is to show that under these assumptions, the 2-category $\mathbf{Cat}(\mathcal{E})$ has coequalisers of pairs of internal functors $F, G : \mathbb{A} \to \mathbb{B}$ which agree on objects in the sense that the morphisms $F_0, G_0 : A_0 \to B_0$ are equal in $\mathcal{E}$. As a corollary, we find that $\mathbf{Cat}(\mathcal{E})$ also has coequifiers under these assumptions.

Proposition 4.1. *Let $\mathcal{E}$ be a category with pullbacks and pullback stable coequalisers. Any pair $F, G : \mathbb{A} \to \mathbb{B}$ of internal functors that agree on objects has a coequaliser in $\mathbf{Cat}(\mathcal{E})$.*

Proof. We define the coequaliser of F and G by first defining $C_0 := B_0$ and defining C_1 as the coequaliser of F_1 and G_1 in $\mathcal{E}$:

$$A_1 \underset{G_1}{\overset{F_1}{\rightrightarrows}} B_1 \xrightarrow{Q_1} C_1.$$

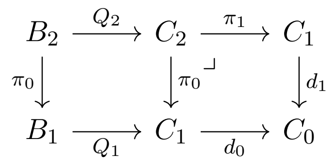
We show that these turn out to be the objects of objects and morphisms for an internal category which has the universal property of the desired coequaliser. We define source and target $d_0, d_1 : C_1 \to C_0$ using the universal property of the coequaliser:

$$\begin{array}{ccccc} A_1 & \underset{G_1}{\overset{F_1}{\rightrightarrows}} & B_1 & \xrightarrow{Q_1} & C_1 \\ \downarrow d_i & & \downarrow d_i & & \downarrow d_i \\ A_0 & \underset{G_0}{\overset{F_0}{\rightrightarrows}} & B_0 & \overset{Q_0}{=\!=} & C_0 \end{array} \qquad i \in \{0, 1\}$$

We define $i : C_0 \to C_1$ as the composite

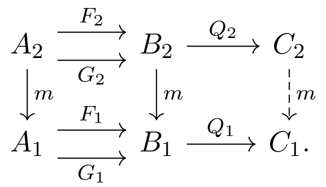
$$C_0 = B_0 \xrightarrow{i} B_1 \xrightarrow{Q_1} C_1.$$

Next, define C_2 as the pullback of $d_0, d_1 : C_1 \to C_0$ and define $Q_2 : B_2 \to C_2$ to be induced by the universal property of the pullback, given the morphisms $Q_1 \cdot \pi_0$ and $Q_1 \cdot \pi_1$. The following exhibits Q_2 as the pullback of $Q_1 : B_1 \to C_1$ by $\pi_0 : C_2 \to C_1$, by the pullback lemma

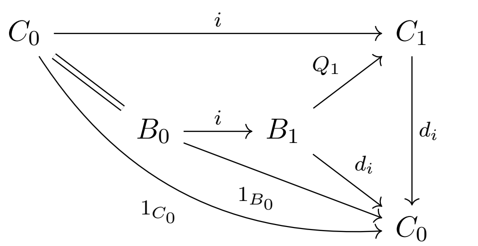
(1)


where the outside is a pullback by definition of B_2 and the string of equalities which follow by definition: $d_0 \cdot Q_1 = d_0 : B_1 \to B_0$ and $d_1 \cdot \pi_1 \cdot Q_2 = d_1 \cdot Q_1 \cdot \pi_1 = d_1 \cdot \pi_1 : B_2 \to B_0$. Note that we can also express Q_2 as the pullback of Q_1 by π_1.

By the assumption that $\mathcal{E}$ has coequalisers which are stable under pullbacks, it follows that upper row of the diagram displayed below is a coequaliser diagram. We can therefore define the dotted arrow $m : C_2 \to C_1$.

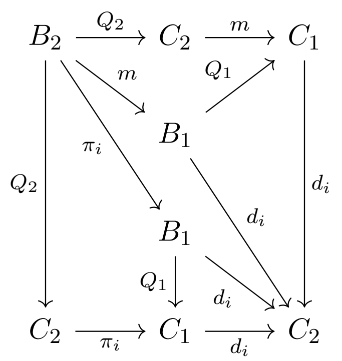


We claim that $\mathbb{C} := (C_0, C_1, d_0, d_1, i, m)$ forms an internal category. The laws specifying the source and target of identity morphisms are satisfied as shown below:



$i \in \{0, 1\}$.

To show that the laws specifying the source and target of composite morphisms are satisfied, we appeal to the universal property of Q_2 as the coequaliser of F_2 and G_2. We show that, for $i \in \{0, 1\}$, the maps $Q_2 d_i m, Q_2 d_i \pi_i : B_2 \to C_0$ are equal in the diagram below. Both maps clearly coequalise $F_2, G_2 : A_2 \to B_2$. By uniqueness aspect of the universal property, it follows that $d_i m = d_i \pi_i$.



The other axioms follow similarly; for example, the left unit law follows from the fact that by the assumption that coequalisers are closed under pullbacks, the following diagram is a coequaliser diagram:

$$B_0 \times_{B_0} A_1 \underset{1_{B_0} \times_{B_0} G_1}{\overset{1_{B_0} \times_{B_0} F_1}{\rightrightarrows}} B_0 \times_{B_0} B_1 \xrightarrow{Q_0 \times_{Q_0} Q_1} C_0 \times_{C_0} C_1$$

and so we can check the left unit law by showing that the maps

$$m \cdot (i \times_{C_0} 1_{C_1}) \cdot (Q_0 \times_{Q_0} Q_1), \qquad \pi_1 \cdot (Q_0 \times_{Q_0} Q_1) : B_0 \times_{B_0} B_1 \to C_1$$

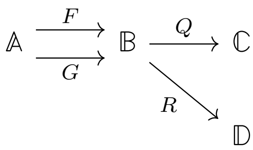
are equal, and since both maps clearly coequalise the diagram above, by uniqueness of the universal property, it follows that $m \cdot (i \times_{C_0} 1_{C_1}) = \pi_1$.

The right unit law and associativity of composition follows using the same method; the details for associativity can be found in appendix A.

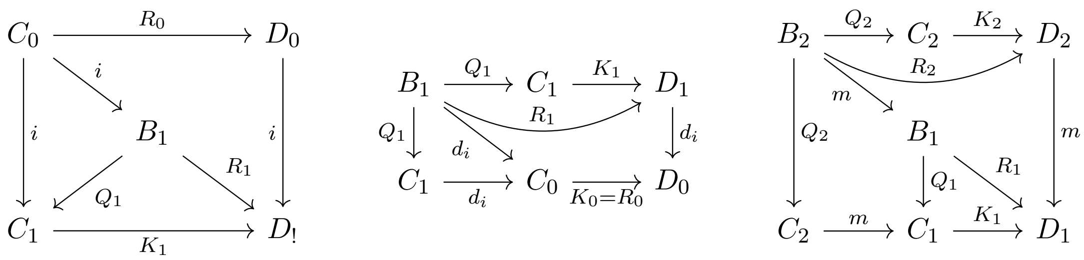
This shows that $\mathbb{C}$ is an internal category.

By definition of $d_0, d_1 : C_1 \to C_0$, $i : C_0 \to C_1$ and $m : C_2 \to C_1$, it also follows that $Q := (Q_0, Q_1)$ is well-defined an internal functor. We now show that it has the universal property of the coequaliser of F and G.

Given



where $RF = RG$ we define a $K_0 := R_0 : C_0 \to D_0$ and $K_1 : C_1 \to D_1$ by the universal property of C_1 as a coequaliser, and the fact that $R_1F_1 = (RF)_1 = (RG)_1 = R_1G_1$. This assembles into a functor $K : \mathbb{C} \to \mathbb{D}$ as witnessed by the following diagrams, in which again we make use of the universal property of Q_1 and Q_2 as coequalisers. Uniqueness of this functor follows from uniqueness of K_1.



□

Coequifiers in $\mathbf{Cat}(\mathcal{E})$, which we show exist in the Corollary to follow, will be used in the construction of coequalisers of parallel pairs of internal functors whose domains are discrete, in Section 6.

Let $\mathcal{K}$ be a 2-category with powers by $\mathbf{2}$. Let $f : g : A \to B$ and $\alpha, \beta : f \Rightarrow g$. Note that by the universal property of the power by $\mathbf{2}$, 2-cells α, β correspond to morphisms $\hat{\alpha}, \hat{\beta} : A \to B^{\mathbf{2}}$. We will use the following well-known result.

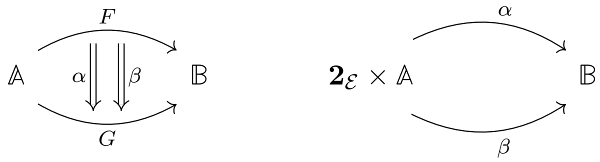
Lemma 4.2. *Let $\mathcal{K}$ be a 2-category. Then the equifier of a parallel pair of 2-cells $\alpha, \beta : f \Rightarrow g$ exists if and only if the equaliser of the corresponding morphisms $\hat{\alpha}, \hat{\beta} : A \to B^{\mathbf{2}}$ exists. In this case, the limits agree.*

Proof. We can check this representably in $\mathbf{Cat}$. Recall that an equaliser of $\hat{\alpha}, \hat{\beta} : \mathcal{A} \to \mathcal{B}^{\mathbf{2}}$ in $\mathbf{Cat}$ is given by the full subcategory of those $a \in \mathcal{A}$ such that $\hat{\alpha}(a) = \hat{\beta}(a)$. Similarly, recall that the equifier of $\alpha, \beta : f \to g$ in $\mathbf{Cat}$ is given by the full subcategory of $a \in \mathcal{A}$

such that $\alpha_a = \beta_a$. By definition, $\hat{\alpha}(a) = \alpha_a$ and $\hat{\beta}(a) = \beta_a$, so these define the same things. ☐

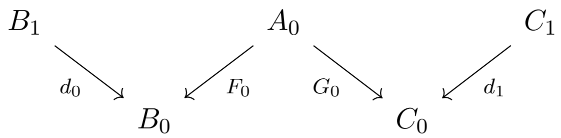
Corollary 4.3. *Let $\mathcal{E}$ be a category with pullbacks and pullback stable coequalisers. The 2-category $\mathbf{Cat}(\mathcal{E})$ has coequifiers.*

Proof. Consider the parallel pair of internal natural transformations displayed below left. By Lemma 4.2 applied to $\mathcal{K} = \mathbf{Cat}(\mathcal{E})^{\mathrm{op}}$, these correspond to the parallel pair of internal functors displayed below right. Observe that both functors are given on objects by the morphism $(F_0, G_0) : A_0 + A_0 \to B_0$. Hence the result follows from 4.1.

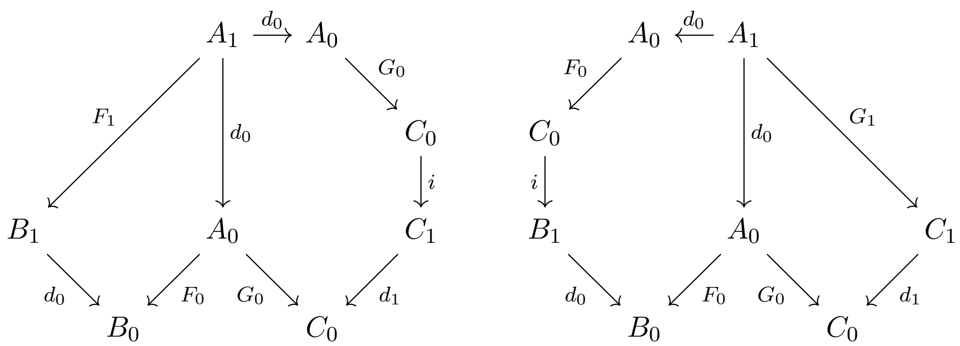


☐

Remark 4.4. We also note that under the assumptions that $\mathcal{E}$ is a pretopos, $\mathbf{Cat}(\mathcal{E})$ also has cocomma objects which are constructed in a similar way. Given a span of functors $\mathbb{A} \xleftarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$, their cocomma has object of objects given by $B_0 + C_0$ and object of morphisms constructed using limits and coequalisers in $\mathcal{E}$. Specifically, first construct the limit L of the diagram displayed below.



When $\mathcal{E} = \mathbf{Set}$ this limit consists of a morphism f in $\mathbb{B}$, a morphism g in $\mathbb{C}$ and a 'heteromorphism' from the target Z of f to the source Y of g whenever there is an object X in $\mathbb{A}$ satisfying $FX = Z$ and $GX = Y$. This heteromorphism will correspond to the component on X of the natural transformation forming part of the cocomma cocone. To ensure that these heteromorphisms form a natural transformation, we next form the coequaliser of a parallel pair of maps from $b, c : A_1 \to L$. These maps are induced by the universal property of L, given the data displayed below left for b and below right for c.



We leave details of the proof that this gives a well-defined internal category which has the universal property of a cocomma to the interested reader. Cocommas in $\mathbf{Cat}(\mathcal{E})$ will not be needed in this paper.

5. The free internal category on an internal graph

Throughout this section, let $\mathcal{E}$ be a list-arithmetic pretopos, with notation as given in Section 2. In this section, we recall the free internal category on an internal graph given in Definition 7.2 of [Mai10]. The description we give is equivalent but uses categorical language to describe the structure rather than the internal type theory of a list-arithmetic pretopos. In Proposition 7.3 of [Mai10], it is proven that this forms a left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$. We will use this result in Section 6 to construct coequalisers of arrows out of a discrete category.

Let $\mathcal{G} = (G_0, G_1, s, t)$. Define $\mathbb{F}\mathcal{G}_0 := G_0$ and $\mathbb{F}\mathcal{G}_1$ as the equaliser of the following diagram:

(2)
$$\begin{array}{ccccc} & & LG_0 \times G_0 & & \\ & \overset{!\times L(t)\times 1_{G_0}}{\nearrow} & & \overset{r_1^{G_0}}{\searrow} & \\ G_0 \times L(G_1) \times G_0 & & & & L(G_0) \\ & \underset{1_{G_0}\times L(s)\times !}{\searrow} & & \underset{r_1^{G_0}\cdot\rho}{\nearrow} & \\ & & G_0 \times L(G_0) & & \end{array}$$

where ρ denotes the symmetry isomorphism of the cartesian product $\rho : G_0 \times L(G_0) \cong L(G_0) \times G_0$ and $! : G_0 \to \mathbf{1}$ is the unique map to the terminal object. The identity assigner $i : \mathbb{F}\mathcal{G}_0 \to \mathbb{F}\mathcal{G}_1$ is induced by the universal property of the equaliser, given that $1_{G_0} \times r_0^{G_1} \cdot ! \times 1_{G_0} : G_0 \to G_0 \times L(G_1) \times G_0$ equalises Diagram 2. We define $d_1, d_0 : \mathbb{F}\mathcal{G}_1 \to G_0$ by the following composites:

$$d_1 := \Big(\mathbb{F}\mathcal{G}_1 \longrightarrow G_0 \times LG_1 \times G_0 \xrightarrow{\pi_0} G_0 \Big)$$
$$d_0 := \Big(\mathbb{F}\mathcal{G}_1 \longrightarrow G_0 \times LG_1 \times G_0 \xrightarrow{\pi_2} G_0 \Big).$$

The following map

$$\begin{array}{c} \mathbb{F}\mathcal{G}_1 \times_{G_0} \mathbb{F}\mathcal{G}_1 \\ \downarrow \\ (G_0 \times LG_1 \times G_0) \times_{G_0} (G_0 \times LG_1 \times G_0) \\ \downarrow \cong \\ G_0 \times LG_1 \times LG_1 \times G_0 \\ \downarrow 1_{G_0} \times \mu_{G_1} \times 1_{G_0} \\ G_0 \times LG_1 \times G_0. \end{array}$$

equalises Diagram 2. This therefore induces a map $m : \mathbb{F}\mathcal{G}_1 \times_{G_0} \mathbb{F}\mathcal{G}_1 \to \mathbb{F}\mathcal{G}_1$.

Definition 5.1 (7.2 of [Mai10])**.** Given an internal graph $\mathcal{G} = (G_0, G_1, s, t)$, we define an internal category $\mathbb{F}\mathcal{G} := (\mathbb{F}\mathcal{G}_0, \mathbb{F}\mathcal{G}_1, d_1, d_0, i, m)$.

Moreover, this internal category is the *free* internal category on an internal graph, forming an adjunction as recorded below. The unit of this adjunction $\eta_{\mathcal{G}}$ is defined by $\eta_{\mathcal{G}_0} := 1_{G_0} : G_0 \to \mathbb{F}\mathcal{G}_0$ and $\eta_{\mathcal{G}_1} : G_1 \to \mathbb{F}\mathcal{G}_1$ which is induced by the universal property of the equaliser, given that $(d_1, \nu_{G_1}, d_1) : G_1 \to G_0 \times LG_1 \times G_0$ equalises Diagram 2. The counit of the adjunction does an internal version of taking a string of composable arrows and composing them.

Theorem 5.2 ([Mai10], Proposition 7.3)**.** *Let $\mathcal{E}$ be a list-arithmetic pretopos. The assignment $\mathcal{G} \mapsto \mathbb{F}\mathcal{G}$ provides a left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$.*

Remark 5.3. If $\mathcal{E}$ has countable coproducts, then it is not too hard to prove that for a graph $\mathcal{G} := (G_0, G_1, s, t)$, the object $\mathbb{F}\mathcal{G}_1 \cong \Sigma_{n\in\mathbb{N}} G_n$, where for $n > 1$, G_n is its object of composable n-arrows:

$$G_n := \underbrace{G_1 \times_{G_0} \ldots \times_{G_0} G_1}_{n \text{ times}}.$$

In this case, the proof of Theorem 5.2 using the internal type theory of $\mathcal{E}$ corresponds to a proof using the universal property of the coproduct; internal induction becomes external universal property. This proof is categorically elegant. We do not ask for $\mathcal{E}$ to have countable coproducts as this is not an elementary condition, despite the fact that arithmetic Π-pretoposes with finite colimits which do not have countable coproducts are hard to construct and do not interact well with other toposes— see, for example, ([Joh02a], D5.1.7).

Remark 5.4. As mentioned, the description we give for the free internal category on an internal graph is different, but equivalent, to the one given by Maietti in [Mai10]. We choose this description as it does not rely on using the internal language of a list-arithmetic pretopos, and it does not use coproducts which are indeed not needed for the construction of free internal categories on graphs. We briefly describe how to see the equivalence between the different descriptions, although a full proof is left to the interested reader. The key to this proof is in noting that the object of non-empty lists of G_1, denoted $L^*(G_1)$ and described in [Mai10] using the internal language of $\mathcal{E}$, is isomorphic to $L(G_1) \times G_1$; the isomorphism between them is given by the maps $r_1^X : L(G_1) \times G_1 \to L^*(G_1)$ and $(\mathrm{Bck}, \mathrm{Las}) : L^*(G_1) \to L(G_1) \times G_1$, where $\mathrm{Las} : L^*(G_1) \to G_1$ internally takes the last element of a non empty list and $\mathrm{Bck} : L^*(G_1) \to L(G_1)$ takes all elements except for the last one. These maps are described inductively using the internal language of $\mathcal{E}$ in ([Mai10], Appenix A). One direction of the isomorphism is shown using the universal property of the product and the list object. The other direction is shown using internal induction on list elements, using the internal language of $\mathcal{E}$. The proof then proceeds by using the fact that $L(G_1) \cong \mathbf{1} + G_1 \times L(G_1)$. This is shown in [Joh02a]. The proof is finished by noticing that the equalising diagrams constructed give the same equaliser.

Remark 5.5. We note that the free category on an internal graph $\mathcal{G} = (G_0, G_1, s, t)$ is also the coinserter of the following diagram in $\mathbf{Cat}(\mathcal{E})$:

$$\mathbf{disc}(G_1) \underset{\mathbf{disc}(t)}{\overset{\mathbf{disc}(s)}{\rightrightarrows}} \mathbf{disc}(G_0) \overset{Q}{\dashrightarrow} \mathbb{F}(\mathcal{G}).$$

This universally coinserts a 2-cell $Q\mathbf{disc}(s) \Rightarrow Q\mathbf{disc}(t)$, which out of a discrete category means that in $\mathbb{F}(\mathcal{G})$, there is an actual 1-cell in $\mathbb{F}(\mathcal{G})$ for any arrow in G_1, with source and target as desired. The universal property of the coinserter in this situation is exactly the same as the universal property of the free category.

This observation is noted in the case when $\mathcal{E} = \mathbf{Set}$ in ([Bou10], Example 2.6).

6. Coequalisers of pairs of arrows out of a discrete category

Throughout this section, we assume that $\mathcal{E}$ is a list-arithmetic pretopos with finite pullback stable coequalisers. The goal of this Section is to prove that $\mathbf{Cat}(\mathcal{E})$ has coequalisers of pairs of arrows $F, G : A_0 \to \mathbb{B}$ where A_0 is a discrete category. Our proof uses the universal property of the free category on a graph, which we state explicitly in Corollary 6.1, to follow.

Corollary 6.1. *Let A_0 be a discrete category internal to $\mathcal{E}$ and let $F, G : A_0 \to \mathbb{B}$ be a parallel pair of internal functors. Form the coequaliser $k_0 : B_0 \to C_0$ of the parallel pair $F_0, G_0 : A_0 \to B_0$ in $\mathcal{E}$. Consider the graph $\mathcal{G} := (B_1, C_0, k_0 \cdot d_0, k_0 \cdot d_1)$ internal to $\mathcal{E}$. There is a category $\mathbb{F}(\mathcal{G})$ and a morphism of graphs $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ with the property that for any internal category $\mathbb{H}$ and morphism of graphs $h : \mathcal{G} \to \mathcal{U}(\mathbb{H})$ there is a unique internal functor $h' : \mathbb{F}(\mathcal{G}) \to \mathbb{H}$ satisfying $\mathcal{U}(h') \cdot \eta_{\mathcal{G}} = h$.*

Proof. The morphism of graphs $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is the component of the unit for the adjunction $\mathbb{F} \dashv \mathcal{U}$ of Theorem 5.2 at the graph $\mathcal{G}$. The property stated for $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is precisely the universal property of the unit. □

Lemma 6.2. *There is a morphism of graphs $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ defined on vertices by the coequaliser $k_0 : B_0 \to C_0$ of F_0 and G_0, and on edges by the edge-assignment $(\eta_{\mathcal{G}})_1 : \mathcal{G}_1 = B_1 \to \mathbb{F}(\mathcal{G})_1$.*

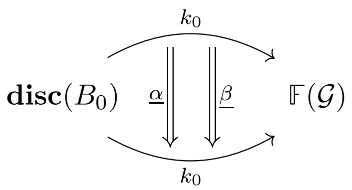
Proof. Since $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is a morphism of graphs, we see that for $i \in \{0, 1\}$, the equation displayed below holds.

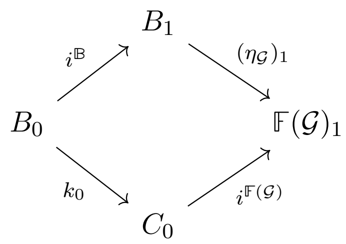
$$d_i^{\mathbb{F}(\mathcal{G})} \cdot (\eta_{\mathcal{G}})_1 = k_0 \cdot d_i^{\mathbb{B}} \tag{3}$$

This is because $k_1 \cdot d_1^{\mathbb{B}} : B_1 \to C_0$ is the source of $\mathcal{G}$ and $k_0 \cdot d_1^{\mathbb{B}} : B_1 \to C_0$ is the target of $\mathcal{G}$. But these equations together with Equation 3 say precisely that $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is well-defined as a morphism of graphs. □

The morphism of graphs $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ of Lemma 6.2 will typically not be compatible with identity or composition structure. This is rectified by constructing a coequifier ensuring each of these conditions is satisfied.

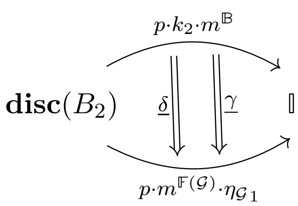
Lemma 6.3. *There is a parallel pair of natural transformations $\underline{\alpha}, \underline{\beta} : k_0 \Rightarrow k_0 : \mathbf{disc}(B_0) \to \mathbb{F}(\mathcal{G})$ as displayed below left, whose component assigning morphisms $\alpha, \beta : B_0 \to \mathbb{F}(\mathcal{G})_1$ are given by $(\eta_{\mathcal{G}})_1 \cdot i^{\mathbb{B}}$ and $i^{\mathbb{F}(\mathcal{G})} \cdot k_0$ respectively, as displayed below right.*

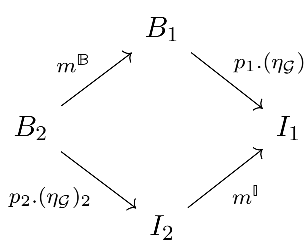




Proof. As $\mathbf{disc}(B_0)$ is discrete, it suffices to show that $\underline{\alpha}$ and $\underline{\beta}$ respect sources and targets. For α this follows from sources and targets for identities for the category $\mathbb{B}$, while for β this follows from the same axioms for the category $\mathbb{F}(\mathcal{G})$. □

Lemma 6.4. *Let $p : \mathbb{F}(\mathcal{G}) \to \mathbb{I}$ be the coequifier of $\underline{\alpha}$ and $\underline{\beta}$. There is a parallel pair of natural transformations $\underline{\gamma}, \underline{\delta} : p \cdot k_2 \cdot m^{\mathbb{B}} \Rightarrow p \cdot m^{\mathbb{F}(\mathcal{G})} \cdot \eta_{\mathcal{G}_1} : \mathbf{disc}(B_2) \to \mathbb{I}$ as displayed below left, whose component assigning morphisms $\underline{\gamma}, \underline{\delta} : B_2 \to \mathbb{I}_1$ are given by $p_1 \cdot (\eta_{\mathcal{G}})_1 \cdot m^{\mathbb{B}}$ and $m^{\mathbb{I}} \cdot p_2 \cdot (\eta_{\mathcal{G}})_2$ respectively, as displayed below right.*



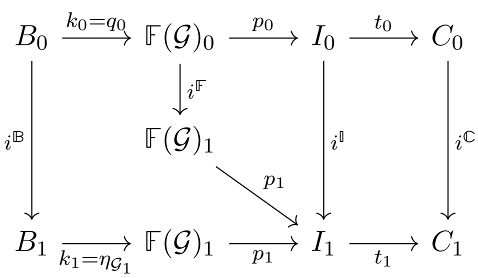


Proof. The proof is similar to that for Lemma 6.3, now using sources and targets for composition for the category $\mathbb{B}$ to prove that γ respects sources and targets, and sources and targets for the category $\mathbb{I}$ to prove that δ respects sources and targets. □

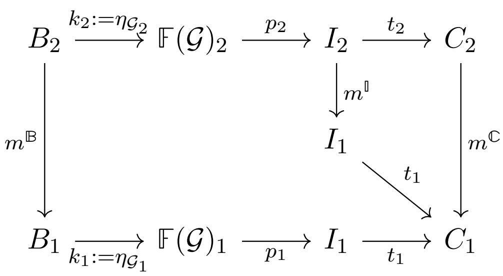
Lemma 6.5. *Let $t : \mathbb{I} \to \mathbb{C}$ be the coequifier of the natural transformations γ and δ of Lemma 6.4. The morphism of graphs displayed below is well-defined as an internal functor.*

$$Q := (\mathbb{B} \xrightarrow{k} \mathbb{F}(\mathcal{G}) \xrightarrow{p} \mathbb{I} \xrightarrow{t} \mathbb{C})$$

Proof. Respect for identities is witnessed by the commutativity of the following diagram, in which the left region commutes by the definition of the coequifier $p : \mathbb{F}(\mathcal{G}) \to \mathbb{I}$, and the other regions commute by functoriality of p and t.



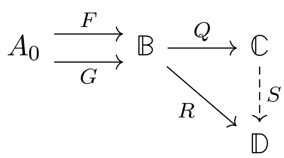
Respect for composition is witnessed by the commutativity of the following diagram, in which the region on the left commutes by definition of the coequifier $t : \mathbb{I} \to \mathbb{C}$ and the region on the right commutes by functoriality of t.



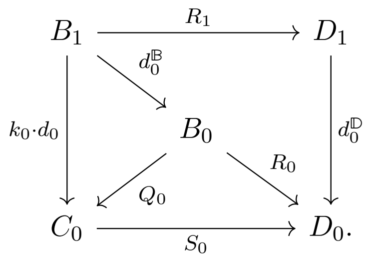
□

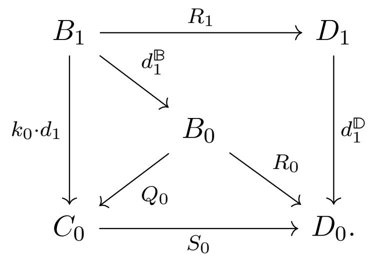
Proposition 6.6. *The internal functors $F, G : A_0 \to \mathbb{B}$ in* **Cat**$(\mathcal{E})$ *have a coequaliser given by $Q : \mathbb{B} \to \mathbb{C}$, where this internal functor is defined as in Lemma 6.5.*

Proof. Given an internal functor $R : \mathbb{B} \to \mathbb{D}$ such that $RF = RG$, we show that there exists a unique internal functor $S : \mathbb{C} \to \mathbb{D}$ satisfying $SQ = R$.

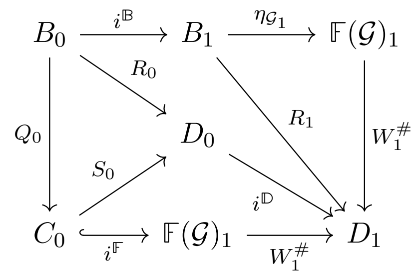


Define $S_0 : C_0 \to D_0$ by the universal property of k_0 as the coequaliser on objects. Note that there is a morphism of graphs $W := (S_0, R_1) : \mathcal{G} \to \mathcal{U}\mathbb{D}$ as exhibited by the commutativity of the following diagrams:

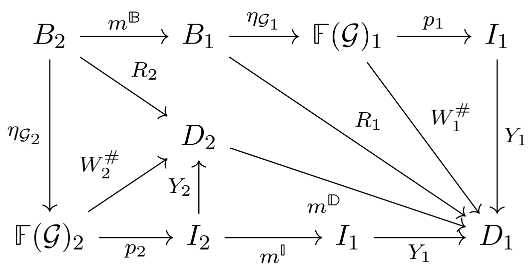




Hence, by the adjunction $\mathbb{F} \dashv \mathcal{U}$, there exists a unique internal functor $W^{\#} : \mathbb{F}(\mathcal{G}) \to \mathbb{D}$ such that $\mathcal{U}(W^{\#})\eta_{\mathcal{G}} = W$. The commutativity of the following diagram shows that $W^{\#}$ coequifies the natural transformations in Equation 6.3, which induces a unique functor $Y : \mathbb{I} \to \mathbb{D}$.



The commutativity of the following diagram shows that Y coequifies the natural transformations in Equation 6.4, which induces a unique functor $Z : \mathbb{C} \to \mathbb{D}$.



By construction, $ZQ = R\mathbb{B} \to \mathbb{C}$ and $R : \mathbb{C} \to \mathbb{D}$ is the unique such functor that does this, as required.

□

7. Coequalisers of arbitrary pairs

In this Section, we put together all the work from previous sections in order to show that $\mathbf{Cat}(\mathcal{E})$ has coequalisers of arbitrary pairs of arrows. Moreover this gives a recipe for how to calculate coequalisers in $\mathbf{Cat}(\mathcal{E})$. We give a proof of this through Lemma 7.1, which is a more general statement about coequalisers in 2-categories $\mathcal{K}$ for which the inclusion of discrete objects $\mathbf{disc}(\mathcal{K}) \to \mathcal{K}$ is sufficiently well-behaved. Our previous results allow us to apply this lemma to the 2-category $\mathbf{Cat}(\mathcal{E})$.

Lemma 7.1. *Let $\mathcal{K}$ be a 2-category for which the inclusion of the full-subcategory of discrete objects $\mathbf{disc} : \mathbf{Disc}(\mathcal{K}) \to \mathcal{K}$ has a left adjoint $(-)_0$ with counit $\varepsilon : \mathbf{disc}((-)_0) \to 1_{\mathcal{K}}$ and unit which is given component-wise by identities. Suppose $\mathcal{K}$ has coequalisers of any parallel pair $f, g : A \to B$ for which either of the following conditions hold.*

(1) $f_0 = g_0$, or
(2) A is in the image of $\mathbf{disc}$.

Then $\mathcal{K}$ has all coequalisers.

Proof. Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be a parallel pair. By condition (2), $\mathcal{K}$ has the coequaliser of $f \cdot \varepsilon_A$ with $g \cdot \varepsilon_A$. Let $q : B \to C$ denote this coequaliser; it has the property that $qf \cdot \epsilon_A = qg \cdot \epsilon_A$. Applying $(-)_0$ to this, and by noting that $A_0 = \mathbf{disc}(A_0)_0$ since the unit has identities as its components and by the triangle identities for the adjunction, it follows that $(\epsilon_A)_0 = 1_{A_0}$, so $(qf)_0 = (qf \cdot \epsilon_A)_0 = (qg \cdot \epsilon_A)_0 = (qg)_0$, so by condition (1), qf and qg have a coequaliser, $p : C \to D$. We claim that $qp : B \to D$ is the required coequaliser of f and g. Certainly, $qpf = qpg$ as they agree on objects and arrows by construction, so it remains to show the universal property of the coequaliser holds. Given $r : B \to E$ such that $rf = rg$, then $rf \cdot \epsilon_A = rg \cdot \epsilon_A$ and so by the universal property of C as a coequaliser of $f.\epsilon_A$ and $g \cdot \epsilon_A$ we get an induced unique arrow $t : C \to E$. But then $t(qf) = rf = rg = t(qg)$ so by the universal property of D as the coequaliser of qf and qg, we get an induced unique arrow $w : D \to E$ such that $wpqf = wpqg$, as required. □

We are now able to verify our main result.

Theorem 7.2. *Let $\mathcal{E}$ be a list-arithmetic pretopos with finite pullback stable coequalisers. Then the 2-category $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits.*

Proof. From the discussion in Section 3, it suffices to show that $\mathbf{Cat}(\mathcal{E})$ has coequalisers. To do this, we verify that Lemma 7.1 applies to $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$. It is well known that $\mathbf{disc} : \mathbf{Disc}(\mathbf{Cat}(\mathcal{E})) = \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ has left adjoint given by $(-)_0 : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$, with $\mathbf{disc}(E)_0 = E$ for any $E \in \mathcal{E}$. By Proposition 4.1, condition (1) of Lemma 7.1 holds while by Proposition 6.6, condition (2) of Lemma 7.1 holds. □

Remark 7.3. In particular, when $\mathcal{E}$ is an elementary topos with a natural numbers object, such as is the case in the setting of [HM24b], the 2-category $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits.

In light of Remark 5.5 which tells us how to use 2-colimits to construct free internal categories on an internal graph, we have the following corollary to Theorem 7.2, which gives a partial characterisation of when $\mathbf{Cat}(\mathcal{E})$ has 2-colimits.

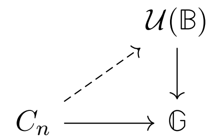
Corollary 7.4. *Let $\mathcal{E}$ be an extensive category with pullback stable coequalisers. Then $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits if and only if there is a left adjoint to $U : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$.*

Proof. If there is a left adjoint to $U : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ then Theorem 7.2 shows that $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits. Conversely, if $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits, then Remark 5.5 tells us how to construct the free category on a graph using coinserters. □

It should be noted that a list-arithmetic pretoposes form the most general known class of a categories that admit free internal categories on internal graphs.

Remark 7.5. Parameterised list objects in $\mathcal{E}$ are needed to form free categories on graphs, which are used in the construction of general coequalisers in $\mathbf{Cat}(\mathcal{E})$. However, it is of interest to describe the coequalisers that exist in $\mathbf{Cat}(\mathcal{E})$ when milder assumptions are made on $\mathcal{E}$, such as just exactness properties between limits and colimits. Let $\mathcal{E}$ have finite limits and colimits and suppose moreover that it is lextensive and has pullback stable coequalisers. Consider a parallel pair of internal functors $F, G : \mathbb{A} \to \mathbb{B}$ and let $Q_0 : B_0 \to C_0$ denote the coequaliser of F_0 and G_0. We briefly describe, without proof, what we believe should be a sufficient condition that is weaker than the existence of the free category on the graph $\mathbb{G} := B_1 \overset{Q_0.d_1}{\underset{Q_0.d_0}{\rightrightarrows}} C_0$ but under which the coequaliser of F and G still exists in $\mathbf{Cat}(\mathcal{E})$. We describe this explicitly when $\mathcal{E} := \mathbf{FinSet}$ and leave the generalisation to the internal setting to the interested reader. Let $C_n \in \mathbf{Gph}(\mathcal{E})$ denote the cycle of length n; this can be built by first constructing the path of length n using the terminal object and coproducts, and then using a coequaliser to identify the source and target of the path. Then the coequaliser of $F, G : \mathbb{A} \to \mathbb{B}$ exists in $\mathbf{Cat}(\mathcal{E})$ if for all $n \in \mathbb{N}$ and any map $C_n \to \mathbb{G}$, the following lifting problem has a solution in $\mathbf{Gph}(\mathcal{E})$.

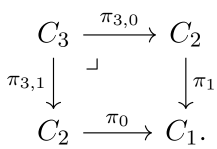
(4)



This is to say that any cycles which appear in the graph produced by taking equivalence classes of objects in $\mathbb{B}$ already exist in the underlying graph of $\mathbb{B}$ itself. This means that the coequaliser of $F.\varepsilon_{\mathbb{A}}$ and $G.\varepsilon_{\mathbb{A}}$ can be formed in $\mathbf{Cat}(\mathcal{E})$, without using parameterised list objects in $\mathcal{E}$. We leave detailed verification of this construction under these milder assumptions to future work.

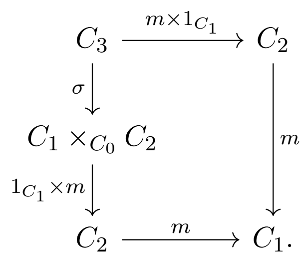
Appendix A. A proof of associativity in Proposition 4.1

We define C_3 as the following pullback.

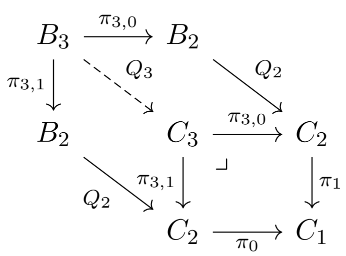


To show associativity, we must show that the following diagram commutes

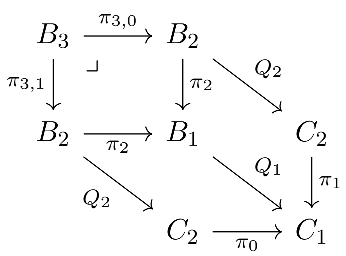
(5)



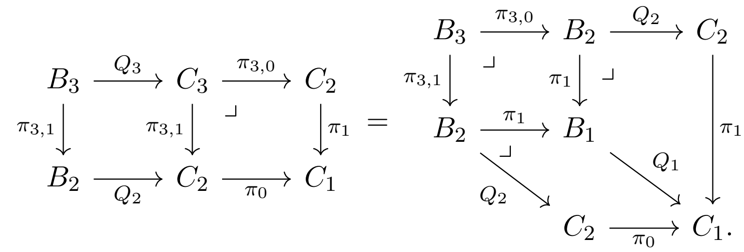
Construct $Q_3 : B_3 \to C_3$ by the universal property of C_3 as a pullback as in the following diagram



in which Q_3 exists by the commutativity of the following diagram:



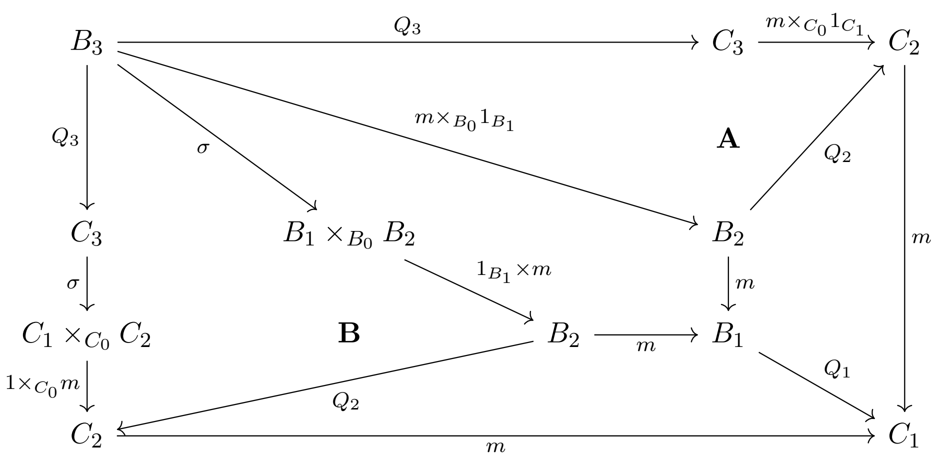
We can express Q_3 as the pullback of Q_2 along Q_2 along $\pi_{1,3}$ by a few applications of the pullback lemma, given the equation of diagrams below, which follows from the definitions of Q_1, Q_2 and Q_3.



Since coequalisers are assumed to be stable under pullback in $\mathcal{E}$, it follows that the following diagram is a coequaliser diagram in $\mathcal{E}$

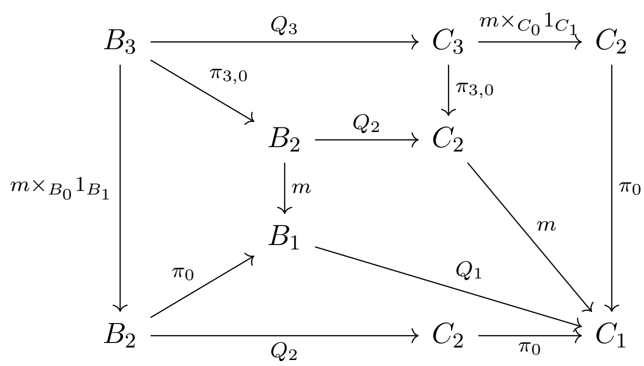
$$A_3 \underset{G_3}{\overset{F_3}{\rightrightarrows}} B_3 \xrightarrow{Q_3} C_3.$$

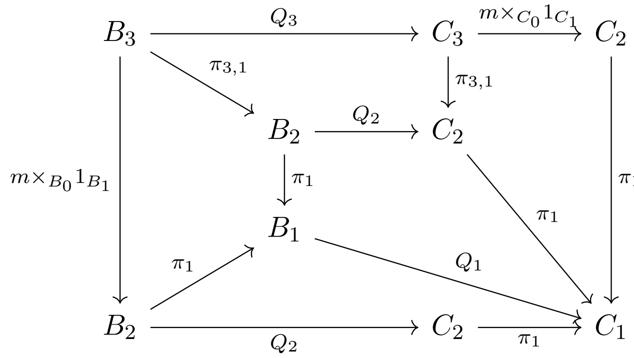
Hence we can appeal to the universal property of the coequaliser: to show that Diagram 5 commutes, it is enough to show that the diagram commutes when precomposed with Q_3. This is witnessed by the following diagram.



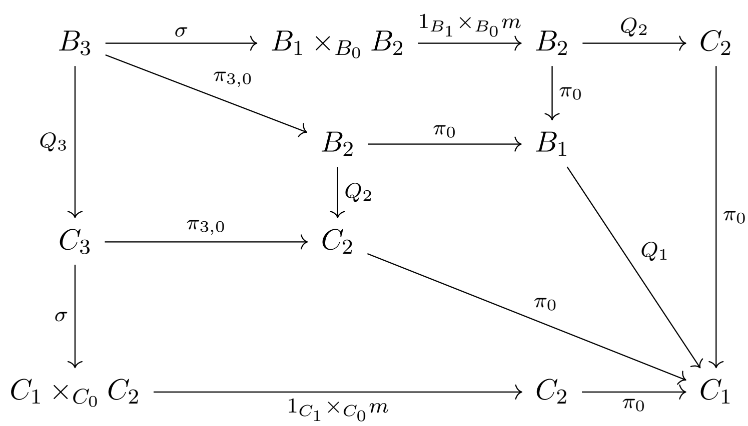
In the above, the regions labelled **A** and **B** are shown to commute by appealing to the universal property of C_2 as a pullback of $\pi_0, \pi_1 : C_2 \mathbb{C}_1$, and showing that the regions commute after postcomposing with these projections.

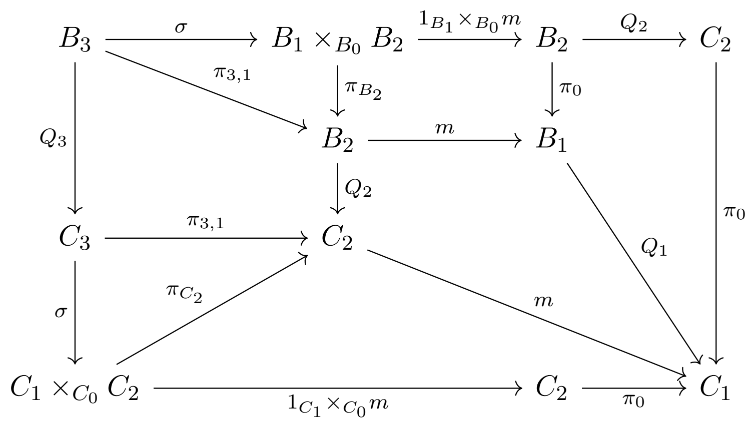
The commutativity of the region **A** is shown by the following pair of commutative diagrams.





The commutativity of the region **B** is shown by the following pair of commutative diagrams.





Putting all the above steps together, we have shown that associativity holds.

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