The Dold-Kan Correspondence

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1. Introduction

Algebraic topology is one of the main areas of modern pure mathematical research, but has also recently found broader applications to the fields of data science [Car09], quantum mechanics [VR19, BV14] and computing [Zom05] due to its ability to solve problems in high dimensions without the need to visualise it.

It was invented to study structure and continuity; a topological space can be thought of as a generalisation of a subset of $\mathbb{R}^n$, and we can ask if two topological spaces are structurally the same by asking if there is a continuous bijection between them with continuous inverse. Such a map is called a *homeomorphism*. Just as in set theory, in which we only really care about sets up to bijection, and in group theory, in which we only really care about groups up to isomorphism, we only really care about topological spaces up to homeomorphism. One of the main goals of algebraic topology is to assign an algebraic object (a sequence of groups for example) to a topological space in some way that is invariant under homeomorphism; therefore, we can tell if two spaces are *not* homeomorphic if the invariant for the spaces is different.

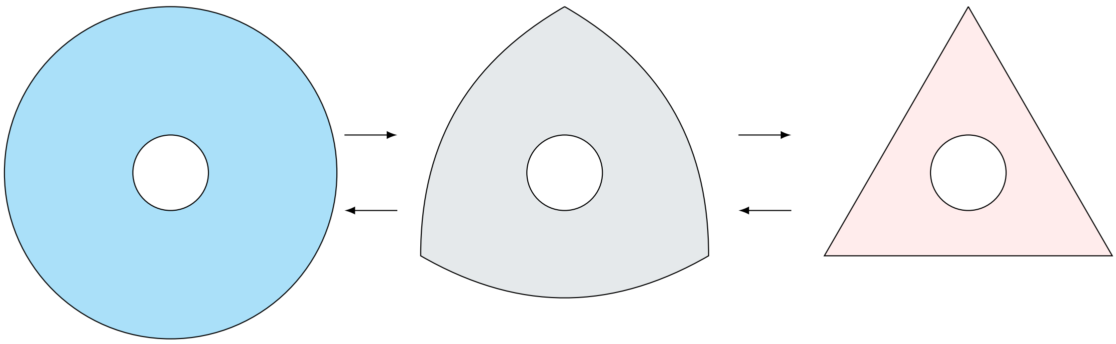


Figure 1. A homeomorphism. This is intuitively continuous, bijective and with continuous inverse.

Broadly speaking, algebraic topology splits naturally into two branches: homology and homotopy [Hat01]. As such, it is important to understand how these two relate. The Dold-Kan correspondence is a theorem which establishes a link between these two areas by showing that certain algebraic objects used to study homology are equivalent in some sense to algebraic objects that are used to study homotopy theory. It does this by showing that there is an *equivalence of categories* between the category of simplicial abelian groups and the category of non-negatively graded chain complexes of abelian groups. In order to understand the Dold-Kan correspondence, then, we must introduce the language of category theory; this is the content of section 2.

Homotopy theory was originally invented to study a topological space by trying to understand maps into this space, from which structural elements can be extracted. One approach to this theory is to translate the space into a *simplicial set*, which is a purely combinatorial object that is meant to have similar geometric properties to n-dimensional tetrahedra, which we call *topological n-simplices*. From a homotopical point of view, a particularly nice type of simplicial set is a *Kan complex*; in order to ensure we are working with these, we restrict our attention to *simplicial abelian groups*. The motivation and details of this are given in section 3.

Homology was invented in the late 1800s in order to study n-dimensional holes in spaces [Wei95]. One approach to this theory is *singular homology*, in which we translate from topological spaces to abelian groups by considering linear combinations of maps from topological n-simplices into a space X, and form the *topological boundary map* between each dimension. By studying the image and kernel of this map, we can form the *nth* *homology group* $H_n(X)$, which is an invariant for the space and, moreover, tells us the number of n-dimensional holes in the space. In order to make this rigorous and more general, we introduce *non-negatively graded chain complexes*, which abstractly capture this idea. More information about these tools is given in section 4.

The Dold-Kan correspondence gives a way of turning a non-negatively graded chain complex into a simplicial abelian group and vice-versa. It is perhaps surprising that these two tools are equivalent, as naively it seems as though simplicial abelian groups have more complexity to them than non-negatively graded chain complexes. Indeed, historically, homotopy theory has been less well understood than homology. As a result of the Dold-Kan correspondence, mathematicians have been able to advance homotopy theory by a great deal using homological methods ([GJ99], section III.2 for example). However, there have recently been a lot of advances in homotopy theory due to the development of areas of study such as ∞-category theory [RV22], and categorical homotopy theory [Rie14]. Due to Dold-Kan, these developments can be used to study homological algebra, and have solved many problems in the area ([Wei95], chapter 8).

The Dold-Kan correspondence is named as such due to it being independently discovered by Albrecht Dold and Daniel Kan in 1958 [Dol58, Kan58]. We present a proof of the abelian group version of the correspondence in section 5. In 1961, Dold and Puppe proved a generalisation of the correspondence to the setting of *abelian categories* [DP61]. Examples of abelian categories include the category of abelian groups, the category of R-modules and the category of vector spaces, but we restrict our attention in this project to abelian groups due to the background required to explain and motivate more general setting in detail.

In 1967, in order to study homotopy theory in a more abstract setting, Daniel Quillen introduced *model categories* and a kind of equivalence between these called a *Quillen equivalence*, which preserves all homotopical relations [Qui67]. In section 6, we upgrade the Dold-Kan correspondence to a Quillen equivalence.

In section 7, we explain some extensions of the Dold-Kan correspondence to more general settings and give concluding remarks.

This project is intended to be an intuitive and self-contained introduction to the Dold-Kan correspondence and the motivation behind it. Due to the background theory needed to motivate this topic, and the diagrams, figures and display mathematics needed to explain the content properly, this project is longer than the 40 page limit. The writer believes a shorter project would have negatively impacted either the clarity or the completeness of explanations. Topological spaces and homeomorphism are not formally introduced due to space constraints, but are used as an intuitive and motivating example throughout; no technical details are needed.

This project provides more direct and explicit proofs of the Dold-Kan correspondence and the associated Quillen equivalence than is found in the literature; as it is aimed at fourth year students who are not necessarily comfortable with category theoretic reasoning, a lot of the machinery used in the proofs in [Wei95] and [GJ99] for example would require too much background. Moreover, subsection 5.2 gives a much more detailed explanation of the functor K than is usually given, including diagrams and examples of how it acts in low dimensions. The proof of the Quillen equivalence (theorem 6.35) is notably much more direct than that found in [GJ99], for example. Most proofs and examples are the authors own; references are given when a proof has been influenced by the literature.

2. Category Theory

Category theory was invented in the 1940s by Eilenberg and Mac Lane in order to study homological algebra, but with the goal of understanding mathematical structure more generally [ML63]. As such, it is quite abstract—indeed it has been referred to, often jokingly, as "abstract nonsense" ([Lan02], p.175). In many ways, it is like learning a new language that has its own 'nouns' and 'verbs' [Rie17]; it has many descriptive definitions in it which allow us to be very expressive and compact with our descriptions of mathematical objects. Whilst learning this language does not require any prerequisites, it formalises mathematical ways of thinking making it applicable to a whole variety of different areas within mathematics.

One example of this is that a category theorist treats the concepts of bijections, group isomorphisms and homeomorphisms between spaces identically. We give one definition for an *isomorphism* in an arbitrary category, and then by specifying that we are working in the category of **Set**, **Group** or **Top** (the categories of sets, groups or topological spaces respectively), and unravelling the definition, we obtain a bijection, a group isomorphism or a homeomorphism. This shows how category theory is useful for packaging up information with the power to understand concepts in different contexts.

Category theory takes the standpoint of understanding an object not by looking at the object itself, but instead by trying to understand maps into and out of the object [FS19]. Despite being invented as a tool for use in algebraic topology, and being highly useful in the area [Rie14], modern research in category theory has applications outsider this, including work on control systems theory [Mye22], analysis [Law73], ecology [Lei21], algebra [Alu21] and is becoming increasingly important to logic with the development of the new field of Homotopy Type Theory [Uni13]. A lot of modern computing languages are written in a category theoretic way, such as the language Haskell [Mil18].

2.1. **Categories.** This chapter explains the category theory necessary to prove the Dold-Kan correspondence; for a more complete introduction to category theory, the reader is referred to [Rie17].

Definition 2.1. A *category* $\mathcal{C}$, consists of

- A class of *objects*, X, Y, Z denoted $\text{ob}(\mathcal{C})$.
- A class of *morphisms* f, g, h between objects, denoted $\text{mor}(\mathcal{C})$. For the class of morphisms between two objects X and Y, we write $\text{Hom}_{\mathcal{C}}(X, Y)$.

such that:

- A morphism has specified *domain* and *codomain* in $\text{ob}(\mathcal{C})$. The notation $f : X \to Y$ means that the domain and codomain of f is X and Y respectively.
- For any $X \in \text{ob}(\mathcal{C})$, there exists an *identity morphism* $1_X : X \to X$.
- For any pair of morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a *composite morphism* $g \circ f : X \to Z$. We sometimes write gf when it is clear that this is a composition.

The morphisms must follow the following composition rules:

- For any $f : X \to Y$ we have $1_X f = f = f 1_Y$.
- For any composable triple $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ we have
$$h(gf) = (hg)f.$$
We therefore forget brackets and write $hgf : X \to W$. This is called *the associative property.*

A lot of this notation will probably be very familiar to the reader, and suggests the notion of sets and functions, or groups and group homomorphisms. We show that a category does indeed generalise these notions.

Example 2.2. The category **Set**, has:

- Sets A, B, C as its objects.
- Functions f, g, h as its morphisms.

There exists an identity function on any set— for any set A, we define a function $id_A : A \to A$ given by $id_A(a) = a$ for all $a \in A$. For the functions $f : A \to B$, $g : B \to C$ and $h : C \to D$ we can certainly form composite function $g \circ f : A \to B$ and this composition is associative: $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$, as proven in [Rod00]. Moreover, it is trivial to see that $Id_A \circ f = f = f \circ Id_B$. Therefore, we see that **Set** is a category.

Example 2.3. The category **Group**, has:

- Groups as as its objects.
- Group homomorphisms as its morphisms.

It is not difficult to check that this satisfies the axioms.

Remark 2.4. We can define the categories **Ring**, **Field**, $\mathbf{Vect}_{\mathbb{K}}$ similarly to **Group**, but replacing homomorphism of *groups* with the correct homomorphism respective to the type of objects.

Example 2.5. The category **Top**, has:

- Topological spaces as as its objects.
- Continuous maps as its morphisms.

These next examples are important categories in this project.

Example 2.6. The category **Ab**, has:

- Abelian groups as as its objects.
- Group homomorphisms between abelian groups as its morphisms.

Example 2.7. Let $\mathbf{\Delta}$ be the category which has:

- Finite, non-empty, totally ordered sets as objects. By this, we mean sets of the form $\{a_0 \leq a_2 \leq a_3 \leq ...\}$. We write $[n] = \{0, 1, 2, ..., n\}$.
- Order-preserving functions as morphisms. That is, functions $\alpha : A \to B$ with $a_i \leq a_j$ in $A \implies \alpha(a_i) \leq \alpha(a_j)$ in B.

Since identities are order-preserving, and the composition of two order-preserving functions is also order-preserving, it is not hard to check that this is a well-defined category.

We have the following properties of morphisms in the category $\mathbf{\Delta}$, which will be used many times in later sections.

Definition 2.8. For each $0 \leq i \leq n$, we define $\epsilon_i : [n] \to [n+1]$ to be the unique injective map in $\mathbf{\Delta}$ so that its image misses out the ith entry:

$$\epsilon_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

We define $\eta_i : [n+1] \to [n]$ to be the unique surjective map in $\mathbf{\Delta}$ such that it has two elements mapping to i:

$$\eta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Example 2.9. The image of $[2]$ under $\epsilon_2 : [2] \to [3]$ is given by $\epsilon_2([2]) = \{0, 1, 3\} \subseteq [3]$.

Remark 2.10. It might be more appropriate to label these maps as ϵ_i^n and η_i^n to make these distinct from ϵ_i^{n-1} and η_i^{n-1}, for example. However, it is usually obvious what the codomain is, and this avoids the over-use of superscripts.

Lemma 2.11. *We have the following identities in* $\mathbf{\Delta}$*:*

(1) $\epsilon_j \epsilon_i = \epsilon_i \epsilon_{j-1}$ *if* $i < j$*,*

(2) $\eta_j\eta_i = \eta_i\eta_{j+1}$ *if* $i < j$,

(3) $\eta_j\epsilon_i = \begin{cases} \epsilon_i\eta_{j-1} & \textit{if } i < j, \\ \mathbb{1}_{[n]} & \textit{if } i \in \{j, j+1\}, \\ \epsilon_{i-1}\eta_j & \textit{if } i > j+1. \end{cases}$

Proof. Let $0 \leq i < j \leq n$. We prove (1); the proofs of (2) and (3) follow a similar line of argument. If $k < i$, then $k < j$ so $\epsilon_j\epsilon_i(k) = k$. If $k > i$ then $\epsilon_i(k) = k+1$ and we have two situations that could occur: $k+1 < j$, in which case $\epsilon_j(k+1) = k+1$, or $k+1 > j$, in which case $\epsilon_j(k+1) = k+2$. Together, we have:

$$\epsilon_j\epsilon_i(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } i \leq k+1 < j, \\ k+2 & \text{if } k+1 \geq j. \end{cases}$$

Similarly, if $k \geq j-1$, then $k+1 \geq j > i$, so $\epsilon_i\epsilon_{j-1}(k) = k+2$. If $k < j-1$ then, again, we get two scenarios: $k < i$, in which case $\epsilon_i\epsilon_{j-1}(k) = k$ and $i \leq k < j-1$, in which case $\epsilon_i\epsilon_{j-1}(k) = k+1$. Together, we have:

$$\epsilon_i\epsilon_{j-1}(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } i \leq k < j-1 \implies i \leq k+1 < j \\ k+2 & \text{if } k \geq j-1 \implies k+1 \geq j. \end{cases}$$

Therefore, these maps are equal.

□

Lemma 2.12. *Let α be a morphism in $\mathbf{\Delta}$. Then there is a unique epi-monic factorisation of $\alpha = \epsilon\eta$ where ϵ is an epimorphism which is the unique combinations of the ϵ_i:*

$$\epsilon = \epsilon_{i_1}...\epsilon_{i_s} \textit{ with } 0 \leq i_s \leq ... \leq i_1 \leq m$$

and η is a monomorphism which is the unique composition of the η_i:

$$\eta = \eta_{j_1}...\eta_{j_t}, \textit{ with } 0 \leq j_1 \leq ... < j_t < n.$$

Proof. Let $\alpha : [n] \to [m]$. An order-preserving function between ordered sets is fully determined by its image and by the set of numbers in $[n]$ which do not increase, i.e. $\alpha(j) = \alpha(j+1)$. The image of α can be written as a list of numbers $v_0 \leq ... \leq v_m$ with some of $\{0, 1, 2, ..., m\}$ possibly missing or repeating. Let $i_1 < ... < i_s$ be the missing numbers, and $j_1 < ... < j_t$ be the numbers such that $\alpha(j) = \alpha(j+1)$. We note that $n+1 = m+1-s+t$ as we start with $n+1$ numbers in the domain, these are sent to $m+1-s$ distinct numbers in $[m]$, but t of these are repeats that make up the rest of the required list size. Set $p = m-s = n-t$. Then $\eta = \eta_{j_1}...\eta_{j_t} : [n] \twoheadrightarrow [p]$ is an epimorphism as it is the composition of epimorphisms (lemma 2.18), and has $\eta(j_r) = \eta(j_{r+1})$ for $j_1 < j_2 < ... < j_t$ by definition of η_j. Moreover, $\epsilon = \epsilon_{i_1}...\epsilon_{i_s} : [p] \hookrightarrow [m]$ is a monomorphism as it is the composition of monomorphisms and it misses out the numbers $0 \leq i_s \leq ... \leq i_1$ by definition of ϵ_i. Hence $\alpha = \epsilon\eta$. Uniqueness follows from specifying the ordering of the i_ks and j_ls. □

Therefore we can see that the ϵ_is and η_is generate all morphisms of $\mathbf{\Delta}$.

2.1.1. *Duality and The Opposite Category.* In linear algebra, we can prove propositions about a finite-dimensional vector space V by proving things about the dual vector space V^*, which is isomorphic to V and has $(V^*)^* = V$. We have a similar notion in category theory; to obtain this dual, we can consider morphisms as arrows and imagine reversing the directions of all the arrows.

Definition 2.13. Let $\mathcal{C}$ be a category. The *opposite category*, denoted $\mathcal{C}^{op}$ has:

- The same objects as $\mathcal{C}$.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $f^{op} : Y \to X$ in $\mathcal{C}^{op}$.

For any composable morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, by defining $f^{op} \circ g^{op} = (g \circ f)^{op}$, we have a composition in $\mathcal{C}^{op}$ and it is clear that we can that $\mathbb{1}_X^{op} = \mathbb{1}_X$ for all X. The other axioms follow from the structure of $\mathcal{C}$. Hence $\mathcal{C}^{op}$ is a well-defined category.

Remark 2.14. For any category $\mathcal{C}$, we have $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Any statement about a category $\mathcal{C}$ has a *dual* statement about the category $\mathcal{C}^{op}$; any proof in category theory simultaneously proves both a proposition and its dual.

2.1.2. *Types of Morphisms.* The philosophy behind category theory is to look at morphisms rather than objects themselves. As such, it makes sense that we should have some words describing kinds of morphisms.

Definition 2.15. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called:

- A *monomorphism* if for any two morphisms $h, k : W \to X$, $fh = fk$ implies that $h = k$.
- An *epimorphism* if for any two morphisms $h, k : Y \to Z$, $hf = kf$ implies that $h = k$.

We call monomorphisms *monic* and epimorphisms *epi.* We sometimes use decorated arrows $f : X \rightarrowtail Y$ to denote that f is a monomorphism and $f : X \twoheadrightarrow Y$ to denote that f is an epimorphism. Monomorphisms are the dual notion to epimorphisms.

Example 2.16. A morphism in **Set** is a monomorphism if and only if it is an injective function. Let $f : X \to Y$ be a monomorphism and consider $x, x' : \mathbf{1} \to X$, where $\mathbf{1}$ denotes a set with a single element. Then $fx = fx' \implies x = x'$ as f is monic. Hence by the one-to-one correspondence between elements $x \in X$ and maps $x : \mathbf{1} \to X$, we see that f is injective. Conversely, if f is an injective function, then for any $h, k : W \to X$, we have $fh = fk \implies f(h(w)) = f(k(w))$ for all $w \in W$. Since f is injective, this implies that $h(w) = k(w)$ for all $w \in W$, which is exactly saying $h = k$ Hence f is monic.

Example 2.17. A morphism in **Set** is an epimorphism if and only if it is a surjective function. The condition $hf = kf$ means that $h(f(x)) = k(f(x))$ for all $x \in X$. This only implies that $h = k$ if $\mathrm{img}(f) = Y$.

Lemma 2.18. *The composition of two monomorphisms is again a monomorphism. Dually, the composition of two epimorphisms is again an epimorphism.*

Proof. Let $g : W \to X$ and $f : X \to Y$ be monic. Then, for $h, k : Y \to Z$, we show that if $(fg)h = (fg)k$ this implies $h = k$. By associativity, we know $f(gh) = f(gk)$. As f is monic, this implies that $gh = gk$, but since g is also monic, this implies that $h = k$, as required.

The proof for epimorphisms is similar. □

Definition 2.19. Let $f : X \to Y$ be a morphism in some category. A *section* of f is a morphism $g : Y \to X$ such that $fg = \mathbb{1}_Y$.

If a section of f exists, then it is clear that f is an epimorphism by applying the right-sided inverse to both sides of $hf = kf$. To acknowledge the existence of a section, we say that f is a *split epimorphism.*

Lemma 2.20. *In $\mathbf{\Delta}$, every epimorphism $\eta : [n] \twoheadrightarrow [p]$ is a split epimorphism, and so there exists a section $\rho : [p] \rightarrowtail [n]$ with $\rho\eta = \mathbb{1}_{[n]}$.*

Proof. For $\eta_i : [n] \to [n-1]$, a right sided inverse is given by ϵ_i; by lemma 2.11, we have $\eta_i \epsilon_i = id$. Now, any other surjection is the composition of the η_i maps— for $\eta = \eta_{i_1} ... \eta_{i_t}$, the right sided inverse is $\epsilon_{i_t} ... \epsilon_{i_1}$. □

Definition 2.21. An *isomorphism* is morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ with $gf = \mathbb{1}_Y$ and $fg = \mathbb{1}_X$. We sometimes use the decorated arrow $f : X \xrightarrow{\sim} Y$ to denote that f is an isomorphism. If $f : X \xrightarrow{\sim} Y$, then we say X is *isomorphic* to Y.

We can see this agrees with our previous notions of isomorphisms but shifts the focus of the definitions away from the objects and onto the morphisms.

Example 2.22. An isomorphism in **Set** is exactly the statement that two sets are in bijection with one another.

Example 2.23. An isomorphism in **Group** is exactly the statement that two groups are isomorphic.

Example 2.24. An isomorphism in **Top** is a homeomorphism.

Lemma 2.25. *A morphism in* **Ab** *is an isomorphism iff it is both epi and monic.*

Proof. A homomorphism of abelian groups f is an isomorphism if and only if it is injective and surjective. But in **Ab**, f is injective and surjective if and only if it is epi and monic. $\square$

This not always the case, as is exemplified in this example from [Rie17].

Example 2.26. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ given by is both epi and monic in the category **Ring**. However, it is clear that $\mathbb{Z}$ is not isomorphic to $\mathbb{Q}$.

2.2. **Functors.** We can take one step backwards and consider the category of categories. However, this causes a problem similar to that of Russel's paradox [Rie17]. In order to solve this, we call a category *locally small* if $Hom_{\mathcal{C}}(X, Y)$ is a set for all objects X and Y in $\mathcal{C}$.

Example 2.27. There is a category **CAT**, of locally small categories. Its objects are locally small categories, and its morphisms are what we call *functors*. ([Rie17], 1.3.13)

A functor is a structure-preserving morphism between categories. The structure it preserves is domains, codomains, composition and identities.

Definition 2.28. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* is a map from $F : \mathcal{C} \to \mathcal{D}$ defined by:

- For every object $X \in \mathrm{ob}(\mathcal{C})$ an object $F(X) \in \mathrm{ob}(\mathcal{D})$.
- For every morphism $f : X \to Y \in \mathrm{mor}(\mathcal{C})$, a morphism $Ff : FX \to FY \in \mathrm{mor}(\mathcal{D})$.

These must be defined in a way that satisfies the *functoriality axioms*:

(F1) For every composable pair $f, g \in Mor(\mathcal{C})$, we have $F(gf) = F(g)F(f)$.
(F2) For each object $X \in \mathrm{ob}(\mathcal{C})$, $F(\mathbb{1}_X) = \mathbb{1}_{FX}$.

We will often drop the brackets to aid reading clarity, for example writing FX instead of $F(X)$.

Example 2.29. Let $\mathcal{C}$ be a category. Then there is a functor called the *identity functor on* $\mathcal{C}$, denoted $\mathbf{1}_{\mathcal{C}}$ that on objects X in $\mathcal{C}$ has $\mathbb{1}_{\mathcal{C}}(X) = X$ and on morphisms f in $\mathcal{C}$ has $\mathbb{1}_{\mathcal{C}}(f) = f$. This trivially satisfies the functoriality axioms. As a result, any inclusion of a category into another category is a functor.

Functors allow us to move between categories.

Example 2.30. The forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$ sends any abelian group to its underlying set.

2.2.1. *Covariant and Contravariant functors.* Functors how we have defined them are also called *covariant functors*. There is another flavour of functor:

Definition 2.31. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* is a map from $F : \mathcal{C} \to \mathcal{D}$ defined by:

- For every object $X \in \text{ob}(\mathcal{C})$ an object $FX \in \text{ob}(\mathcal{D})$.
- For every morphism $f : X \to Y \in \text{mor}(\mathcal{C})$, a morphism $Ff : FY \to FX \in \text{mor}(\mathcal{D})$.

This must be defined such that they satisfy the *cofunctoriality axioms*:

(CF1) For every composable pair of morphisms $f, g \in \text{mor}(\mathcal{C})$, we have $F(gf) = F(f)F(g)$.
(CF2) For each object $X \in \text{ob}(\mathcal{C})$, $F(\mathbb{1}_X) = \mathbb{1}_{FX}$.

Note that this is the same data as a covariant functor, but with arrows reversed by the functor. This causes the change to the order of composition seen in the difference between (CF1) and (F1). As such, it makes sense to view this as a covariant functor from the dual category.

Example 2.32. Let $\mathcal{C}$ be a category. There is a contravariant functor $(-)^{op} : \mathcal{C} \to \mathcal{C}^{op}$, which does nothing on objects and for $f : X \to Y$, $f^{op} : Y \to X$. By the definition of the opposite category and how composition and identities were defined, it is clear that $(-)^{op}$ is a contravariant functor.

Lemma 2.33. *A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}$.*

Proof. Compose the contravariant functor $(-)^{op} : \mathcal{C}^{op} \to \mathcal{C}$ with the contravariant functor $F : \mathcal{C} \to \mathcal{D}$ to get a map $F^{op} : \mathcal{C}^{op} \to \mathcal{D}$. This swaps around the arrows and direction of composition again to make F^{op} covariant. $\square$

2.2.2. *Diagrams in Category Theory.* An incredibly important method of proof in category theory is "the art of the diagram chase" ([Rie17], 1.6). This method allows for arguments to be well-organised and visually clear, rather than tedious and verbose.

Definition 2.34. A *diagram* in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$ such that $\mathcal{J}$ is a small category.

This is the formal definition, but we depict this graphically with arrows between nodes.

Example 2.35. The *empty diagram* is a diagram indexed by the category with no objects and no morphisms— this is called *the empty category.*

Example 2.36. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. Then, by considering the category $\mathcal{J}$ which has $\text{ob}(\mathcal{J}) = \{X, Y\}$ and $\text{mor}(\mathcal{J}) = \{f, \mathbb{1}_X, \mathbb{1}_Y\}$, then we have a diagram given by the inclusion functor $\mathcal{J} \hookrightarrow \mathcal{C}$. This satisfies the formal definition of a diagram, but graphically this looks like $X \xrightarrow{f} Y$.

Definition 2.37. A diagram is said to *commute* if any morphisms in the diagram with the same domain and codomain are equal.

Example 2.38. Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ be morphisms in some category. Then by the associativity axiom for categories, we must have that $h(gf) = (hg)f$. Another way to say this is that the diagram:

$$\begin{tikzcd} X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[d, "h"] \\ Y \arrow[r, "hg"] & W \end{tikzcd}$$

commutes.

Lemma 2.39. *Suppose we have a pair of composable commutative diagrams:*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & Y \arrow[d, "k"] & Y \arrow[r, "b"] \arrow[d, "k"] & Z \arrow[d, "c"] \\ U \arrow[r, "g"] & V & V \arrow[r, "h"] & W \end{tikzcd}$$

Then the diagram:

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & Y \arrow[r, "b"] \arrow[d, "k"] & Z \arrow[d, "c"] \\ U \arrow[r, "g"] & V \arrow[r, "h"] & W \end{tikzcd}$$

commutes too.

Proof. The only morphisms where this might not be the case are morphisms from X to W. However, in this case $cba = (cb)a = (hk)a = h(ka) = hgf$ and so the diagram commutes. $\square$

Remark 2.40. Note that the above proof relied heavily on the associativity of composition of morphisms. Indeed, the reason that we can express things in terms of diagrams at all is equivalent to the fact that we have associativity.

Lemma 2.41. *Functors preserve commutative diagrams.*

Proof. Let $F : \mathcal{J} \to \mathcal{C}$ be a commutative diagram and suppose we have a functor $G : \mathcal{C} \to \mathcal{D}$. Then we can form $GF : \mathcal{J} \to \mathcal{D}$; this is clearly a diagram in $\mathcal{D}$, and preserves relationships in the original diagram by functoriality, so is commutative. $\square$

Definition 2.42. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. A *cone* to F is an object M of $\mathcal{C}$ together with a family of morphisms $\psi_X : M \to FX$ indexed by objects X in $\mathcal{J}$ such that for any morphism $f : X \to Y$ in $\mathcal{J}$, we have $Ff \circ \psi_X = \psi_Y$. Another way to say this is there exists morphisms ψ_X, ψ_Y for all objects X, Y and morphisms $f : X \to Y$ in $\mathcal{J}$ such that the following diagram commutes.

$$\begin{tikzcd} & M \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\ FX \arrow[rr, "Ff"] & & FY \end{tikzcd}$$

We use the notation (M, ψ) to denote a cone to F.

Definition 2.43. The *limit* of the diagram F is a cone (L, ϕ) to F such that for any other cone (M, ψ) to F, there exists a unique morphism $u : M \to L$ such that $\phi_X \circ u = \psi_X$ for all $X \in \text{ob}(\mathcal{J})$.

The dual notions of these are *cocones* and *colimits*.

Remark 2.44. We can package definition 2.43 in the following digram, where $\exists! u$ and the dotted arrow indicate that in this scenario there exists a unique u that makes the diagram commute.

$$\begin{tikzcd} & M \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\ & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\ FX \arrow[rr, "Ff"] & & FY \end{tikzcd}$$

This kind of construction is common in category theory, and is called a *universal property.*

Example 2.45. Let $\mathcal{C}$ be a category. A *terminal object* is the limit over the empty diagram (when this exists). That is, it is an object $\mathbf{1}$ in $\mathcal{C}$ such that for any object X in $\mathcal{C}$, there is a unique morphism $X \to \mathbf{1}$.

Similarly, an *initial object* is the colimit over the empty diagram (when this exists); it is an object $\mathbf{0}$ in $\mathcal{C}$ such that for any other object X there is a unique morphism $\mathbf{0} \to X$.

Both of these are unique up to isomorphism, and so we can talk about *the* initial/terminal object of a category.

Note that limits do not always exist in categories; however, it is a nice property if they do.

Definition 2.46. Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is *complete* if every diagram in $\mathcal{C}$ has a limit in $\mathcal{C}$.

Dually, we say $\mathcal{C}$ is *cocomplete* if every diagram in $\mathcal{C}$ has a colimit in $\mathcal{C}$.

2.2.3. *Natural Transformations.* We can also define a notion of morphism between functors.

Definition 2.47. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_X : FX \to GX$ in $\mathcal{D}$ for each object X in $\mathcal{C}$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes in $\mathcal{D}$.

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd}$$

If α_X is an isomorphism for every X, we call this a *natural isomorphism*

Example 2.48. Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. We can define a natural transformation $\mathbb{1}_F : F \Rightarrow F$ for each object X in $\mathcal{C}$ by $\mathbb{1}_{F_X} = \mathbb{1}_{F(X)}$. From the functoriality axioms, it is clear that the following diagram commutes.

$$\begin{tikzcd} FX \arrow[r, "\mathbb{1}_{F_X}"] \arrow[d, "Ff"] & FX \arrow[d, "Ff"] \\ FY \arrow[r, "\mathbb{1}_{F_Y}"] & FY \end{tikzcd}$$

Hence, any functor is naturally isomorphic to itself.

Since we have morphisms between functors, we can now define a category of functors.

Definition 2.49. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category*, denoted $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ has:

- Functors $\mathcal{C} \to \mathcal{D}$ as objects.
- Natural transformations as morphisms.

The previous example gives the identity morphisms. For $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, the composite determines a natural transformation with $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$. By gluing together the commutative diagrams it is clear that $\beta \circ \alpha$ is a natural transformation.

$$
\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] & GX \arrow[r, "\beta_X"] \arrow[d, "Gf"] & HX \arrow[d, "Hf"] & \rightsquigarrow & FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"] \arrow[rr, bend left, "(\beta\circ\alpha)_X"] & GX \arrow[r, "\beta_X"] \arrow[d, "Gf"] & HX \arrow[d, "Hf"] \\
FY \arrow[r, "\alpha_Y"] & GY & GY \arrow[r, "\beta_Y"] & HY & & FY \arrow[r, "\alpha_Y"] \arrow[rr, bend right, "(\beta\circ\alpha)_Y"'] & GY \arrow[r, "\beta_Y"] & HY
\end{tikzcd}
$$

Example 2.50. The functor category $\mathrm{Func}(\mathbf{\Delta}^{op}, \mathbf{Ab})$ will be extremely useful later.

2.3. **Equivalence of Categories.** Natural isomorphisms are precisely the isomorphisms in $\mathrm{Func}(\mathcal{C}, \mathcal{D})$. Therefore, it makes sense to only care about functors up to natural isomorphism. Now, if we consider a functor $F : \mathcal{C} \to \mathcal{C}$ that is naturally isomorphic to the identity functor $\mathbb{1}_{\mathcal{C}}$, then this map has shuffled the objects and morphisms around a little bit, but we can always use the inverse natural isomorphism to un-shuffle it. As such, it has not changed the structure of the category in a meaningful way. This idea motivates the concept of an equivalence of categories.

Definition 2.51. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *equivalence of categories* consists of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that GF is naturally isomorphic to $\mathbb{1}_{\mathcal{C}}$ and FG is naturally isomorphic to $\mathbb{1}_{\mathcal{D}}$.

Remark 2.52. This is a slightly weaker definition than an *isomorphism* of categories, which would require us to have $GF = \mathbb{1}_{\mathcal{C}}$ and $FG = \mathbb{1}_{\mathcal{D}}$— an example of such an isomorphism is given by the $(-)^{op}$ functor. This condition would be unreasonably strict [Lei14]. However, an equivalence of categories says that GF is isomorphic to $\mathbb{1}_{\mathcal{C}}$ in $\mathrm{Func}(\mathcal{C}, \mathcal{C})$ and FG is isomorphic to $\mathbb{1}_{\mathcal{D}}$ in $\mathrm{Func}(\mathcal{D}, \mathcal{D})$.

3. Simplicial Abelian Groups

3.1. **Introduction.** Homotopy theory was first designed as a way of understanding a topological space by identifying paths in the space that could be continuously deformed into each other. Roughly speaking, holes can be detected by homotopical information.

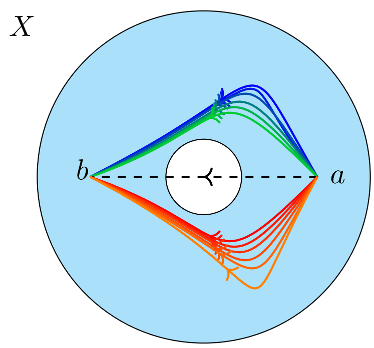


Figure 2. Looking at paths from a to b in X. Paths on the top half of the hole can be continuously morphed into one another, but cannot be continuously morphed into any path below the hole.

One approach to understanding the homotopical structure of topological spaces is through the use of *simplices*. The idea is to study the properties of a topological space by 'breaking it up' in some sense into n-dimensional triangles, and then examining how these fit back together to make the whole space. We can then turn the n-dimensional triangles into ordered lists of integers, in order to obtain a purely combinatorial model of the space called a *simplicial set*. This correspondence may seem like a loose analogy, but is actually made incredibly precise by showing that the homotopical data of a topological space is exactly the same as the homotopical data of a simplicial set. This is expanded upon insection 6, but not proved this in this project, as it would be a diversion from the main topic of this project; the interested reader is instead recommended [GJ99] or [May67].

3.2. **Simplices.**

3.2.1. *Whence Triangles?* Historically, simplices have not always been the way in which algebraic topologists break up spaces; instead of triangles, we could break up spaces into n-dimensional circles (leading to the concept of CW complexes) or n-dimensional squares (leading to the concept of cubical sets) for example.

Indeed, the concept of cell complexes has been used to prove many results in algebraic topology [Hat01]. One problem with these is that they are not closed under taking boundaries: the boundary of a sphere is not a finite collection of circles for example. This problem is not shared with simplices: the boundary of a tetrahedron really is 4 triangles. This is important as for homology, we need to be able to take boundaries.

Cubical sets were originally used by Daniel Kan to study homotopy theory, but it was realised that cubical abelian groups are not automatically fibrant, there is no notion of a normalised chain complex of a cubical set and there is an issue in formulating a geometric realisation formula for cubical sets. None of these problems are shared with simplicial sets, as was proven in [ML63] and [Kan58]. Cubical sets have, however, found recent uses in type theory, for example in [BCH14]. Moreover, in ([SHB11], chapter 14, section 8), Brown et al. provide a cubical version of the Dold-Kan correspondence. However, this is slightly less straightforward than the version for simplicial sets; it is not that the category of cubical abelian groups is equivalent to the category of non-negatively graded chain complexes of abelian groups, but rather that the category of cubical abelian groups *with connections* is. In ([SHB11], Remark 14.8.3), they cite this as another reason for abandoning the use of cubical sets in favour of simplicial sets. Therefore we use simplicial sets as they have the desired properties that make calculations and proofs most elegant in this context.

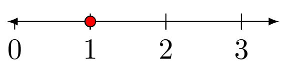
3.2.2. *Topological Simplices.* In order to motivate simplicial sets, it is useful to understand the topological origins of simplicial theory. This approach is inspired by [Fri12]. We begin by giving the definition of the topological n-simplex:

Definition 3.1. We define the topological n-simplex by

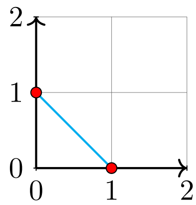
$$\Delta_n = \left\{ (t_0, ..., t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \text{ for each } 0 \leq i \leq n \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}$$

Example 3.2. We unravel this definition to show that it produces n-dimensional analogues of triangles.

- The topological 0-simplex is $\Delta_0 = \{(t_0) \in \mathbb{R} : t_0 \geq 0 \text{ and } t_0 = 1\} = \{1\}$.



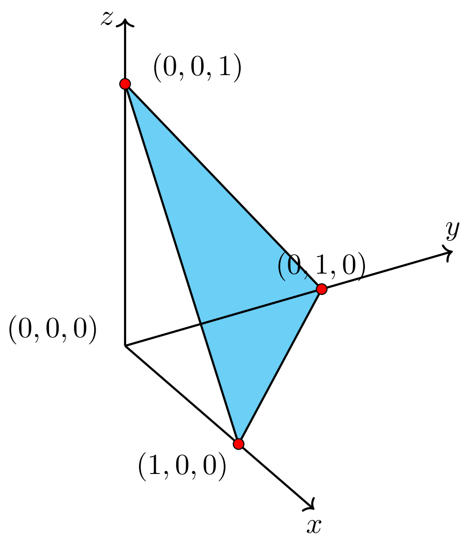
- The topological 1-simplex is $\Delta_1 = \{(t_0, t_1) \in \mathbb{R}^2 : t_0, t_1 \geq 0 \text{ and } t_0 + t_1 = 1\}$. So it the straight line from $(0, 1)$ to $(1, 0)$. We often associate it with the interval $[0, 1]$; we can do this by defining a homeomorphism $f : [0, 1] \to \Delta_1$ by $f(t) = (t, 1 - t)$.



- The topological 2-simplex is

$$\Delta_2 = \left\{(t_0, t_1, t_2) \in \mathbb{R}^3 : t_0, t_1, t_2 \geq 0 \text{ and } t_0 + t_1 + t_2 = 1\right\}.$$

It a triangle with vertices $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$.

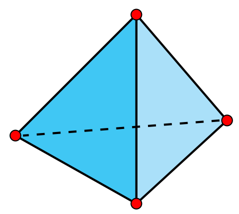


We often visualise this in 2 dimensions.

- The topological 3-simplex is

$$\Delta_3 = \left\{(t_0, t_1, t_2, t_3) \in \mathbb{R}^4 : t_0, t_1, t_2, t_3 \geq 0 \text{ and } t_0 + t_1 + t_2 + t_3 = 1\right\}.$$

Whereas we cannot visualise this in $\mathbb{R}^4$, we visualise this in 3 dimensions as a tetrahedron.



We want to move on to thinking about multiple simplices fitting together in some way. If we wanted to 'glue' two triangles together, we would need to specify which edges are attached. Hence we need a way of talking about the *faces* of a simplex.

Definition 3.3. For $0 \leq i \leq n-1$, We define $\delta^i : \Delta_{n-1} \to \Delta_n$ by

$$\delta^i(t_0, t_1, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}).$$

This is the inclusion of the $(n-1)$-simplex into the ith face of the n-simplex.

Example 3.4. In Figure 3, we show how the maps δ^i act on Δ_1 and Δ_2. For example $\delta^0(\Delta_1) = \{(0, t_0, t_1) : t_0, t_1 \geq 0 \text{ and } t_0 + t_1 = 0\}$ and so is the straight line from $(0, 1, 0)$ to $(0, 0, 1)$ in the (y, z) plane.

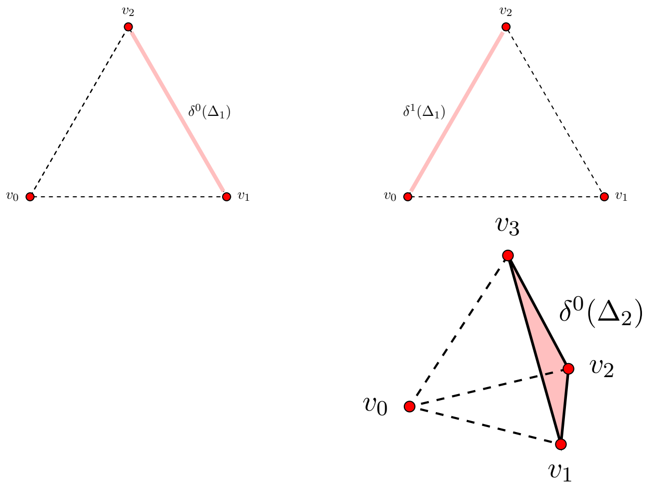


FIGURE 3. The image of the map δ^i is shown in pink.

Lemma 3.5. *For* $0 \leq i < j \leq n$, *we have* $\delta^j \delta^i = \delta^i \delta^{j-1} : \Delta_{n-2} \to \Delta_n$.

Proof. Consider the point $\mathbf{t} = (t_0, ..., t_{n-2}) \in \Delta_{n-2}$. Now, $\delta^i(\mathbf{t})$ inserts a 0 in the ith position, and since $i < j$ when we form $\delta^j \delta^i(\mathbf{t})$, by adding a 0 in the jth position, this does not move the first 0, so we have 0 in positions i and j.

On the other hand, $\delta^{j-1}(\mathbf{t})$ has 0 in position $j-1$. As $i \leq j-1$, forming $\delta^i \delta^{j-1}(\mathbf{t})$ by adding a 0 in the ith position shifts up the other 0 by one place, causing it to be in the jth position, as required. (Proof inspired by [Str21], section 10) □

We also have maps going the other way:

Definition 3.6. For $0 \leq i \leq n$, We define $s^i : \Delta_n \to \Delta_{n-1}$ by

$$s^i(t_0, t_1, ..., t_n) = (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_{n-1}).$$

This is the projection of the topological n-simplex onto the $(n-1)$-simplex.

Example 3.7. The map $s^1 : \Delta_2 \to \Delta_1$ is defined by $s^1(t_0, t_1, t_2) = (t_0, t_1 + t_2)$ and can be visualised with Figure 4.

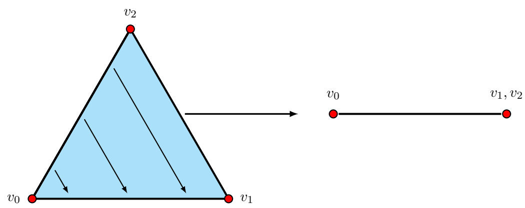


FIGURE 4. The map $s^1 : \Delta_2 \to \Delta_1$.

These maps satisfy the following relations.

Lemma 3.8. *The* s_i *and* δ_i *satisfy the following relations:*

(1) $s^j s^i = s^i s^{j+1}$ *if* $i \leq j$.

(2) $s^j \delta^i = \begin{cases} \delta^i s^{j-1} & \text{if } i < j \\ \mathbb{1}_{\Delta_n} & \text{if } i \in \{j, j+1\} \\ \delta^{i-1} s^j & \text{if } i > j+1. \end{cases}$

Proof. (1) $s^i(t_1, ..., t_n)$ has $t_i + t_{i+1}$ in the ith position and shifts t_{j+1} so that it is in the jth position. Applying s^j then has $t_{j+1} + t_{j+2}$ in the jth position. Equivalently, $s^{j+1}(t_0, ..., t_n)$ has $t_{j+1} + t_{j+2}$ in the $(j+1)$th position and t_i in the ith. Applying s^i to this gives $t_i + t_{i+1}$ in the ith position and $t_{j+1} + t_{j+2}$ in the jth, as required.

(2) These all follow a similar line of reasoning; for example $s^j \delta^j(t_0, ..., t_n)$ first inserts a zero in the jth position, shifting t_j to the $(j+1)$th position and then combines the jth and $(j+1)$th places to give t_j in the jth position. It is therefore the identity. This heuristically makes sense as it is saying that embedding the n-simplex as the jth face of the $(n+1)$-simplex and then taking the projection of this onto the j-th face will give you back what you started with.

□

3.3. **Simplicial Sets.** A simplicial set aims to have the same structural properties as topological simplices, without unneeded geometric information such as the embedding in space or the length of lines. The properties it turns out are important to preserve in homotopy theory are the properties of the δ^i and s^i given in lemmas 3.5 and 3.8. A good introduction to the theory of simplicial sets is given by [Rie11].

Definition 3.9. A *simplicial set* is a sequence $(X_n)_n$ of sets with maps $\partial_i : X_n \to X_{n-1}$ and $\sigma_j : X_n \to X_{n+1}$ which satisfy the *simplicial identities*:

(S1) $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i < j$,

(S2) $\sigma_i \sigma_j = \sigma_{j+1} \sigma_i$ if $i \leq j$,

(S3) $\partial_i \sigma_j = \begin{cases} \sigma_{i-1} \partial_j & \text{if } i < j, \\ \mathbb{1}_{X_n} & \text{if } i \in \{j, j+1\}, \\ \partial_j \sigma_{i-1} & \text{if } i > j+1. \end{cases}$

These maps are called *face* and *degeneracy* maps respectively. A simplex $x \in X_n$ is called *degenerate* if it in the image of a degeneracy map i.e. $x = \sigma_i(y)$ for some i and $y \in X_{n-1}$. Otherwise, it is called *non-degenerate*.

The definition we have given for simplicial sets is very useful when doing computations. However, we have an alternative definition of a simplicial set:

Definition 3.10. (Alternative definition of simplicial set) A simplicial set is a contravariant functor $X : \mathbf{\Delta} \to \mathbf{Set}$. Equivalently, a simplicial set is a covariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$.

We should probably check that this agrees with our previous definition.

Proposition 3.11. *The two definitions of a simplicial set, definitions 3.9 and 3.10, agree.*

Proof. Suppose $\tilde{X}$ is a contravariant functor $\mathbf{\Delta} \to \mathbf{Set}$. Then we can define a non-negative sequence of sets by assigning $X_n := \tilde{X}[n]$. Recall from definition 2.8 that in $\mathbf{\Delta}$ we have the maps $\epsilon_i : [n-1] \to [n]$ and $\eta_i : [n+1] \to [n]$. We define face and degeneracy maps for $0 \leq i \leq n$ by $\partial_i := \tilde{X}(\epsilon_i) : X_n \to X_{n-1}$ and $\sigma_i := \tilde{X}(\eta_i) =: X_n \to X_{n+1}$. These satisfy the simplicial identities due to the contravariance of $\tilde{X}$ and lemma 2.11. Hence we have a simplicial set in the sense of definition 3.9.

Now, conversely, suppose that we have a non-negative sequence $(X_n)_{n \in \mathbb{N}}$ of sets with face and degeneracy maps ∂_i and σ_i which is a simplicial set X in the sense of definition 3.9. We define a map $\tilde{X} : \mathbf{\Delta} \to \mathbf{Set}$ given by $\tilde{X}[n] := X_n$. For any morphism $\alpha \in \mathbf{\Delta}$, recall that we can factorise α into a unique composition of the ϵ_i and η_i by lemma 2.12. Therefore, it is enough to determine what $\tilde{X}$ does on these maps, and extend this cofunctorially so that for any composeable maps α and β we have $\tilde{X}(\alpha\beta) = \tilde{X}(\beta)\tilde{X}(\alpha)$. We define $\tilde{X}(\epsilon_i) := \partial_i$ and $\tilde{X}(\eta_i) := \sigma_i$. The simplicial identities on X forces $\tilde{X}$ to to be well-defined.

□

By definition 3.10, the morphisms between simplicial sets must be morphisms between functors— or in other words, natural transformations. This means that for every object X_n, we must define a map $f_n : X_n \to Y_n$ such that it commutes with maps $X_n \to X_m$. Since all maps are generated by face and degeneracy maps, it is enough to say that f_n needs to commute with faces and degeneracies. Moreover, this justifies our desire to have degeneracy maps; without degeneracy maps, we would not be able to have this neat description of simplicial maps.

Definition 3.12. A simplicial map $f : X \to Y$ is a collection of set maps $X_n \to Y_n$ that commute with the face and degeneracy maps.

Definition 3.13. There is a category **sSet** which has:

- Simplicial sets as objects.
- Simplicial maps as morphisms.

This is a well-defined category by the following proposition.

Proposition 3.14. *As categories,* $\mathbf{sSet} = \mathrm{Func}(\mathbf{\Delta}^{op}, \mathbf{Set})$.

Proof. This is a direct corollary to proposition 3.11, and by noting that a simplicial map commuting with face and degeneracy maps means that f is a natural transformation. □

Moreover, by this categorical definition, we can define the simplest examples of simplicial sets, which are fundamental to the theory that follows.

Definition 3.15. Define the simplicial set $\Delta^n : \mathbf{\Delta}^{op} \to \mathbf{Set}$ by $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$, which is the functor that sends an object $[m] \in \mathbf{\Delta}$ to the set of order-preserving maps $\mathrm{Hom}_{\mathbf{\Delta}}([m], [n])$ and that takes an order-preserving function $g : [l] \to [m]$ and sends it to a function $g^* : \mathrm{Hom}_{\mathbf{\Delta}}([m], [n]) \to \mathrm{Hom}_{\mathbf{\Delta}}([l], [n])$ defined by pre-composition— for $f : [m] \to [n]$ we have $g^*(f) = g \circ f : [l] \to [n]$. It is not hard to check that this defines a functor. We call Δ^n the *standard n simplex.*

We write $\Delta^n_k = \mathrm{Hom}_{\mathbf{\Delta}}([k], [n])$, so k-simplices in Δ^n are maps from $[k]$ to $[n]$ in $\mathbf{\Delta}$. Let $f : [k] \to [n]$ be a k-simplex. Then for $0 \le i \le n$ we can define the face map $\delta_i(f) : [k-1] \to [n]$ by the composite:

$$[k-1] \xrightarrow{\epsilon_i} [k] \xrightarrow{f} [n]\,.$$

Similarly, we define the degeneracy map $s_i(f) : [k+1] \to [n]$ by the composite:

$$[k+1] \xrightarrow{\eta_i} [k] \xrightarrow{f} [n]\,.$$

The standard n-simplex has a unique non-degenerate simplex in Δ^n_n that corresponds to the identity map $[n] \to [n]$. More generally, non-degenerate k-simplices in Δ^n_k are precisely the injective maps $[k] \to [n]$.

Example 3.16. We give descriptions of the standard-simplices for small n It is traditional to denote the elements of Δ^n_k by the images of the maps $[k] \to [n]$, and to keep track of degeneracies by repeating elements in the set; this gives a one-to-one correspondence between these sets and the maps. For example, the set $\{0, 0, 0\}$ corresponds to the map $\eta_0\eta_0\epsilon_1\epsilon_2 : [2] \to [2]$ as:

$$\begin{array}{ccccccccc} [2] & \xrightarrow{\epsilon_2} & [1] & \xrightarrow{\epsilon_1} & [0] & \xrightarrow{\eta_0} & [1] & \xrightarrow{\eta_0} & [2] \\ \\ \{0,1,2\} & \longmapsto & \{0,1\} & \longmapsto & \{0\} & \longmapsto & \{0,0\} & \longmapsto & \{0,0,0\} \end{array}$$

If we left out the repeated zeroes, we would not be able to tell which map this set corresponded to. We choose this notation rather than the function notation in order to emphasise the connection to topological simplices. We can think of each number as being a vertex.

- Δ^0 is the simplicial set with $\Delta^0_0 = \{\{0\}\}$ and $\Delta^0_k = \{\{0, 0, ..., 0\}\}$.
- Δ^1 is given by $\Delta^1_0 = \{\{0\}, \{1\}\}$, $\Delta^1_1 = \{\{0, 1\}, \{0, 0\}, \{1, 1\}\}$, and for $k > 1$, Δ^1_k is full of degenerate simplices.
- Δ^2 is given by:

$$\begin{aligned}\Delta^2_0 &= \{\{0\}, \{1\}, \{2\}\}, \\ \Delta^2_1 &= \{\{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 0\}, \{1, 1\}, \{2, 2\}\}, \\ \Delta^2_2 &= \left\{\begin{matrix}\{0, 0, 0\}, \{0, 0, 1\}, \{0, 0, 2\}, \{0, 1, 1\}, \{0, 1, 2\}, \\ \{0, 2, 2\}, \{1, 1, 1\}, \{1, 1, 2\}, \{1, 2, 2\}, \{2, 2, 2\}\end{matrix}\right\},\end{aligned}$$

  and for $k > 2$, Δ^2_k is full of degenerate simplices.

As we can see, these quickly become very large because of all of the degenerate simplices.

The standard n-simplices are fundamental objects in the theory of simplicial sets by the following result.

Proposition 3.17. *Let $[n] \in \mathbf{\Delta}$ and let $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$. Simplicial maps $\Delta^n \to X$ correspond bijectively to elements in X_n.*

This is direct from the Yoneda lemma as explained in ([Rie11], 3), which we shall not prove here due to space constraints; a proof of the lemma is given in ([Rie17], Theorem 2.2.4). This proposition tells us that the Δ^n represent in some way all n-simplices in a simplicial set X, and by considering maps from these standard simplices we can fully understand X. It also tells us that any simplex $x \in X_n$ can be thought of as a simplicial map $\widetilde{x} : \Delta^n \to X$.

Definition 3.10 allows us to define the more general notion of a *simplicial object*.

Definition 3.18. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\mathbf{\Delta} \to \mathcal{C}$.

There is a category of simplicial objects which we denote $\mathbf{s}\mathcal{C}$. The morphisms $f : X \to Y$ in $\mathbf{s}\mathcal{C}$ are determined by a sequence of morphisms $f_n : X_n \to Y_n$ in $\mathcal{C}$ which commute with the face and degeneracy maps, or equivalently, natural transformations between the contravariant functors X and Y.

Example 3.19. A simplicial set is a simplicial object in the category **Set**.

Example 3.20. Define a covariant functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ by $\Delta[n] = \Delta_n$, and for each $0 \leq i \leq n$, $\Delta(\epsilon_i) = \delta^i$ and $\Delta(\eta_i) = s^i$. By lemmas 3.5 and 3.8, these satisfy the dual simplicial relations, and so by a dual argument to proposition 3.11 is a well-defined functor. Hence the set of topological simplices is a *co*simplicial object in the category **Top**. This makes precise the connection between simplicial sets and topological simplices.

3.3.1. *The Singular Simplicial Set.* We said that simplicial sets were made to provide a combinatorial model of topological spaces. We can now roughly state how that is done. In much the same way that we can think of n-simplices in X_n as maps $\Delta^n \to X$, we can think of n-simplices in a topological space X as continuous maps $\Delta_n \to X$.

Definition 3.21. Let X be a topological space. We define

$$S_n(X) = \{u : \Delta_n \to X \text{ such that } u \text{ is continuous}\} = \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X).$$

We write S_* to be the collection of these. We call this *the singular simplicial set of X.*

Example 3.22. We depict what S_nX roughly looks like for low dimensional n on the annulus in Figure 5. Note that there are infinitely many elements in each set in this case, and we only depict a sub-collection of them.

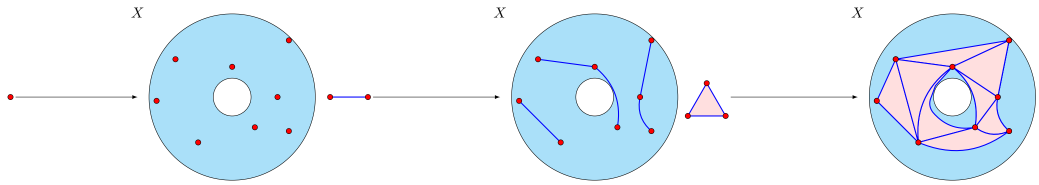


FIGURE 5. Where X is the annulus, we show $S_0(X)$ (left), $S_1(X)$ (centre), and $S_2(X)$ (right). The elements of these sets are the *maps* indicated by the arrows.

As the name suggests, S_*X turns out to be a simplicial set. This is because the maps δ^i and s^i of Δ_n are transferred over by pre-composition (see Figure 6). It turns out that S_* is a functor **Top** $\to$ **sSet** that has the special property that all homotopical data is preserves.

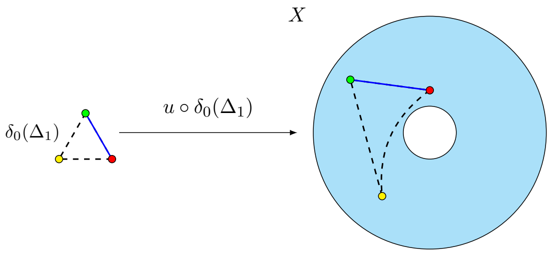


FIGURE 6. A visualisation of how the singular simplicial set inherits face maps through pre-composition with u.

There is a partial converse of this functor $|-|$: **sSet** $\to$ **Top** called *geometric realisation*, which roughly sends an n-simplex in a simplicial set to a copy of the topological simplex Δ_n, and then glues these all together in a way that is determined by the face and degeneracy maps. For any topological space X, $|S_*X|$ is homeomorphic to X. For more details, see ([May99],16.2).

3.4. **Simplicial Abelian Groups.** Simplicial abelian groups, in contrast to simplicial sets, have the special property of being *Kan complexes* introduced in subsubsection 3.4.1, which are essential for doing a lot of the theory in section 6.

Definition 3.23. A simplicial abelian group is a simplicial object in the category of abelian groups, **Ab**. There is a category **sAb** which has:

- Simplicial abelian groups as objects.
- Morphism $f : A \to B$ which are a collection of abelian group homomorphism $f_n : A_n \to B_n$ that commute with the face and degeneracy maps.

This category is one of the categories that appears in the Dold-Kan correspondence. We can send any simplicial set to a simplicial abelian group by the *free functor*.

Definition 3.24. Let X be a set. Then the *free abelian group generated by* X is the set $\mathbb{Z}\{X\}$ is the set of $\mathbb{Z}$ linear combinations of elements of X.

Example 3.25. Let $X = \{a, b, c\}$. Then $3a + 7b - 4c \in \mathbb{Z}\{X\}$.

Remark 3.26. It is clear that for any set X, $\mathbb{Z}\{X\}$ is an abelian group— the structure is inherited from the structure of $\mathbb{Z}$. We could rephrase this to say that $\mathbb{Z}$ is a functor from **Set** to **Ab**, with its effect on morphisms just functions just linearly extending them. However, what we are interested in is simplicial abelian groups, so it would be useful if this restricted to a functor from simplicial sets to simplicial abelian groups. This is the content of the next lemma.

Lemma 3.27. *There is a well-defined functor $\mathbb{Z} : \mathbf{sSet} \to \mathbf{sAb}$:*

- *A simplicial set X is sent to $\mathbb{Z}X$ with $(\mathbb{Z}X)_n = \mathbb{Z}\{X_n\}$.*
- *A simplicial map $f : X \to Y$ is extended linearly to become a map $\mathbb{Z}f : \mathbb{Z}X \to \mathbb{Z}Y$.*

We call this the free functor.

Proof. Let X be a simplicial set. Then X is a functor $X : \Delta^{op} \to \mathbf{Set}$. Compose this with the functor $\mathbb{Z} : \mathbf{Set} \to \mathbf{Ab}$, to get a functor $\mathbb{Z}X : \Delta^{op} \to \mathbf{Ab}$. Hence $\mathbb{Z}X$ is a simplicial abelian group. □

Example 3.28. There is a simplicial abelian group $\mathbb{Z}\Delta^n$ for each $n \in \mathbb{N}$.

- For $n = 0$, $\mathbb{Z}\Delta^0_0$ is the free abelian group with generator as a point, and so is isomorphic to $\mathbb{Z}$. For $k > 0$ we have $\mathbb{Z}\Delta^0_k = \mathbb{Z}\{\{0, 0, ..., 0\}\}$, the free abelian group generated by the one degenerate simplex we have and so is isomorphic to $\mathbb{Z}$ at every level.
- For $n = 1$, $\mathbb{Z}\Delta^1$ is given by $\mathbb{Z}\Delta^1_0 = \mathbb{Z}\{\{0\}, \{1\}\}$, $\mathbb{Z}\Delta^1_1 = \mathbb{Z}\{\{0, 1\}, \{0, 0\}, \{1, 1\}\}$ and for $k > 1$, $\mathbb{Z}\Delta^1_k$ is full of degenerate simplices. This is a simplicial abelian group with face maps and degeneracies given by extending δ_i and s_i linearly.

Example 3.29. For any topological space X, there is a simplicial abelian group given by $\mathbb{Z}S_*X$.

This following lemma will be useful in section 6.

Lemma 3.30. *Let A and B be simplicial abelian groups, and let $f : A \to B$ be a simplicial map between them. Then* $\ker f$ *is a simplicial abelian group too.*

Proof. For each n, $f_n : A_n \to B_n$ is a homomorphism of abelian groups. Hence $\ker f_n$ is an abelian subgroup of A_n. The elements in $\ker f_n$ are closed under face and degeneracy maps, since f commutes with them: for $x \in \ker f_n$, we have $f_{n-1}(\partial_i(x)) = \partial_i(f_n(x)) = \partial_i(0) = 0$, so $\partial_i(x) \in \ker(f_{n-1})$. Similarly, $f_{n+1}(\sigma_i(x)) = \sigma_i(f_n(x)) = \sigma_i(0) = 0$, so $\sigma_i(x) \in \ker(f_{n+1})$. Hence $(\ker f_n)_n$ forms a simplicial abelian group. □

3.4.1. *Horns.* In this section, we define the notion of a horn, which will be useful later. These can be thought of as a simplicial set missing a face and its interior. Certain simplicial sets have the property that these faces can always be filled back in; such simplicial sets are called *Kan complexes.* This property is inspired by the homotopy lifting condition of CW complexes in topology, which is important in understanding topological homotopies. For more details of topological lifting, see ([Str21], 22) or ([Hat01], 1.3). In the context of simplicial sets, these are studied in ([Rie11], 5) and ([Fri12], 7). Importantly, the singular simplicial set of a topological space S_*X and any simplicial abelian group are Kan complexes. This allows us to do homotopy theory in these settings.

Definition 3.31. The *simplicial horn* Λ^n_k is the union of all the faces of Δ^n except the k-th face:

$$\Lambda^n_k := \bigcup_{\substack{i=0 \\ i \neq k}}^{n} \delta_i(\Delta^n).$$

Example 3.32. The three horns Λ^2_0, Λ^2_1 and Λ^2_2 of Δ^2 can be visualised as in Figure 7.

We can generalise this notion to an arbitrary simplicial set:

Definition 3.33. A horn is a map of simplicial sets $\Lambda^n_k \to X$. Equivalently, it is a collection of $(n-1)$-simplices $(y_0, ..., y_{k-1}, -, y_{k+1}, ..., y_n)$, such that $\partial_i x_j = \partial_{j-1} x_i$ wherever this is defined. This condition should be thought of as saying that these simplices fit together and are not disjoint. A horn $\Lambda^n_k \to X$ is sometimes referred to as *an (n, k)-horn.*

We now isolate the special property of being able to fill these horns.

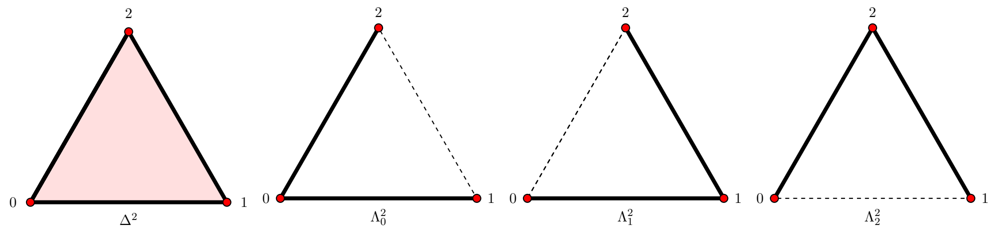


FIGURE 7. The 3 horns of Δ^2.

Definition 3.34. A *Kan complex* is a simplicial set X such that any horn $\Lambda^n_k \to X$ in X can be extended along the inclusion $\Lambda^k_n \hookrightarrow \Delta^n$. Equivalently, there exists a map $\Delta^n \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

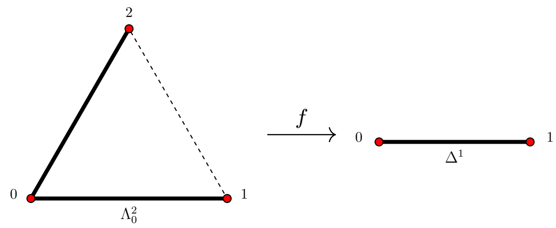
This is sometimes called *the Kan condition.*

There is an equivalent reformulation of this definition which is less conceptual, but more combinatorial.

Definition 3.35. A simplicial set X is said to be a Kan complex if for any collection of $(n-1)$-simplices $(y_0, ..., y_{k-1}, -, y_{k+1}, ..., y_n)$, such that $\partial_i x_j = \partial_{j-1} x_i$ wherever this is defined, there exists an n-simplex y such that $d_i y = y_i$ for all $i \neq k$.

Not all simplicial sets have this property; in fact, not even the standard simplices have this property.

Example 3.36. (Inspired by [Fri12], example 7.4)
Λ^2_0 consists of the 1-simplices $\{0,1\}$, $\{0,2\}$ and degeneracies. Consider the map $f : \Lambda^2_0 \to \Delta^1$ given by sending $\{0,1\}$ to $\{0,1\}$ and $\{0,2\}$ to $\{0,0\}$. This is order-preserving, so is a well-defined simplicial map.



We note that this map has the assignments $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0$. Any extension of this map $x : \Delta^2 \to \Delta^1$ with $xi = f$ would have to respect this; therefore $\{1,2\}$ would have to be sent to $\{1,0\}$, but this is not order-preserving, and therefore not a simplicial map. Therefore, Δ^1 is not a Kan complex.

We have the following lemma, which will be very important in section 6. This result is originally due to [Moo57], and motivates our study of simplicial abelian groups. The proof of this is constructive and gives an explicit algorithm for computing these horn fillers.

Lemma 3.37. *The underlying simplicial set of a simplicial abelian group is a Kan complex.*

Proof. Let A be a simplicial abelian group and let $(x_0, ..., x_{k-1}, -, x_{k+1}, ..., x_{n+1})$ be a horn in A_n, so that $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ and $i, j \neq k$ (*). We use induction on r

to find an $a_r \in A_{n+1}$ with $\partial_i(a_r) = x_i$ for all $i \leq r, i \neq k$. We start the induction with $a_0 = \sigma_0(x_0) \in A_{n+1}$. This has $\partial_0(a_0) = x_0$ by (S3), as required. Now, suppose we have an a_{r-1} such that $\partial_i a_{r-1} = x_i$ for all $i \leq r-1$, $i \neq k$. If $r = k$, we set $a_r = a_{r-1}$, and then $\partial_i(a_r) = x_i$ for all $i \leq r$ and $i \neq k$; otherwise, let $u_r = -x_r + \partial_r(a_{r-1})$. For $i < r, i \neq k$, we have:

$$\begin{aligned}
\partial_i(u_r) &= \partial_i(-x_r + \partial_r(a_{r-1})) \\
&= -\partial_i(x_r) + \partial_i\partial_r(a_{r-1}) && \text{as } \partial_i \text{ is a group homomorphism,} \\
&= -\partial_i(x_r) + \partial_{r-1}\partial_i(a_{r-1}) && \text{by (S1),} \\
&= -\partial_i(x_r) + \partial_{r-1}(x_i) && \text{by the inductive hypothesis,} \\
&= -\partial_i(x_r) + \partial_i(x_r) && \text{by (*),} \\
&= 0.
\end{aligned}$$

Therefore, by (S3), we have $\partial_i\sigma_r(u_r) = \sigma_{r-1}\partial_i(u_r) = \sigma_{r-1}(0) = 0$, and so if we define $a_r = a_{r-1} - \sigma_r(u_r)$, then for each $0 \leq i \leq r-1, i \neq k$, we have

$$\partial_i(a_r) = \partial_i(a_{r-1}) - \partial_i\sigma_r(u_r) = x_i.$$

Moreover, for $i = r \neq k$, we have:

$$\begin{aligned}
\partial_r(a_r) &= \partial_r(a_{r-1}) - \partial_r\sigma_r(u_r) \\
&= \partial_r(a_{r-1}) - u_r, && \text{by (S3),} \\
&= \partial_r(a_{r-1}) - (-x_r + \partial_r(a_{r-1})) \\
&= x_r.
\end{aligned}$$

In either case, the element $a_r \in A_{n+1}$ satisfies the induction step, and so we can find a $y = a_n \in A_{n+1}$ such that $\partial_i(y) = x_i$ for all $i \neq k$, as required. Hence A is a Kan complex.
[Proof inspired and adapted from ([Wei95], Lemma 8.2.8)] □

By definition 3.34, we can rephrase this in the following way.

Corollary 3.38. *Let A be a simplicial abelian group and suppose we have a horn given by a map $f : \Lambda^n_k \to A$. Then f extends to an n-simplex of A, i.e. there exists a map $x : \Delta^n \to A$ such that $xi = f$. Equivalently, there exists an $x : \Delta^n \to A$ such that the following diagram commutes.*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"] & A \\
\Delta^n \arrow[ur, dashed, "\exists x"'] &
\end{tikzcd}$$

Note that in this proof, we required the use of elements having additive inverses, so it cannot be generalised to arbitrary simplicial sets. We now compute an example to show how the algorithm can be implemented.

Example 3.39. Suppose we have a $(2,2)$-horn $(x_0, x_1, -)$ in a simplicial abelian group A. This means we have the relation $\partial_0 x_1 = \partial_0 x_0$ (**). We calculate $y \in A_3$ such that $\partial_0(y) = x_0$ and $\partial_1(y) = x_1$. First, we set $a_0 = \sigma_0(x_0)$. This clearly has the property that $\partial_0(a_0) = x_0$. Now, let $u_1 = \partial_1(a_0) - x_1 = x_0 - x_1$. Then, we proceed by setting

$$a_1 = \sigma_0(x_0) - \sigma_1(x_0 - x_1) = \sigma_0(x_0) - \sigma_1(x_0) + \sigma_1(x_1).$$

The next step is the case $r = k$, so we set $y = a_2 = a_1$, and we have:

$$\begin{aligned}
\partial_0(y) &= \partial_0\sigma_0(x_0) - \partial_0\sigma_1(x_0) + \partial_0\sigma_1(x_1) \\
&= x_0 - \sigma_0\partial_0(x_0) + \sigma_0\partial_0(x_1) && \text{by (S3),} \\
&= x_0 - \sigma_0\partial_0(x_0) + \sigma_0\partial_0(x_0) && \text{by (**),} \\
&= x_0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_1(y) &= \partial_1\sigma_0(x_0) - \partial_1\sigma_1(x_0) + \partial_1\sigma_1(x_1) \\
&= x_0 - x_0 + x_1 && \text{by (S3),} \\
&= x_1.
\end{aligned}$$

Therefore, y is our desired filling.

3.5. **Simplicial Homotopy Groups.** Our aim is to describe an algebraic notion of homotopy through the use of simplicial abelian groups. This section explains how we do this. We present simplicial homotopy groups in the setting of abelian groups; these notions can be generalised to any Kan complex.

Definition 3.40. Let A be a simplicial abelian group. We define

$$\begin{aligned}
\widetilde{Z}_n(A) &= \{x \in X_n : \partial_i(x) = \sigma_0^{n-1}(0) \text{ for all } i = 0, ..., n\} \\
&= \bigcap_{i=0}^{n} \ker(\partial_i : X_n \to X_{n-1}).
\end{aligned}$$

Remark 3.41. We often just write 0 instead of $\sigma_0^{n-1}(0)$ for ease of notation. This makes more sense as $\sigma^{n-1}(0)$ is the zero element of the abelian group A_n anyway.

We now define the notion of simplicial homotopy:

Definition 3.42. Let A be a simplicial abelian group. We say that two elements $x, x' \in A_n$ are *homotopic* if there is a $y \in A_{n+1}$ such that

$$\partial_i(y) = \begin{cases} 0 & \text{if } i < n \\ x & \text{if } i = n \\ x' & \text{if } i = n+1. \end{cases}$$

We call y a homotopy from x to x'.

This definition may seem to contrast with the intuitive notion of homotopy given in subsection 3.1. However, by examining the lowest-dimensional case— homotopies (or paths) between points— we can see why this definition is used. Let X be a topological space and let a and b be two points that are homotopic, i.e., there is a path between them u. We consider u as a map from the 1-simplex: $u : \Delta_1 \to X$. Therefore u is an element of $S_1(X)$ which has $\partial_0(u) = a \in S_0(X) = X$ and $\partial_1(u) = b \in S_0(X)$ and so is a simplicial homotopy between a and b in S_1X. Similarly, figure 8 gives a visual reasoning behind why homotopies between paths correspond to maps from Δ_2 to X.

Lemma 3.43. *Let A be a simplicial abelian group, and consider the group $Z_n(A)$. For $x, x' \in Z_n$, we say $x \sim x'$ if there is a homotopy from x to x'. Then $\sim$ is an equivalence relation on $Z_n(A)$.*

Proof. For any $x \in Z_n(A)$, $y = \sigma_n(x)$ is a homotopy from x to x, since:

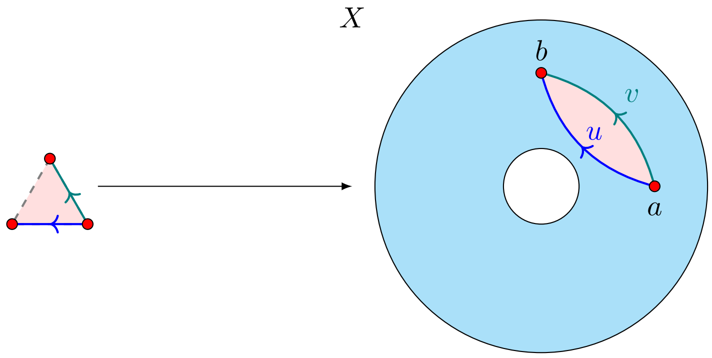


FIGURE 8. A simplicial homotopy between u and v in X corresponds to a map from Δ_2 into X. As we can fill in the space between u and v, we can imagine continuously deforming u into v through the pink region, giving our intuitive notion of homotopy between paths.

$$\begin{aligned} \partial_{n+1} y &= \partial_{n+1}\sigma_n x, \\ &= x && \text{by (S3)}, \end{aligned}$$

and similarly:

$$\begin{aligned} \partial_n y &= \partial_n \sigma_n x, \\ &= x && \text{by (S3)}. \end{aligned}$$

Moreover, for $i < n$:

$$\begin{aligned} \partial_i y &= \partial_i \sigma_n x, \\ &= \sigma_{n-1}\partial_i x && \text{by (S3) with } i < n, \\ &= \sigma_{n-1}(0) && \text{since } x \in Z_n(A), \\ &= 0. \end{aligned}$$

Hence, this is a simplicial homotopy from x to x, so $x \sim x$ and the reflexivity condition holds.

Now, let $x \sim x'$ and $x \sim x''$ in A_n. Let $y \in A_{n+1}$ be a homotopy from x to x' and let $y' \in A_{n+1}$ be a homotopy from x to x''. Then

$$\partial_i(y) = \begin{cases} 0 & \text{if } i < n \\ x & \text{if } i = n \\ x' & \text{if } i = n+1, \end{cases}$$

and

$$\partial_i(y') = \begin{cases} 0 & \text{if } i < n \\ x & \text{if } i = n \\ x'' & \text{if } i = n+1. \end{cases}$$

By applying the Kan condition to the horn $(0, 0, ..., 0, y, y', -)$, there exists a $z \in A_{n+2}$ with $\partial_n z = y$, $\partial_{n+1} z = y'$ and $\partial_i z = 0$ for all $i < n$. The element $\partial_{n+2} z$ provides the desired homotopy from x to x'':

$$\partial_n \partial_{n+2} z = \partial_{n+1} \partial_n z = \partial_{n+1} y'' = x',$$
$$\partial_{n+1} \partial_{n+2} z = \partial_{n+1} \partial_{n+1} z = \partial_{n+1} y' = x'',$$

and for $i < n$:

$$\partial_i \partial_{n+2} z = \partial_{n+1} \partial_i z = \partial_{n+1} 0 = 0.$$

So $x \sim x'$ and $x \sim x'' \implies x' \sim x''$. Taking $x'' = x$, we see that the symmetry condition holds, and this also implies that the transitivity condition holds.

□

Definition 3.44. Let A be a simplicial abelian group. We define $\pi_n(A) = \widetilde{Z}_n(A)/\sim$, where $\sim$ denotes the equivalence relation of being simplicial homotopic.

Remark 3.45. This is an invariant for simplicial abelian groups— if two spaces X and Y are homeomorphic to each other, then the simplicial homotopy groups $\pi_*(\mathbb{Z}SX)$ and $\pi_*(\mathbb{Z}SY)$ are equivalent. The converse of this is often useful: if $\pi_*(\mathbb{Z}SX) \neq \pi_*(\mathbb{Z}SY)$, then X is not homeomorphic to Y.

4. Chain Complexes

4.1. **Introduction.** Homology theory was developed in the late 1800s by Poincaré and others as a tool to study holes in topological spaces [Wei95]. Informally, it does this by making a distinction between cycles on the surface which are the boundary of a hole and cycles which are not. For example, consider an annulus A, as pictured in Figure 9. Any cycle u around the centre hole can be shrunk until it fits around the hole, but not further whilst staying within the space. We identify all such cycles. We can also imagine a cycle which goes around the hole n times; this is genuinely different from the first kind and cannot be continuously shrunk to a cycle which goes around the hole fewer times. Any other cycle v can be shrunk to a point, or alternatively can be viewed as a cycle which goes round the hole 0 times; thus there is a bijection between classes of cycles and $\mathbb{Z}$.

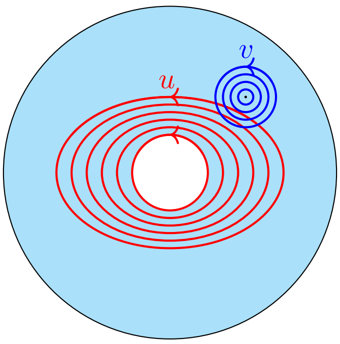


Figure 9. Cycles on an annulus. Cycles such as v can be continuously deformed until they are a point (shown in blue), whereas cycles such as u can only be shrunk down until they surround the hole (shown in red).

4.2. **The Idea Behind Simplicial Homology.** We aim to capture this idea algebraically with simplicial theory; in this subsection we give an informal introduction on how to do this. A 1-cycle in a space corresponds to a map $u \in S_1 X$ with the endpoints glued together as seen in Figure 10. This translates to $u \circ \delta_0(\Delta_1) = u \circ \delta_1(\Delta_1)$, or rearranging we obtain $u \circ \delta_0(\Delta_1) - u \circ \delta_1(\Delta_1) = 0$. This idea generalises to higher dimensions; for $u \in S_n X$ we define the *algebraic boundary map* $d_n : S_n X \to S_{n-1} X$ by $d_n(u) = \sum_{i=0}^{n} (-1)^i u \circ \delta_i$; we

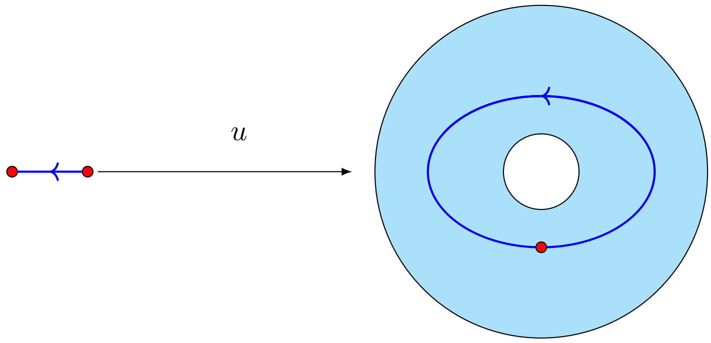


FIGURE 10. A 1-cycle corresponds to a map from the 1-simplex into the space which glues the endpoints together.

extend this concept linearly to get a map $d_n : \mathbb{Z}S_nX \to \mathbb{Z}S_{n-1}X$ So an *n-cycle* is a map $u \in \mathbb{Z}S_nX$ with $u \in \ker(d_n)$.

Similarly, two 1-cycles that go around the hole once can be continuously deformed into each other if and only if the space in between them can be filled in with 2-simplices, as seen in Figure 11.

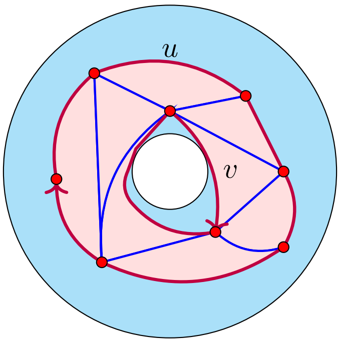


FIGURE 11. the 1-cycle u can be continuously deforms into the 1-cycle v. Equivalently, we can fill the space between them with 2-simplices.

Again, we generalise this idea to higher dimensions; two n-cycles are to be thought of as the same if they can be filled in by a collection of $(n+1)$-simplices. Now, if we calculate the algebraic boundary of the complex made up by these 2-simplices, the inner blue lines end up cancelling each other out due to the alternating nature of the sum, and we are left with $u - v$. So $u - v \in \operatorname{img}(d_2)$, and more generally two n-cycles are the same if and only if their difference is in $\operatorname{img}(d_{n+1})$.

Recall that the images and kernels of group homomorphisms are subgroups. We can see now another reason why we choose to work with $\mathbb{Z}S_n(X)$ rather than just $S_n(X)$; since the former is an abelian group, $\operatorname{img}(d_{n+1})$ and $\ker(d_n)$ are both *normal* subgroups of $\mathbb{Z}S_n(X)$. Moreover, $d_n \circ d_{n+1} = 0$, and so $\operatorname{img}(d_{n+1})$ is a subgroup of $\ker(d_n)$, allowing us to form the quotient group $H_n(\mathbb{Z}SX) := \ker(d_n)/\operatorname{img}(d_{n+1})$. This is called the *nth homology group* of the space, and is invariant under homeomorphism. Moreover, $H_n(\mathbb{Z}SX)$ tells us information about how many holes there are. For example, if $H_n(\mathbb{Z}SX) \cong \mathbb{Z}^m$, this tells us there are m n-dimensional holes in X. For the annulus in Figure 9, we can calculate that $H_2(\mathbb{Z}SX) \cong \mathbb{Z}^1$. The exponent 1 tells us that there is one 2-dimensional hole, which of course we already knew.

4.3. **Chain Complexes.** In order to study this idea rigorously and in a more general setting, we introduce the notion of *chain complexes*.

Definition 4.1. A *chain complex* (A_*, d) of abelian groups is a sequence of abelian groups $\{A_n\}_{n\in\mathbb{Z}}$ equipped with linear homomorphisms $\{d_n : A_n \to A_{n-1}\}$ such that $d_n \circ d_{n+1} = 0$. These maps are called *differentials*.

$$... \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} ...$$

We sometimes write $d^2 = 0$.

The condition $d_n \circ d_{n+1} = 0$ captures the idea that the boundary of a boundary is zero.

Remark 4.2. To illustrate the diversity of this abstract concept, we introduce an application to differential geometry and physics. Let ψ be any scalar field and let $\mathbf{V}$ be any vector field. Recall that the curl of a gradient is zero i.e. $\nabla\times(\nabla\psi) = 0$, and the divergence of a curl is again zero i.e. $\nabla\cdot(\nabla\times\mathbf{V}) = 0$. In fact, we can rephrase this as $\nabla^2 = 0$, and so ∇ satisfies the conditions to be a differential. From this, we can construct the *de Rham cohomology*, which has applications in constructing invariants on smooth manifolds and also in Lagrangian Mechanics [Lee13].

Now that we have defined what a chain complex is, it is sensible to define a map between them. This will allow us to form a category of chain complexes.

Definition 4.3. Let C_* and D_* be chain complexes with differentials d_* and e_* respectively. A *chain map* $u_* : C_* \to D_*$ is a family of abelian group homomorphisms $u_n : C_n \to D_n$ which commute with the differentials, i.e.

$$u_{n-1} \circ d_n = e_n \circ u_n$$

for all n.

This is equivalent to saying that the following diagram commutes.

$$\begin{tikzcd}
... \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & ... \\
... \arrow[r, "e_{n+2}"] & D_{n+1} \arrow[r, "e_{n+1}"] & D_n \arrow[r, "e_n"] & D_{n-1} \arrow[r, "e_{n-1}"] & ...
\end{tikzcd}$$

Proposition 4.4. *The composition of chain maps is well-defined and associative.*

Proof. Let C_*, D_* and E_* be chain complexes. Let $u_* : C_* \to D_*$, and $v_* : D_* \to E_*$ be chain complex maps. We call the differentials of each chain complex d_n, but be aware that these are potentially different for each complex. We have the following diagram:

$$\begin{tikzcd}
... \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & ... \\
... \arrow[r, "d_{n+2}"] & D_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "v_{n+1}"] & D_n \arrow[r, "d_n"] \arrow[d, "v_n"] & D_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "v_{n-1}"] & ... \\
... \arrow[r, "d_{n+2}"] & E_{n+1} \arrow[r, "d_{n+1}"] & E_n \arrow[r, "d_n"] & E_{n-1} \arrow[r, "d_{n-1}"] & ...
\end{tikzcd}$$

This commutes by definition of chain maps and by being able to glue together commutative diagrams (lemma 2.39), and so $v_* \circ u_* : C_* \to E_*$ is a chain map. Associativity follows from the fact that each of the maps is a homomorphism of abelian groups, which have associative composition. $\square$

We can define the notion of an isomorphism too.

Definition 4.5. Let C_* and D_* be chain complexes. An *isomorphism* of chain complexes is a chain map $u_* : C_* \to D_*$ such that u_n is an isomorphism of groups at each level.

Remark 4.6. It is *not* enough to show that there is an isomorphism of groups at every level, and we do really need to show that this is a chain map. Consider any chain complex (C, d). There is also a chain complex with differential the zero map, $(C, 0)$. In this case we have an isomorphism at every level (since $C_n = C_n$), but not a chain map unless $d = 0$.

$$\begin{tikzcd}
\dots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, equal] & C_n \arrow[r, "d_n"] \arrow[d, equal] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, equal] & \dots \\
\dots \arrow[r, "0"] & C_{n+1} \arrow[r, "0"] & C_n \arrow[r, "0"] & C_{n-1} \arrow[r, "0"] & \dots
\end{tikzcd}$$

Often in pure mathematics, after we define what an object is, we go on to define a suitable notion of a *sub*object. It is useful for us to define the notion of a subcomplex.

Definition 4.7. Let C_* be a chain complex of abelian groups. A chain complex C'_* is a *subcomplex* of C_* if each C'_n is a subgroup of C_n, and if the differential on C'_* is the restriction of the differential on C. Equivalently, this says that that C'_n is a subcomplex of C_n whenever the inclusion maps $i_n : C'_n \hookrightarrow C_n$ form a chain map.

As subgroups of an abelian group are all normal, then we know that C_n/C'_n is a well defined abelian group for each n.

Definition 4.8. Let C_* be a chain complex and C'_* be a subcomplex of this. Then we can form the *quotient complex* $(C/C')_*$ with $(C/C')_n = C_n/C'_n$ and differential given by $\tilde{d}_n(x + C'_n) = d_n(x) + C'_n$.

Remark 4.9. It is clear that this defines a chain complex, with $\tilde{d}^2 = 0$ coming from $d^2 = 0$.

Just as we are able to take the disjoint union of sets or direct sum of groups, we want a sensible notion of have a direct sum of chain complexes. These ideas are all generalised under the categorical notion of a coproduct ([Rie17], definition 3.1.23); we shall state the definition explicitly here.

Definition 4.10. Let $(C_*, d^C), (D_*, d^D)$ be chain complexes. The *direct sum* is defined to be the chain complex $(C \oplus D, d^C \oplus d^D)$, where level-wise $(C \oplus D)_n := C_n \oplus D_n$, is the direct sum of abelian groups and $(d^C \oplus d^D)_n : (C \oplus D)_n \to (C \oplus D)_{n-1}$ is given by $(d^C \oplus d^D)_n(x, y) = (d^C_n(x), d^D_n(y))$.

Remark 4.11. It is clear that this defines a chain complex; recall that the direct sum of abelian groups is again an abelian group, and also $(d^C \oplus d^D)^2 = ((d^C)^2, (d^D)^2) = (0, 0)$.

Proposition 4.12. *Let N and D be subcomplexes of C such that level-wise we have $N_n \oplus D_n \cong C_n$ as abelian groups. Then we have an isomorphism of chain complexes $N \oplus D \cong C$.*

Proof. Let N, D and C be as above, with $f_n : C_n \to N_n \oplus D_n$ the isomorphism of groups at every level. We must show that f_n is a chain map, i.e. that $f_{n-1}d^C_n = (d^N \oplus d^D)_n f_n$, or that the diagram

$$\begin{tikzcd}
\dots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r, "(d^N \oplus d^D)_{n+2}"] & (N \oplus D)_{n+1} \arrow[r, "(d^N \oplus d^D)_{n+1}"] & (N \oplus D)_n \arrow[r, "(d^N \oplus d^D)_n"] & (N \oplus D)_{n-1} \arrow[r, "(d^N \oplus d^D)_{n-1}"] & \dots
\end{tikzcd}$$

commutes. For any $x \in C_n$, we write $f_n(x) = (y, z)$, so $x = y + z$ is the unique way to split x into a sum with $y \in N_n$ and $z \in D_n$. Then, by definition, it follows that

$$(d^N \oplus d^D)_n(f_n(x)) = (d_n^N(y), d_n^D(z)) = (d_n^C(y), d_n^C(z)),$$

as N and D are subcomplexes, so have differentials which are just the restriction of d^C. Also, by linearity:

$$d_n^C(x) = d_n^C(y + z) = d_n^C(y) + d_n^C(z) = d_n^N(y) + d_n^D(z),$$

with $d_n^N(y) \in N_{n-1}$ and $d_n^D(z) \in D_{n-1}$. Therefore, this is the unique way to split $d_n^C(x)$, and so $f_{n-1}(d_n^C(x)) = (d^N \oplus d^D)_n(f_n(x))$, as required. $\square$

As our goal is to relate chain complexes to simplicial abelian groups, and as simplicial abelian groups are defined by a *non-negative* sequence of abelian groups $(A_n)_{n\geq 0}$, it makes sense to restrict our attention to chain complexes which look like this.

Definition 4.13. A chain complex C_* is called *non-negatively graded* if $C_n = 0$ for $n < 0$.

Example 4.14. (Exercise 1.1.1 from [Wei95])

There is a non-negatively graded chain complex C_* given by

$$C_n = \begin{cases} \mathbb{Z}/8\mathbb{Z} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases}$$

with differential $d_n(x) = 4x \bmod 8$ for $n > 0$ and $d_n = 0$ for $n \leq 0$. Indeed, this is a family of abelian groups with $d_n \circ d_{n+1} = 0$. This is clear for $n \leq 1$, as one of the maps will be the zero map. For $n > 1$, pick arbitrary $x \in C_n$. Then

$$d_n \circ d_{n+1}(x) = d_n(4x) = 16x = 8(2x) \equiv 0 \bmod 8,$$

as required.

Since we have chain complexes and maps between them, and we have shown that these maps are composable by proposition 4.4, we can form a category. The identity maps are identity maps at every level; this is clearly a chain map.

Definition 4.15. There is a category $Ch^+(\mathbf{Ab})$ with non-negatively graded chain complexes of abelian groups as its objects, and chain maps as its morphisms.

4.3.1. *A Differential on Simplicial Abelian Groups.* Our overall goal is to supply functors between the categories of simplicial abelian groups and non-negatively graded chain complexes. We can now define a differential on simplicial abelian groups.

Definition 4.16. Let A be a simplicial abelian group with face maps ∂_i. We define the *alternating sum of face maps* $d_n : A_n \to A_{n-1}$ by $d_n = \sum_{i=0}^{n}(-1)^i \partial_i$.

Proposition 4.17. *The alternating sum of face maps, d_n is a differential.*

Proof. We must show that $d_n \circ d_{n+1} = 0$. We have

$$\begin{aligned}
d_n \circ d_{n+1} &= d_n \left(\sum_{j=0}^{n+1} (-1)^j \partial_j \right) \\
&= \sum_{i=0}^{n} (-1)^i \partial_i \left(\sum_{j=0}^{n+1} (-1)^j \partial_j \right) \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n+1} (-1)^{i+j} \partial_j \partial_i && \text{by linearity,} \\
&= \sum_{0 \le i < j \le n} (-1)^{i+j} \partial_i \partial_j + \sum_{0 \le j \le i < n} (-1)^{i+j} \partial_i \partial_j && \text{breaking the sum apart,} \\
&= \sum_{0 \le i < j \le n} (-1)^{i+j} \partial_{j-1} \partial_i + \sum_{0 \le j \le i < n} (-1)^{i+j} \partial_i \partial_j && \text{by (S1),} \\
&= - \sum_{0 \le i \le j < n} (-1)^{i+j} \partial_j \partial_i + \sum_{0 \le j \le i < n} (-1)^{i+j} \partial_i \partial_j && \text{by reindexing } j \text{ for } j+1, \\
&= 0.
\end{aligned}$$

□

Example 4.18. For any topological space X, S_*X can be considered a chain complex with differential given by $d_n(u) = \sum_{i=0}^{n} (-1)^i \delta_i \circ u$. We call this the *topological boundary map*. This is the alternating sum of face maps for this simplicial set.

4.4. **Homology of Chain Complexes.** The rough definition of homology for topological spaces we gave in subsection 4.1 is generalised to the notion of homology of chain complexes.

Definition 4.19. Let (C_*, d_n) be a chain complex of abelian groups. We define:

- $Z_n(C_*) = \ker(d_n : C_n \to C_{n-1})$.
- $B_n(C_*) = \operatorname{img}(d_{n+1} : C_{n+1} \to C_n)$.

The elements of $Z_n(C_*)$ are called *cycles* and the elements of $B_n(C_*)$ are called *boundaries*. Any $y \in B_n(C)$ is of the form $y = d_{n+1}(x)$. We note that as $d_n \circ d_{n+1} = 0$, we have $d_n y = d_n \circ d_{n+1}(x) = 0$, so $y \in Z_n(C_*)$. Hence $B_n(C_*)$ is a subgroup of $Z_n(C_*)$. It follows that there is a well-defined quotient group:

$$H_n(C_*) = Z_n(C_*)/B_n(C_*).$$

We call this the n-th *homology group*.

Remark 4.20. We often write $Z_*(C), B_*(C)$ and $H_*(C)$ to refer to the system of groups as a whole.

Proposition 4.21. *Z_*, B_* and H_* form functors $Ch(\mathbf{Ab}) \to \mathbf{Ab}$.*

inspired by ([Str21], Proposition 13.11). Let $u_* : C_* \to C'_*$ be a chain map. We show that:

(i) for all n, $u(Z_n(C))$ is a subgroup of $Z_n(C')$.
(ii) for all n, $u(B_n(C))$ is a subgroup of $B_n(C')$.
(iii) there is a well-defined map $H_*(u) : H_*(C) \to H_*(C')$ given level-wise by

$$H_n(u)(z + B_n(C)) = u(z) + B_n(C').$$

This is an essential step in showing that the functors are well-defined.

(i) Let $z \in Z_n(C)$. Then $d_n^C(z) = 0$, and so $d_n^{C'}(u_n(z)) = u_{n-1}(d_n^C(z)) = u_n(0) = 0$ as u_* is a chain map and so commutes with the differentials. So $u_n(z) \in Z_n(C')$, as required.

(ii) Let $b \in B_n(C)$. Then $b = d^C_{n+1}(x)$ for some $x \in C_{n+1}$. So we have

$$u_n(b) = u_n\left(d^C_{n+1}(x)\right) = d^{C'}_{n+1}(u_{n+1}(x)),$$

so $u_n(b) \in B_n(C')$, as required.

(iii) If $z + B_n(C) = z' + B_n(C)$ in $H_n(C)$, we must check that

$$u_n(z) + B_n(C') = u_n(z') + B_n(C')$$

in $H_n(C')$. In this case, we have $z - z' \in B_n(C)$ and so

$$u_n(z) - u_n(z') = u_n(z - z') \in B_n(C')$$

by (ii) and the fact that u is a group homomorphism. Hence

$$u_n(z) + B_n(C') = u_n(z') + B_n(C')$$

in $H_n(C')$, so $H_*(u)$ is well-defined.

Now we can define the functors; we have shown what they do on objects of $Ch(\mathbf{Ab})$, so it remains to show what they do on chain maps. We define

$$Z_n(u) = u|_{Z_n(C)} : Z_n(C) \to Z_n(C').$$

By (i), this is well-defined. Similarly, we define $B_n(u) = u|_{B_n(C)}, : B_n(C) \to B_n(C')$ which is well-defined by (ii). As these are just restrictions, these are compatible with identity morphisms and composition, and hence are functors. Finally, in (iii) we defined what H_* does to chain maps; it is clear from this definition that if u_* is the identity on C_*, then $H_*(u)$ is the identity on $H_*(C)$. It remains to show that this is compatible with composition. Suppose we have chain maps:

$$C_* \xrightarrow{u} C'_* \xrightarrow{v} C''_*.$$

Then

$$\begin{aligned} H_n(v \circ u)(z + B_n(C) &= (v \circ u)(z) + B_n(C'') \\ &= v(u(z)) + B_n(C'') \\ &= H_n(v)(u(z) + B_n(C')) \\ &= (H_n(v) \circ H_n(u))(z + B_n(C)), \end{aligned}$$

so H_* is compatible with composition, and therefore a functor.

□

Remark 4.22. We have defined a functor $\mathbb{Z}S_*$ from **Top** to $Ch^+(\mathbf{Ab})$. We can therefore define the homology of a topological space by the functor $H_*\mathbb{Z}S_* : \mathbf{Top} \to \mathbf{Ab}$. The groups given as a result of this functor are invariant under homeomorphism. Similarly, for a chain complex C_*, the groups $H_*(C)$ are invariant under many transformations of chain complexes for example chain homotopies. For more details and a more precise proof of the content of this remark, the interested reader is referred to [Str21, Hat01].

5. The Dold-Kan Correspondence

The main result of this chapter and indeed this project is proven in this section. We have seen that both chain complexes and simplicial abelian groups are useful for understanding topological spaces, and that there are many similarities between them. The Dold-Kan correspondence makes this precise.

Theorem 5.1 (The Dold-Kan Correspondence)**.** *There is an equivalence of categories between $Ch^+(\mathbf{Ab})$ and* **sAb***.*

As per the definition of equivalence of categories, to prove this we need to construct functors:

$$K : Ch^+(\mathbf{Ab}) \rightleftarrows \mathbf{sAb} : N$$

such that KN is naturally isomorphic to $\mathbb{1}_{\mathbf{sAb}}$ and NK is naturally isomorphic to $\mathbb{1}_{Ch^+(\mathbf{Ab})}$. The functor N is constructed in subsection 5.1. The functor K is constructed and elaborated on in subsection 5.2. The natural isomorphisms are proven in subsection 5.3. We then use this result to prove a relationship between the homology groups and the homotopy groups in subsection 5.5.

5.1. **From Simplicial Abelian Groups to Non-negatively Graded Chain Complexes.** We construct a functor from simplicial abelian groups to non-negatively graded chain complexes, and prove that this is a functor by checking all the conditions. As part of the proof of the Dold-Kan correspondence, it will be helpful to introduce two other functors from simplicial abelian groups to non-negatively graded chain complexes. These functors will be related in theorem 5.10.

5.1.1. *The Un-normalised Chain Complex.* We start by introducing the most simple of the three functors from $\mathbf{sAb}$ to $Ch^+(\mathbf{Ab})$ that we will introduce.

Lemma 5.2. *There is a functor $C : \mathbf{sAb} \to Ch^+(\mathbf{Ab})$ given level-wise:*

- *On objects, $C_n(A) := A_n$.*
- *On morphisms $f : A \to B$, we let $C(f) : C(A) \to C(B)$ be given by $C(f) := f$.*

The resulting image $C(A)$ is a non-negatively graded chain complex with differential $d_n^C = \sum_{i=0}^{n}(-1)^i \partial_i$.

Proof. Proposition 4.17 proves that the image of A is a non-negatively graded chain complex of abelian groups. As f commutes with degeneracies, $N(f)$ commutes with the differential, and so is a chain map. It is trivial to check the functoriality axioms. □

5.1.2. *The Degenerate Chain Complex.* We now introduce a functor which is the restriction of C to only the degenerate simplices of A.

Proposition 5.3. *There is a functor $D : \mathbf{sAb} \to Ch^+(\mathbf{Ab})$ defined level-wise:*

- *On objects, $D_n(A) := \sum_{i=0}^{n-1} \sigma_i(C_{n-1}A)$.*
- *On morphisms, $D(f) := f|_{D(A)}$.*

The differential is defined by $d_n^D := d_n^C|_D$.

Proof. Since simplicial maps commute with degeneracies, the image of a degeneracy under a simplicial map is a degeneracy. Moreover, as the differential is defined in terms of degeneracies, commuting with the differential is the same as commuting with degeneracies. Hence, $D(f)$ is a chain map, and so this map is well-defined on morphisms. To show that it is well-defined on objects, we show that the image is a chain complex, by showing that d^D does actually define a differential. First, we prove that $d_n^D(D_n(A)) \subseteq D_{n-1}(A)$. We do this by showing that the image under this map is a sum of degeneracies. It is enough to show this for $y = \sigma_j(x) \in D_n(A)$ for some $0 \leq j \leq n-1$ and $x \in C_{n-1}(A)$, and the result follows by linearity.

$$
\begin{aligned}
d_n^D(y) &= \sum_{i=0}^{n-1}(-1)^i \partial_i \sigma_j(x), \\
&= \sum_{i=0}^{j-1}(-1)^i \partial_i \sigma_j(x) + (-1)^j \partial_j \sigma_j(x) + (-1)^{j-1}\partial_{j+1}\sigma_j(x) + \sum_{i=j+1}^{n-1}(-1)^i \partial_i \sigma_j(x), \\
&= \sum_{i=0}^{j-1}(-1)^i \sigma_{j-1}\partial_i(x) + (-1)^j x + (-1)^{j-1} x + \sum_{i=j+1}^{n-1}(-1)^i \sigma_j \partial_{i-1}(x), \\
&= \sigma_{j-1}\left(\sum_{i=0}^{j-1}\partial_i(x)\right) + \sigma_j\left(\sum_{i=j+1}^{n-1}\partial_{i-1}(x)\right) \in D_{n-1}(A),
\end{aligned}
$$

where the third equality follows from (S3). The proof that d^D is a differential follows the same line of reasoning as proposition 4.17. As D is defined by restrictions, it is compatible with composition and identities, and so is a functor.

□

We also have:

Lemma 5.4. *For any simplicial abelian group A, $D(A)$ is a subcomplex of $C(A)$.*

Proof. Since $0 = \sigma_i(0)$ it follows that $0 \in D_n(A)$. By linearity it also follows that $D_n(A)$ is closed under addition and inverses are contained in $D_n(A)$. Hence $D_n(A)$ is a subgroup of $C_n(A)$. By definition, $d_n^C|_D = d_n^D$, which completes the proof. □

5.1.3. *The Normalised Chain Complex.* We can now define the functor N that appears in the Dold-Kan correspondence. It will turn out that $N(A)$ is isomorphic as a chain complex to $C(A)/D(A)$, as is proved in theorem 5.10. This means that $N(A)$ can be thought of as the complex $C(A)$ modulo all degeneracies.

Proposition 5.5. *There is a functor $N : \mathbf{sAb} \to Ch^+(\mathbf{Ab})$ given level-wise:*

- *On objects $A \in \mathbf{sAb}$, we define $N_n(A) := \bigcap_{i=0}^{n-1} \ker(\partial_i : A_n \to A_{n-1})$ for $n > 0$, and $N_0(A) := A_0$.*
- *On morphisms $f : A \to B$, we define $N_n(f) := f_n|_{N_n(A)}$.*

The differential on $N_n(A)$ is given by $d_n^N := (-1)^n \partial_n$.

Proof. We show that this is well-defined by showing the image of a simplicial group is a non-negatively graded chain complex, and the image of a simplicial map is well-defined and a chain map. We then show that this is a functor.

Let $A \in \mathbf{sAb}$. Recall that the kernel of a group is a subgroup, and that the intersection of groups is another group. Therefore, for each n, $N_n(A)$ is an abelian group. We start by showing that d^N is a well-defined map by showing $d^N(N_n(A)) \subseteq N_{n-1}(A)$. Let $y \in N_n(A)$. We want to show that $d_n^N(y) \in N_{n-1}(A)$. We do this by showing that $\partial_i d_n(y) = 0$ for all $0 \leq i \leq n-2$. This follows from the simplicial identity (S1): for any $0 \leq i \leq n-2$, we have:

$$\partial_i d_n^N(y) = (-1)^n \partial_i \partial_n(y) = (-1)^n \partial_{n-1}\partial_i(y) = 0,$$

so $d_n^N(y) \in N_{n-1}(A)$, as required. To show that $N(A) \in Ch^+(\mathbf{Ab})$, it remains to show that $d_n \circ d_{n+1} = 0$. This too follows from (S1); indeed, $d_n \circ d_{n+1} : N_{n+1}(A) \to N_{n-1}(A)$ and $N_{n+1}(A) = \bigcap_{i=0}^{n} \ker(\partial_i : A_{n+1} \to A_n)$. In particular, $\partial_n(N_{n+1}(A)) = 0$. Now,

$$d_n \circ d_{n+1}(N_{n+1}(A)) = (-1)^{2n+1}\partial_n \partial_{n+1}(N_{n+1}(A)) = -\partial_n \partial_n(N_{n+1}(A)) = -\partial_n(0) = 0,$$

as required.

To show the image of a simplicial map is well-defined under N, we must show that its image commutes with the differential. Let A and B be simplicial sets with face maps ∂_i^A and ∂_i^B respectively, and let $f : A \to B$ be a simplicial map. Now, suppose $x \in N_n(A)$. Then $\partial_i^A x = 0$ for all $0 \leq i \leq n-1$. Since f_n commutes with face maps, then for all $0 \leq i \leq n-1$,

$$\partial_i^B(f_n(x)) = f_n\left(\partial_i^A x\right) = f_n(0) = 0,$$

and so $f_n(x) \in \bigcap_{i=0}^{n-1} \ker\left(\partial_i^B : B_n \to B_{n-1}\right) = N_n(B)$. To see that it is a chain map, we note that the condition for a simplicial map to commute with face maps is the same as the condition for a chain map to commute with differentials when $d^N = (-1)^n \partial_n$. Therefore, f commutes with d^N and so $N(f) = f|_{N(A)}$ commutes with d^N, and so is a chain map. Hence $N(f)$ is well-defined.

Let A, B and C be simplicial sets, and let $f : A \to B$ and $g : B \to C$ be simplicial maps. To show functoriality of N, we need to show it satisfies (F1) and (F2).

(F1) We can see that for $x \in N_n(A)$,

$$\begin{aligned} N((g \circ f))(x) &:= (g \circ f)|_{N_n(A)}(x) && \\ &= g(f(x)) && \text{as } x \in N_n(A), \\ &= g(f|_{N_n(A)}(x)) && \\ &= g|_{N_n(B)}(f|_{N_n(A)}(x)) && \text{as } f|_{N_n(A)}(x) \in N_n(B), \\ &= (g|_{N_n(B)} \circ f|_{N_n(A)})(x) && \\ &= N(g)N(f) && \text{as required.} \end{aligned}$$

(F2) By noting that $\mathbb{1}_A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{1}_{\mathbb{A}_n}$ we can easily see that since

$$N(\mathbb{1}_{A_n}) := \mathbb{1}|_{N_n(A)} = \mathbb{1}_{N_n(A)},$$

for all $n \geq 0$, we have $N(\mathbb{1}_A) = \mathbb{1}_{N(A)}$.

We have therefore shown that N is a functor.

□

$N(A)$ is referred to as *the Moore chain complex* or *the normalised chain complex* of a simplicial abelian group A [GJ99].

Lemma 5.6. *For any simplicial abelian group A, $N(A)$ is a subcomplex of $C(A)$.*

Proof. It is clear that for each n, $N_n(A) \subseteq C_n(A)$. Now, for any $y \in N_n(A)$, we have $d_n^C(y) = \sum_{i=0}^n (-1)^i \partial_i(y)$. However, since $y \in N_n(A) \subseteq \ker(\partial_i)$ for $0 \leq i \leq n-1$, then we can simplify this as $d_n^C(y) = (-1)^n \partial_n(y) = d_n^N(y)$, as required. Hence $N(A)$ is a subcomplex of $C(A)$.

□

We can now relate the functors C, D and N.

Lemma 5.7. *Let A be a simplicial abelian group. Any element $y \in C_n(A)$ is congruent to an element in $N_n(A)$ modulo $D_n(A)$, i.e. $y = x + z$ with $x \in N_n(A)$ and $z \in D_n(A)$.*

Proof. Clearly if $\partial_j(y) = 0$ for all j, then $y \in N_n(A)$ and so $y = y + 0$ is our desired splitting. We prove that if j is the smallest integer such that $\partial_j(y) \neq 0$ then we can write $y = x + z$ as required. We prove this by downwards induction on j.

For our base case, we let $j = n$. We construct the element $y' = y - \sigma_n \partial_n(y)$. Clearly, $y = y' + \sigma_n \partial_n(y)$ and $\sigma_n \partial_n(y) \in D_n(A)$. We show that for all $0 \leq i < n$, we have $\partial_i(y') = 0$ and so $y' \in N_n(A)$, giving us our desired split. Let $i < n$. Now:

$$
\begin{aligned}
\partial_i(y') &= \partial_i(y - \sigma_n\partial_n(y)), \\
&= \partial_i(y) - \partial_i\sigma_n\partial_n(y) && \text{by linearity,} \\
&= \partial_i(y) - \sigma_{n-1}\partial_i\partial_n(y) && \text{by (S3),} \\
&= \partial_i(y) - \sigma_{n-1}\partial_{n-1}\partial_i(y) && \text{by (S1).}
\end{aligned}
$$

However, since n was the smallest integer such that $\partial_j(y) \neq 0$ and $i < n$, then $\partial_i(y) = 0$. Hence $\partial_i(y') = 0$ for all $i \leq n-1$. Therefore $y' \in N_n(A)$.

Now, suppose that this holds for all $j = n, n-1, ..., k+1$. If the smallest integer such that $\partial_j(y) \neq 0$ is $j = k$, then we construct the element $y' = y - \sigma_k\partial_k(y)$. Then for $i < k$ we have identical calculations as our base case, and $\partial_i(y') = 0$. Suppose then that the smallest integer j such that $\partial_j(y') \neq 0$ is bigger than k. But then by the inductive hypothesis, $y' = y'' + \sigma_j\partial_j(y')$ with $y'' \in N_n(A)$. This gives us:

$$
\begin{aligned}
y &= y' + \sigma_k\partial_k(y) \\
&= y'' + \sigma_j\partial_j(y') + \sigma_k\partial_k(y),
\end{aligned}
$$

with $y'' \in N_n(A)$ and $\sigma_j\partial_j(y') + \sigma_k\partial_k(y) \in D_n(A)$. Hence by induction we have proved our claim.

□

Lemma 5.8. $N_n(A) \cap D_n(A) = \{0\}$.

Proof. Let $y \in N_n(A) \cap D_n(A)$. Clearly, $0 \in N_n(A) \cap D_n(A)$ as $0 \in \ker(\partial_i)$ and $0 = \sigma_i(0)$ for all i. Now, suppose $y \neq 0$. Then $y = \sum_{j=0}^{n-1} \sigma_j(x_j)$ with $x_j \in C_{n-1}(A)$, since $y \in D_n(A)$. As $y \neq 0$, there exists some j such that $\sigma_j(x_j) \neq 0$. Let $0 \leq k \leq n-1$ be the smallest such integer, so we write $y = \sum_{j=k}^{n-1} \sigma_j(x_j)$. We note that $\partial_k y = 0$ as $y \in N_n(A) \subseteq \ker \partial_k$, so we also have $\sigma_k\partial_k y = 0$, and so $y - \sigma_k\partial_k y = y$. We note that:

$$
\begin{aligned}
\sigma_k\partial_k y &= \sigma_k\partial_k\left(\sum_{j=k}^{n-1}\sigma_j(x_j)\right), \\
&= \sigma_k\left(\sum_{j=k}^{n-1}\partial_k\sigma_j(x_j)\right) && \text{by linearity,} \\
&= \sigma_k\left(\partial_k\sigma_k x_k + \sum_{j=k+1}^{n-1}\partial_k\sigma_j(x_j)\right), \\
&= \sigma_k\left(x_k + \sum_{j=k+1}^{n-1}\sigma_{j-1}\partial_k(x_j)\right) && \text{by (S3),} \\
&= \sigma_k x_k + \sum_{j=k+1}^{n-1}\sigma_k\sigma_{j-1}\partial_k(x_j) && \text{again, by linearity,} \\
&= \sigma_k x_k + \sum_{j=k+1}^{n-1}\sigma_j\sigma_k\partial_k(x_j) && \text{by (S2).}
\end{aligned}
$$

This means that:

$$
\begin{aligned}
y &= y - \sigma_k \partial_k y, \\
&= \sum_{j=k}^{n-1} \sigma_j(x_j) - \left(\sigma_k x_k + \sum_{j=k+1}^{n-1} \sigma_j \sigma_k \partial_k (x_j) \right), \\
&= \sum_{j=k+1}^{n-1} \sigma_j \left(x_j - \sigma_k \partial_k (x_j) \right), \\
&= \sum_{j=k+1}^{n-1} \sigma_j (x_j'),
\end{aligned}
$$

which is a contradiction to our assumption on k. Hence, no such y exists and therefore $N_n(A) \cap D_n(A) = \{0\}$, as required. ☐

Corollary 5.9. *Let A be a simplicial abelian group. For each $n > 0$, there is an isomorphism of abelian groups $C_n(A) \cong N_n(A) \oplus D_n(A)$.*

Proof. By lemma 5.7 and lemma 5.8, it is clear that for any $x \in C_n(A)$ we can define a map $f : C_n(A) \to N_n(A) \oplus D_n(A)$ given by writing x uniquely as $x = y + z$ with $y \in N_n(A)$ and $z \in D_n(A)$ and letting $f(x) = (y, z)$. This has inverse map $g : N_n(A) \oplus D_n(A) \to C_n(A)$ given by $g(y, z) = y + z$. This is clearly an isomorphism of abelian groups due to the uniqueness of the expression and the fact that both $N_n(A)$ and $D_n(A)$ are subgroups of $C_n(A)$. ☐

Theorem 5.10. *Let A be a simplicial abelian group. $C(A)$ is isomorphic as a chain complex to $N(A) \bigoplus D(A)$. Therefore, $N(A)$ is isomorphic as a chain complex to $C(A)/D(A)$.*

Proof. Proposition 4.12 tells us that it is enough to prove that both $N_n(A)$ and $D_n(A)$ are subcomplexes of $C_n(A)$ and that we have level-wise isomorphisms of abelian groups. These are proved in lemmas 5.4, 5.6 and corollary 5.9. ☐

We have therefore found our functor from simplicial abelian groups to non-negatively graded chain complexes of abelian groups. We move on to finding a functor from non-negatively graded chain complexes of abelian groups to simplicial abelian groups.

5.2. **From Non-negatively Graded Chain Complexes to Simplicial Abelian Groups.** This section provides a functor going from $Ch^+(\mathbf{Ab})$ to $\mathbf{sAb}$. In order to understand the structure more clearly, we provide examples of how K works on the lower levels of a chain complex.

Proposition 5.11. *There is a functor $K : Ch^+(\mathbf{Ab}) \to \mathbf{sAb}$:*

- *On objects $C_* \in Ch^+(\mathbf{Ab})$,*

$$K_n(C) = \bigoplus_{p=0}^{n} \bigoplus_{\eta:[n]\twoheadrightarrow[p]} C_p[\eta],$$

where $C_p[\eta]$ is a copy of C_p which we index with the order-preserving epimorphism η.

- *On morphisms $u_* : C_* \to D_*$, we have*

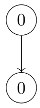
$$K_n(u) = (u_0, \underbrace{u_1, ..., u_1}_{n \text{ times}}, \underbrace{u_2, ..., u_2}_{\binom{n}{2} \text{ times}}, ..., \underbrace{u_{n-1}, ..., u_{n-1}}_{\binom{n}{n-1} \text{ times}}, u_n).$$

We show that K is well-defined by showing the image of a non-negatively graded chain complex is a simplicial abelian group. We prove this by showing that for any object $C_* \in Ch^+(\mathbf{Ab})$, $K(C_*)$ is a contravariant functor from $\mathbf{\Delta}$ to $\mathbf{Ab}$. We then show that K is a functor from $Ch^+(\mathbf{Ab})$ to $\mathbf{sAb}$.

First, however, let us unravel the definition.

Example 5.12. To understand what $K_n(C)$ looks like in low degrees, we must understand surjections from $[n]$ to $[p]$. We note that in $\mathbf{\Delta}$, a surjection is an epimorphism, so we will write $[n] \twoheadrightarrow [p]$ to show a surjection.

- In degree 0 we have just one possible value of $p = 0$ and thus only one possible map:

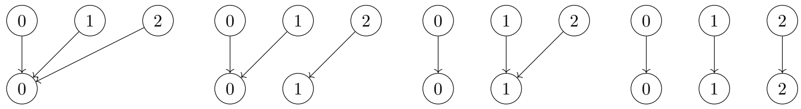


Hence $K_0(C) = \bigoplus_{p=0}^{0} \bigoplus_{\eta:[n]\twoheadrightarrow[p]} C_p[\eta] = C_0[\downarrow]$.

- In degree 1, we can have $p = 0$ or $p = 1$, giving us 2 order-preserving surjections:

So $K_1(C) = C_0[\downarrow\swarrow] \bigoplus C_1[\downarrow\downarrow]$.

- In degree 2, we have $p \in \{0, 1, 2\}$ and we get 4 order-preserving surjections:



Therefore:

$$K_2(C) = C_0[\downarrow\swarrow\swarrow] \bigoplus C_1[\downarrow\swarrow\swarrow] \bigoplus C_1[\downarrow\downarrow\swarrow] \bigoplus C_2[\downarrow\downarrow\downarrow] \cong C_0 \bigoplus C_1^2 \bigoplus C_2.$$

We can see from this example that when $p = 0$, the only map we can get is the map where everything is sent to 0, and so C_0 always appears exactly once as a summand in $K_n(C)$. Similarly, we see that when $p = n$, the only order-preserving surjection we can have is the identity map, and so C_n also always appears exactly once as a summand in $K_n(C)$. To understand what happens between these, we must understand for fixed n and p, how many copies of C_p there are in $K_n(C)$.

Lemma 5.13. *There are $\binom{n}{p}$ order-preserving surjections $[n] \twoheadrightarrow [p]$.*

Proof. An order-preserving surjective map $[n] \twoheadrightarrow [p]$ is the same as a partition of $(n+1)$ elements into $(p+1)$ parts including only adjacent elements. To see this, pick a partition of $\{0, 1, 2, ..., n\}$ into $(p+1)$ parts in this way. We send the first part to 0, the second part to 1,..., the $(p+1)$th part to p. This defines a map $[n] \to [p]$ which is clearly surjective and order-preserving:

$$\{\underbrace{0,1}_{0}, \underbrace{2,3,4}_{1}, \underbrace{5,6}_{2}, ..., \underbrace{n-3, n-2, n-1, n}_{p}.\}$$

Conversely, suppose we had an order-preserving surjection $\eta : [n] \twoheadrightarrow [p]$. By looking at $\eta^{-1}(i)$ for each $0 \le i \le p$, because of the order-preserving property we get a partition of n, and because this is a surjection, we get $(p+1)$ parts:

$$\{\underbrace{0,1}_{\eta^{-1}(0)},\underbrace{2,3,4,}_{\eta^{-1}(1)}\underbrace{5,6}_{\eta^{-1}(2)},...,\underbrace{n-3,n-2,n-1,n}_{\eta^{-1}(p)}\}.$$

There are $\binom{n}{p}$ ways to partition $(n+1)$ elements into $(p+1)$ parts in this way. This is because it is the same as picking p elements out of the numbers $0,...,n$ as the last element of the part. We do not need to specify where the $(p+1)$th part ends as it ends at the $(n+1)$th element since it is a surjection. This fully determines the partition.

□

Corollary 5.14. *There are $\binom{n}{p}$ copies of C_p in $K_n(C)$.*

Proof. This is direct from the previous lemma and the definition of K. □

We can then understand what an element of $K_n(C)$ looks like for general n:

Lemma 5.15. $\sum_{p=0}^{n}\binom{n}{p}=2^n$.

Proof. Recall the binomial expansion formula for $a,b\in\mathbb{R}$ and $n\geq 0$:

$$(a+b)^n=\sum_{p=0}^{n}\binom{n}{p}a^{n-p}b^p.$$

By letting $a=b=1$, we see that:

$$2^n=\sum_{p=0}^{n}\binom{n}{p}.$$

□

Corollary 5.16. *The direct sum decomposition of $K_n(C)$ has 2^n summands.*

Proof. This follows from the way K was defined in proposition 5.11, and from corollary 5.14 and lemma 5.15. □

This fits with our calculations from example 5.12. Now that we have got more intuition for what K is doing, we move onto proving that it is a functor.

Lemma 5.17. *Fix $C_*\in Ch^+(\mathbf{Ab})$. There is a contravariant functor $\widetilde{K}(C_*):\mathbf{\Delta}\to\mathbf{Ab}$:*

- *On objects $[n]\in\mathbf{\Delta}$,*

$$\widetilde{K}(C_*)[n]=\bigoplus_{p=0}^{n}\bigoplus_{\eta:[n]\twoheadrightarrow[p]}C_p[\eta].$$

- *On morphisms in $\mathbf{\Delta}$, $\alpha:[m]\to[n]$ it is enough to specify $K(C_*)(\alpha)|_\eta$ for each surjection $\eta:[n]\twoheadrightarrow[p]$. Find the unique epi-monic factorisation of $\eta\alpha=\epsilon\mu$, as displayed in the commutative diagram below.*

$$\begin{tikzcd}
{[n]} \arrow[r, "\alpha"] \arrow[d, two heads, "\mu"] & {[m]} \arrow[d, two heads, "\eta"] \\
{[q]} \arrow[r, tail, "\epsilon"] & {[p]}
\end{tikzcd}$$

  Now:
  - *if $q=p$, then we take $K(C_*)(\alpha)|_\eta$ to be the natural association of $C_p[\eta]$ with $C_p[\mu]$.*
  - *If $p=q+1$ and $\epsilon=\epsilon_p$, then we take $K(C_*)(\alpha)|_\eta=d_p:C_p\to C_{p-1}$.*
  - *Otherwise, we set $K(C_*)(\alpha)|_\eta=0$.*

Proof. We must show that $K(C_*)$ is well-defined by showing that $K_n(C_*)$ is an abelian group for each n and that for any morphism α, $K(C_*)(\alpha)$ is a homomorphism of abelian groups. We then show that $K(C_*)$ is cofunctorial.

The former follows directly from the definition of chain complexes, which tells us that each $C_n[\eta]$ is an abelian group, and the fact that the direct sum of abelian groups is again an abelian group. Also, $K(C_*)(\alpha)$ is a group homomorphism as it is defined to be made up of the zero map, an identity map, and the map d, all of which are group homomorphisms.

Now, let $\alpha : [n] \to [m]$ and $\beta : [m] \to [l]$. We show that K satisfies (CF1) and (CF2).

(CF1) Fix some order-preserving surjection $\eta : [l] \twoheadrightarrow [p]$. Let $\eta(\beta\alpha) = \epsilon\eta'$ be the unique epi-monic factorisation of $\beta\alpha$:

$$\begin{tikzcd} {[n]} \arrow[r, "\alpha"] \arrow[d, two heads, "\eta'"] & {[m]} \arrow[r, "\beta"] & {[l]} \arrow[d, two heads, "\eta"] \\ {[q']} \arrow[rr, tail, "\epsilon"] & & {[p]}\,. \end{tikzcd} \tag{1}$$

Also, let $\eta\beta = \rho\mu$ be the epi-monic factorisation of $\eta\beta$:

$$\begin{tikzcd} {[m]} \arrow[r, "\beta"] \arrow[d, two heads, "\mu"] & {[l]} \arrow[d, two heads, "\eta"] \\ {[q]} \arrow[r, tail, "\rho"] & {[p]}\,. \end{tikzcd} \tag{2}$$

We can also find an epi-monic factorisation of $\mu\alpha = \iota\kappa$:

$$\begin{tikzcd} {[n]} \arrow[r, "\alpha"] \arrow[d, two heads, "\kappa"] & {[m]} \arrow[d, two heads, "\mu"] \\ {[r]} \arrow[r, tail, "\iota"] & {[q]}\,. \end{tikzcd} \tag{3}$$

We can glue (2) and (3) together by lemma 2.39 to get the following commutative diagram.

$$\begin{tikzcd} {[n]} \arrow[r, "\alpha"] \arrow[d, two heads, "\kappa"] & {[m]} \arrow[r, "\beta"] \arrow[d, two heads, "\mu"] & {[l]} \arrow[d, two heads, "\eta"] \\ {[r]} \arrow[r, tail, "\iota"] & {[q]} \arrow[r, tail, "\rho"] & {[p]}\,. \end{tikzcd} \tag{4}$$

Since the composition of two monomorphism is a monomorphism, we see that $(\rho\iota)\kappa$ is an epi-monic factorisation of $\eta\beta\alpha$. Since epi-monic factorisations are unique, then we must have $\kappa = \eta', \rho\iota = \epsilon$ and $r = q'$.

If $q' = p$, then $K(C)(\beta\alpha)|_\eta$ is by definition the natural association of $C_p[\eta]$ with $C_p[\eta']$. But if $q' = p$ then we must also have $q = p$, as monics in $\mathbf{\Delta}$ must either be the identity of have domain less than or equal to the codomain, which is not possible in this situation. Therefore, by considering (2) we see that $K(C)(\beta)|_\eta$ is the natural association of $C_p[\eta]$ with $C_p[\mu]$, and that $K(C)(\alpha)|_\mu$ is the natural association of $C_p[\mu]$ with $C_p[\eta']$. Hence $K(C)(\alpha)|_\mu K(C)(\beta)|_\eta$ is the natural association of $C_p[\eta]$ with $C_p[\eta']$ and so

$$K(C)(\beta\alpha)|_\eta = K(C)(\alpha)|_\mu K(C)(\beta)|_\eta.$$

If $p = q' + 1$ and $\epsilon = \epsilon_p$ then

$$K(C)(\beta\alpha)|_\eta = d_p : C_p[\eta] \to C_{p-1}.$$

In this case, by looking at (4), we see that either $q = q' + 1$ or $q = q'$. In the former case, by looking at (3), we see that $K(C)(\alpha)|_\mu = d_p : C_p[\mu] \to C_{p-1}$. But then $q = p$ and so $K(C)(\beta)|_\eta$ is the natural identification of $C_p[\eta]$ with $C_p[\mu]$. Hence $K(C)(\alpha)|_\mu K(C)(\beta)|_\eta = d : C_p[\eta] \to C_{p-1} = K(C)(\beta\alpha)|_\eta$. The latter case is similar.

Otherwise, $K(C)(\beta\alpha)|_\eta = 0$. If $q \neq p-1$ or $q = p-1$ but $\rho \neq \epsilon_p$, then $K(C)(\beta)|_\eta = 0$ and so $0 = K(C)(\alpha)|_\mu K(C)(\beta)|_\eta = K(C)(\beta\alpha)|_\eta = 0$. Similarly, if $q' \neq q-1$ or $q = q-1$ but $\iota \neq \epsilon_q$, then we get $0 = K(C)(\alpha)|_\mu K(C)(\beta)|_\eta = K(C)(\beta\alpha)|_\eta = 0$.

Note that if either $q = p$ or $q' = q$ then we must get a situation we have described before.

This only leaves us with one more type: if $q = p-1$ and $q' = q-1 = p-2$:

$$\begin{tikzcd}
{[n]} \arrow[r, "\alpha"] \arrow[d, two heads, "\kappa"] & {[m]} \arrow[r, "\beta"] \arrow[d, two heads, "\mu"] & {[l]} \arrow[d, two heads, "\eta"] \\
{[p-2]} \arrow[r, tail, "\iota=\epsilon_q"] & {[p-1]} \arrow[r, tail, "\rho=\epsilon_p"] & {[p]}\,.
\end{tikzcd}$$

In this case, we have $K(C)(\alpha)|_\mu = d_{p-1} : C_{p-1} \to C_{p-2}$ and $K(C)(\beta)|_\eta = d_p : C_p \to C_{p-1}$, and so

$$K(C)(\alpha)|\mu K(C)(\beta)|_\eta = d_{p-1} \circ d_p = 0 = K(C)(\beta\alpha)|_\eta,$$

since d is a differential.

Hence we have shown that for any surjection η, we have

$$K(C)(\beta\alpha)|_\eta = K(C)(\alpha)|\mu K(C)(\beta)_\eta,$$

where μ is determined by η. This gives us the result we require:

$$K(C)(\beta\alpha) = K(C)(\alpha)K(C)(\beta).$$

(CF2) We note that $\mathbb{1}_{K(C_*)[n]} = (\mathbb{1}_{C_0}, \underbrace{\mathbb{1}_{C_1}, ..., \mathbb{1}_{C_1}}_{n \text{ times}}, \underbrace{\mathbb{1}_{C_2}, ..., \mathbb{1}_{C_2}}_{\binom{n}{2} \text{ times}}, ..., \underbrace{\mathbb{1}_{C_{n-1}}, ..., \mathbb{1}_{C_{n-1}}}_{\binom{n}{n-1} \text{ times}}, \mathbb{1}_{C_n})$.

Now, the epi-monic factorisation of $\eta\mathbb{1}_{[n]}$ is $\mathbb{1}_{[p]}\eta$. So we are in the case where $q = p$ and so we obtain the natural association of $C_p[\eta]$ with $C_p[\eta]$, which is the identity. Therefore, we get the identity for every surjection, and

$$K(C_*)(\mathbb{1}_{[n]}) = (\mathbb{1}_{C_0}, \underbrace{\mathbb{1}_{C_1}, ..., \mathbb{1}_{C_1}}_{n \text{ times}}, \underbrace{\mathbb{1}_{C_2}, ..., \mathbb{1}_{C_2}}_{\binom{n}{2} \text{ times}}, ..., \underbrace{\mathbb{1}_{C_{n-1}}, ..., \mathbb{1}_{C_{n-1}}}_{\binom{n}{n-1} \text{ times}}, \mathbb{1}_{C_n}) = \mathbb{1}_{K(C_*)[n]},$$

as required.

Therefore we have shown that $K(C)$ is a contravariant functor, as required. □

Proposition 5.18. *There is a functor $K : Ch^+(\mathbf{Ab}) \to \mathbf{sAb}$, which we define level-wise:*

- *On objects $C_* \in Ch^+(\mathbf{Ab})$,*

$$K_n(C_*) = \widetilde{K}(C_*)[n].$$

- *On morphisms $u_* : C_* \to D_*$, we define*

$$K_n(u) = (u_0, \underbrace{u_1, ..., u_1}_{n \text{ times}}, \underbrace{u_2, ..., u_2}_{\binom{n}{2} \text{ times}}, ..., \underbrace{u_{n-1}, ..., u_{n-1}}_{\binom{n}{n-1} \text{ times}}, u_n).$$

Proof. By lemma 5.17, for any non-negatively graded chain complex C_*, $K(C_*)$ is a simplicial abelian group. To show that this is a well-defined map, we must also show that for any chain map u_*, that $K(u_*)$ is a map of simplicial abelian groups i.e a natural transformation. To show this, we must show that given a chain map $u_* : C_* \to D_*$ and any morphism $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$, the diagram:

$$\begin{tikzcd}
K_n(C_*) \arrow[r, "K_n(u_*)"] \arrow[d, "K(C_*)(\alpha)"] & K_n(D) \arrow[d, "K(D_*)(\alpha)"] \\
K_m(C) \arrow[r, "K_m(u_*)"] & K_m(D),
\end{tikzcd}$$

commutes. This follows from the fact that u_* commutes with the maps d, the zero map and the identity map which make up $K(C_*)(\alpha)$. Hence K is well-defined.

Now, to prove functoriality, let C_*, D_*, E_* be non-negatively graded chain complexes, and let $u_* : C_* \to D_*$, $v_* : D_* \to E_*$ be chain maps. We show (F1) $K(v_*u_*) = K(v_*)K(u_*)$ and (F2) $K(\mathbb{1}_{C_*}) = \mathbb{1}_{K(C_*)}$ and are satisfied.

(F1) By definition:

$$\begin{aligned}
K_n(v_*u_*) &= (v_0 \circ u_0, \underbrace{v_1 \circ u_1, ..., v_1 \circ u_1}_{n \text{ times}}, ..., \underbrace{v_{n-1} \circ u_{n-1}, ..., v_{n-1} \circ u_{n-1}}_{\binom{n}{n-1} \text{ times}}, v_n \circ u_n) \\
&= (v_0, \underbrace{v_1, ..., v_1}_{n \text{ times}}, ..., \underbrace{v_{n-1}, ..., v_{n-1}}_{\binom{n}{n-1} \text{ times}}, v_n) \circ (u_0, \underbrace{u_1, ..., u_1}_{n \text{ times}}, ..., \underbrace{u_{n-1}, ..., u_{n-1}}_{\binom{n}{n-1} \text{ times}}, u_n) \\
&= K(v_*)K(u_*).
\end{aligned}$$

(F2) We note that $\mathbb{1}_{\mathbb{C}_*} = (\mathbb{1}_{C_0}, \mathbb{1}_{C_1}, \mathbb{1}_{C_3}, ...)$ and hence for each n,

$$K(\mathbb{1}_{C_n}) = (\mathbb{1}_{C_0}, \underbrace{\mathbb{1}_{C_1}, ..., \mathbb{1}_{C_1}}_{n \text{ times}}, ..., \underbrace{\mathbb{1}_{C_{n-1}}, ..., \mathbb{1}_{C_{n-1}}}_{\binom{n}{n-1} \text{ times}}, \mathbb{1}_{C_n}) = \mathbb{1}_{K(C_n)},$$

by corollary 5.14. Therefore, $K(\mathbb{1}_{C_*}) = \mathbb{1}_{K(C_*)}$, as required.

Therefore we have shown that K is a functor. □

This proves proposition 5.11. We can also reformulate this in order to understand the face and degeneracy maps.

Corollary 5.19. *$K(C_*)$ is a simplicial abelian group where the face maps are given by $\partial_i = K(C_*)(\epsilon_i)$ and the degeneracy maps are given by $\sigma_i = K(C_*)(\eta_i)$.*

5.3. **The Equivalence.** To finish off the proof of the Dold-Kan correspondence, we must prove that NK is naturally isomorphic to $\mathbb{1}_{Ch^+(\mathbf{Ab})}$ and KN is naturally isomorphic to $\mathbb{1}_{\mathbf{sAb}}$.

5.3.1. *NK is naturally isomorphic to $\mathbb{1}_{Ch^+(\mathbf{Ab})}$.* We prove a couple of lemmas which make this result a straightforward corollary of theorem 5.10.

Lemma 5.20. *For $0 \le p \le n-1$, for any $\eta : [n] \twoheadrightarrow [p]$, we have*

$$N_n\left(C_p[\eta]\right) = 0 \text{ in } N_n(KC).$$

Proof. Let $p \neq n$. For arbitrary $\eta : [n] \twoheadrightarrow [p]$, we can find a factorisation of η into degeneracy maps: $\eta = \eta_{i_1}\eta_{i_2}...\eta_{i_t}$. The epi-monic factorisation of $\eta\mathbb{1}_{[p]}$ is $\mathbb{1}_{[n]}\eta$, and so $K(C)(\eta)|_{\mathbb{1}_{[p]}}$ is the natural association of $C_p\left[\mathbb{1}_{[p]}\right]$ with $C_p[\eta]$, by definition of K. But also, by the contravariance of $K(C)$:

$$\begin{aligned}K(C)(\eta) &= K(C)(\eta_{i_1}\eta_{i_2}...\eta_{i_t}) \\ &= K(C)(\eta_{i_t})K(C)(\eta_{i_{t-1}})...K(C)(\eta_{i_1}) \\ &= \sigma_{i_t}\sigma_{i_{t-1}}...\sigma_{i_2}\sigma_{i_1}.\end{aligned}$$

Therefore, we have $C_p[\eta] = \sigma_{i_t}\sigma_{i_{t-1}}...\sigma_{i_2}\sigma_{i_1}C_p\left[\mathbb{1}_{[p]}\right]$, and so $C_p[\eta] \subseteq D_n(K(C_*))$. By lemma 5.8, $N_n(A) \cap D_n(A) = \{0\}$, and so $N_n(C_p[\eta]) = 0$, as required. □

The remaining case to look at is when $p = n$. In this case, the only possible order-preserving surjection $[n] \twoheadrightarrow [n]$ is the identity map, $\mathbb{1}_{[n]}$.

Lemma 5.21.

$$N_n\left(C_n\left[\mathbb{1}_{[n]}\right]\right) = C_n\left[\mathbb{1}_{[n]}\right] \text{ in } N_n(KC).$$

Proof. Consider $\partial_i = K(C)(\epsilon_i)$. By the way we have defined this, $\partial_i = 0$ unless $i = n$. Therefore, $C_n\left[\mathbb{1}_{[n]}\right] \subseteq N_n(K(C_*))$. It is clear that $N_n(K(C_*)) \subseteq C_n\left[\mathbb{1}_{[n]}\right]$, by definition of N and K. □

Proposition 5.22. *Let C_* be a chain complex. Then $NK(C_*)$ is isomorphic as a chain complex to C_*.*

Proof. By lemma 5.21 and corollary 5.20, $N_n(KC) = C_n\left[\mathbb{1}_{[n]}\right]$, so $N_n(K(C))$ is a non-negatively graded sequence of abelian groups. Its differential is given in degree n by $\tilde{d_n} = (-1)^n\partial_n = (-1)^n K(C)(\epsilon_n)$. As $N_n(KC) = C_n\left[\mathbb{1}_{[n]}\right]$, we can look at this restricted to $\mathbb{1}_{[n]}$. We obtain the epi-monic factorisation $\epsilon_n\mathbb{1}_{[n]} = \mathbb{1}_{[n-1]}\epsilon_n$ as displayed below:

$$\begin{tikzcd} {[n-1]} \arrow[r, tail, "\epsilon_n"] \arrow[d, two heads, "\mathbb{1}_{[n-1]}"] & {[n]} \arrow[d, two heads, "\mathbb{1}_{[n]}"] \\ {[n-1]} \arrow[r, tail, "\epsilon_n"] & {[n]}. \end{tikzcd}$$

Hence $K(C)(\epsilon_n) = d_n : C_n \to C_{n-1}$. Therefore, $\tilde{d_n} = (-1)^n d_n$. Clearly $\tilde{d}^2 = 0$, so $NK(C_*)$ is a chain complex. This chain complex is isomorphic to C_* by the chain map $f_* : C_* \to NK(C_*)$ given by

$$f_n = \begin{cases} \mathbb{1}_{C_n} & \text{if } n \text{ is congruent to 0 or 3 mod 4,} \\ -\mathbb{1}_{C_n} & \text{otherwise.} \end{cases}$$

This is a well-defined chain map as it is clear that the diagram:

$$\begin{tikzcd} \dots \arrow[r, "d_{4(k+1)}"] & C_{4k+3} \arrow[r, "d_{4k+3}"] \arrow[d, "\mathbb{1}_{C_{4k+3}}"] & C_{4k+2} \arrow[r, "d_{4k+2}"] \arrow[d, "-\mathbb{1}_{C_{4k+2}}"] & C_{4k+1} \arrow[r, "d_{4k+1}"] \arrow[d, "-\mathbb{1}_{C_{4k+1}}"] & C_{4k} \arrow[r, "d_{4k}"] \arrow[d, "\mathbb{1}_{C_{4k}}"] & \dots \\ \dots \arrow[r, "d_{4(k+1)}"] & C_{4k+3} \arrow[r, "-d_{4k+3}"] & C_{4k+2} \arrow[r, "d_{4k+2}"] & C_{4k+1} \arrow[r, "-d_{4k+1}"] & C_{4k} \arrow[r, "d_{4k}"] & \dots \end{tikzcd}$$

commutes. Moreover, this map is self inverse as $f_n^2 = \mathbb{1}_{C_n}$ for all n, so is therefore an isomorphism between the chain complexes. Hence, $NK(C_*) \cong C_*$ as required. We call this isomorphism Φ^C. □

Lemma 5.23. *For any chain map u_*, $NK(u_*) = u_*$.*

Proof. Let u_* be a chain map. Then, by definition, for each n we have

$$K_n(u_*) = (u_0, \underbrace{u_1, ..., u_1}_{n \text{ times}}, \underbrace{u_2, ..., u_2}_{\binom{n}{2} \text{ times}}, ..., \underbrace{u_{n-1}, ..., u_{n-1}}_{\binom{n}{n-1} \text{ times}}, u_n).$$

By proposition 5.22, $NK_n(u_*) := K(u_*)|_{N_n(K(C))} = K(u_*)|_{C_n} = u_n$. It follows that $NK(u_*) = u_*$. □

Corollary 5.24. *NK is naturally isomorphic to $\mathbb{1}_{Ch^+(\mathbf{Ab})}$.*

Proof. We construct the natural isomorphism Φ object-wise by the isomorphism given by proposition 5.22. Given $u_* : C_* \to D_*$ the diagram

$$\begin{tikzcd} NK(C) \arrow[r, "\sim", "\Phi^C"'] \arrow[d, "NK(u_*)"] & \mathbb{1}_{Ch^+(\mathbf{Ab})}(C) \arrow[d, "\mathbb{1}_{Ch^+(\mathbf{Ab})}(u_*)"] \\ NK(D) \arrow[r, "\sim", "\Phi^D"'] & \mathbb{1}_{Ch^+(\mathbf{Ab})}(D), \end{tikzcd}$$

commutes by lemma 5.23. Hence by definition of natural isomorphism, NK is naturally isomorphic to $\mathbb{1}_{Ch^+(\mathbf{Ab})}$.

□

5.3.2. *KN is naturally isomorphic to $\mathbb{1}_{\mathbf{sAb}}$.* This proof is inspired by the proofs given in ([GJ99], III prop. 2.2) and in ([Wei95], 8.4.4).

Proposition 5.25. *Let A be a simplicial abelian group. Then $KN(A)$ is isomorphic to A as a simplicial abelian group.*

Proof. We define a map by $\Psi^A : KN(A) \to A$ by defining restrictions of it at the nth level corresponding to surjections $\eta : [n] \twoheadrightarrow [p]$. We define $\Psi^A_n|_\eta$ by the composite:

$$\begin{tikzcd} N_p(A) \arrow[r, hook] \arrow[rr, bend left, "\Psi^A_n|_\eta"] & A_p \arrow[r, "A(\eta)"] & A_n. \end{tikzcd}$$

First we show that Ψ^A is a simplicial map, which is a natural transformation. In $\mathbf{\Delta}$, let $\alpha : [m] \to [n]$, $\eta : [n] \twoheadrightarrow [p]$, and let $\epsilon\eta'$ be the epi-mono factorisation of $\eta\alpha$, so $\eta' : [m] \twoheadrightarrow [q]$ and $\epsilon : [q] \rightarrowtail [p]$ such that:

$$\begin{tikzcd} {[m]} \arrow[r, "\alpha"] \arrow[d, two heads, "\eta'"] & {[n]} \arrow[d, two heads, "\eta"] \\ {[q]} \arrow[r, tail, "\epsilon"] & {[p]} \end{tikzcd}$$

commutes. By applying the contravariant functor A to this which reverses arrows, we get the following diagram, which commutes by lemma 2.41.

(5)
$$\begin{tikzcd} A_m & A_n \arrow[l, "A(\alpha)"'] \\ A_q \arrow[u, "A(\eta')"] & A_p \arrow[l, "A(\epsilon)"'] \arrow[u, "A(\eta)"'] \end{tikzcd}$$

Furthermore, the diagram:

$$
\begin{tikzcd}
N_p(A) \arrow[r, hook, "i"] \arrow[d, "N_p(A(\epsilon))"] & A_p \arrow[d, "A(\epsilon)"] \\
N_q(A) \arrow[r, hook, "i"] & A_q
\end{tikzcd}
\tag{6}
$$

commutes due to the fact that $A(\epsilon)|_{N_p(A)} = N_p(\epsilon)$ by definition.

Therefore, the diagram:

$$
\begin{tikzcd}
N_p(A) \arrow[r, hook, "i"] \arrow[d, "N_p(\epsilon)"] \arrow[rr, bend left, "\Psi_A|_\eta"] & A_p \arrow[r, "A(\eta)"] \arrow[d, "A(\epsilon)"] & A_n \arrow[d, "A(\alpha)"] \\
N_q(A) \arrow[r, hook, "i"] \arrow[rr, bend right, "\Psi_A|_{\eta'}"'] & A_q \arrow[r, "A(\eta')"] & A_m
\end{tikzcd}
$$

commutes by gluing together the commutative squares given in (5) (rotated) and (6). This is enough to show that Ψ^A is a simplicial map from $KN(A)$ to A, by the fact that maps out of a direct sum are defined by maps on each part.

Next, we prove that Ψ_n^A is an isomorphism of abelian groups for all n. We do this by induction. For our base case, we note that for $n = 0$, the only possible map we can consider is $\mathbb{1}_{[0]} : [0] \twoheadrightarrow [0]$. Moreover, $N_0(A) := A_0$. Hence $\Psi_0^A(N(A)) = A_0$, so we have our required isomorphism.

Now assume that Ψ_n^A is an isomorphism at every degree less than n. By restricting Ψ_n^A to $\mathbb{1}_{[n]}$, which is the map:

$$
\begin{tikzcd}
N_n(A) \arrow[r, hook] \arrow[rr, bend left, "\Psi_n^A|_{\mathbb{1}_{[n]}}"] & A_n \arrow[r, equal, "A(\mathbb{1}_{[n]})"] & A_n.
\end{tikzcd}
$$

we see that $N_n(A) \subseteq Im(\Psi_n^A)$.

Consider an arbitrary degeneracy $\sigma_j x \in D_n(A)$. Then $x \in A_{n-1}$. Let $y = \left(\Psi_{n-1}^A\right)^{-1}(x)$, which exists by inductive hypothesis. As Ψ is a simplicial map, it commutes with face and degeneracy maps of A, so we have:

$$\sigma_j x = \sigma_j \Psi_{n-1}^A(y) = \Psi_n^A(\sigma_j y),$$

and so $\sigma_j x \in Im(\Psi_n^A)$. As this was an arbitrary element of $D_n(A)$, it follows that $D_n(A) \subseteq Im(\Psi_n^A)$. Hence $N_n(A) \bigoplus D_n(A) \subseteq Im(\Psi_n^A)$. By theorem 5.10, it follows that $A_n \subseteq Im(\Psi_n^A)$. Therefore, Ψ_n^A is surjective.

It remains to show that Ψ_n^A is injective, as this will show that it is an isomorphism by lemma 2.25. We write elements of $KN_n(A)$ as (x_η), in which each part x_{η_0} is labelled by some surjection $\eta : [n] \twoheadrightarrow [p]$ with $0 \le p \le n$. We show that $\Psi_n^A|_\eta(x_\eta) = 0 \iff x_\eta = 0$. This will show $\ker\left(\Psi_n^A\right) = 0$ and that Ψ_n^A is injective.

Fix an $\eta : [n] \twoheadrightarrow [p]$ and suppose that $\Psi_n^A|_\eta(x_\eta) = 0$. Recall from lemma 2.20 that there exists a section $\mu : [p] \to [n]$ such that $\eta\mu = \mathbb{1}_{[p]}$. Then we have:

$$
\begin{aligned}
0 &= \Psi_n^A|_\eta(x_\eta) \\
&= A(\eta)(x_\eta) \\
&= A(\mu)A(\eta)(x_\eta) && \text{applying } A(\mu) \text{ to both sides,} \\
&= A(\eta\mu)(x_\eta) && \text{by (CF1),} \\
&= A\left(\mathbb{1}_{[p]}\right)(x_\eta) && \text{as } \mu \text{ was the section of } \eta, \\
&= x_\eta && \text{by (CF2).}
\end{aligned}
$$

Therefore $\Psi_n^A|_\eta(x_\eta) = 0 \iff x_\eta = 0$ and so Ψ_n^A is injective. Hence Ψ_n^A is an isomorphism, and by induction we have proved that it is an isomorphism for every $n \geq 0$. Hence Ψ^A is a natural isomorphism, so $KN(A)$ is isomorphic to A as a simplicial abelian group. $\square$

Corollary 5.26. *KN is naturally isomorphic to $\mathbb{1}_{\mathbf{sAb}}$.*

Proof. We provide a natural transformation $\Psi : KN \Rightarrow \mathbb{1}_{\mathbf{sAb}}$ defined object-wise by $\Psi^A : KN(A) \to A$, as given in proposition 5.25. The diagram:

$$
\begin{tikzcd}
KN(A) \arrow[r, "\sim", "\Psi(A)"'] \arrow[d, "KN(f)"] & \mathbb{1}_{\mathbf{sAb}}(A) \arrow[d, "\mathbb{1}_{\mathbf{sAb}}(f)"] \\
KN(B) \arrow[r, "\sim", "\Psi(B)"'] & \mathbb{1}_{\mathbf{sAb}}(B),
\end{tikzcd}
$$

commutes by the definition of natural transformation applied to Ψ^A and Ψ^B. Hence Ψ is itself is a natural transformation, in which each component is an isomorphism, and is therefore a natural isomorphism. $\square$

We have therefore proven the Dold-Kan Correspondence.

5.4. **Examples.** Explicit examples of the Dold-Kan correspondence in action quickly become very technical and mostly an exercise in tracking combinatorial information. We give here some simple examples.

Example 5.27. Recall $\mathbb{Z}\Delta^1$, the simplicial abelian group discussed in example 3.28. For an element $a\{0,1\} + b\{0,0\} + c\{1,1\} \in \mathbb{Z}\Delta^1_1$, we have

$$\partial_0(a\{0,1\} + b\{0,0\} + z\{1,1\}) = a\partial_0\{0,1\} + b\partial_0\{0,0\} + c\partial_0\{1,1\} = a\{1\} + b\{0\} + c\{1\},$$

and similarly

$$\partial_1(a\{0,1\} + b\{0,0\} + c\{1,1\}) = a\{0\} + b\{0\} + c\{1\}.$$

It follows that, $N\mathbb{Z}\Delta^1_0 = \mathbb{Z}\Delta^1_0$ and $N\mathbb{Z}\Delta^1_1 = \ker \partial_0 = \{a\{0,1\} - a\{1,1\} : a \in \mathbb{Z}\}$. From the definition, $KN\mathbb{Z}\Delta^1_0 = \mathbb{Z}\Delta^1_0$. Moreover,

$$
\begin{aligned}
KN\mathbb{Z}\Delta^1_1 &= \bigoplus_{p=0}^{1} \bigoplus_{\eta:[1]\twoheadrightarrow[p]} N\mathbb{Z}\Delta^1_p[\eta] \\
&= N\mathbb{Z}\Delta^1_0[\,\diagup\,] \bigoplus N\mathbb{Z}\Delta^1_1[\,\vert\,\vert\,],
\end{aligned}
$$

as was worked out in example 5.12. Elements in here are of the form

$$(x\{0\} + y\{1\}, a\{0,1\} - a\{1,1\}).$$

for $x, y, a \in \mathbb{Z}$. We show that $\Psi_1^{\mathbb{Z}\Delta^1}$ sends this to an element of $\mathbb{Z}\Delta^1_1$. We calculate the restrictions to the maps $\diagup$ and $\vert\,\vert$:

$$\begin{tikzcd} N\mathbb{Z}\Delta^1_0 \arrow[r, hook] \arrow[rr, bend left, "\Psi_1^{\mathbb{Z}\Delta^1}|"] & \mathbb{Z}\Delta^1_0 \arrow[r] & \mathbb{Z}\Delta^1_1, \end{tikzcd}$$

where the map $\mathbb{Z}\Delta^1_0 \to \mathbb{Z}\Delta^1_1$ is σ_0 as there is no other choice. Also, we have:

$$\begin{tikzcd} N\mathbb{Z}\Delta^1_1 \arrow[r, hook] \arrow[rr, bend left, "\Psi_1^{\mathbb{Z}\Delta^1}|"] & \mathbb{Z}\Delta^1_1 \arrow[r, equal] & \mathbb{Z}\Delta^1_1. \end{tikzcd}$$

Hence

$$\begin{aligned} \Psi_1^{\mathbb{Z}\Delta^1}(x\{0\} + y\{1\}, a\{0,1\} - a\{1,1\}), &= \sigma_0(x\{0\} + y\{1\}) + a\{0,1\} - a\{1,1\}, \\ &= x\{0,0\} + y\{0,1\} + a\{0,1\} - a\{1,1\}, \\ &= p\{0,1\} + q\{0,0\} + r\{1,1\} \in \mathbb{Z}\Delta^1_1, \end{aligned}$$

for $p = a + y$, $q = x$ and $r = -a$. It is clear that there is a bijection between these elements, uniquely determined by choosing 3 parameters. Hence we can see the natural isomorphism of these sets.

We also have an example in the other direction.

Example 5.28. Recall example 4.14, which shows the non-negatively graded chain complex C_* given by

$$\ldots \xrightarrow{\;4\;} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\;4\;} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\;4\;} \mathbb{Z}/8\mathbb{Z} \longrightarrow 0$$

Now, $KC_0 = \mathbb{Z}/8\mathbb{Z}$, $KC_1 = C_0[\;] \bigoplus C_1[\;] = \mathbb{Z}/8\mathbb{Z}[\;] \bigoplus \mathbb{Z}/8\mathbb{Z}[\;]$,

$$K_2(C) = \mathbb{Z}/8\mathbb{Z}[\;] \bigoplus \mathbb{Z}/8\mathbb{Z}[\;] \bigoplus \mathbb{Z}/8\mathbb{Z}[\;] \bigoplus \mathbb{Z}/8\mathbb{Z}[\;],$$

and so on. As explained in proposition 5.22, applying N to these picks out the top bit of these. Hence

$$NKC_1 = \mathbb{Z}/8\mathbb{Z}[\;] \;\cong\; \mathbb{Z}/8\mathbb{Z},$$

$$NKC_2 = \mathbb{Z}/8\mathbb{Z}[\;] \cong \mathbb{Z}/8\mathbb{Z}.$$

This is therefore the non-negatively graded chain complex

$$\ldots \xrightarrow{\;-4\;} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\;4\;} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\;-4\;} \mathbb{Z}/8\mathbb{Z} \longrightarrow 0$$

which is isomorphic as a chain complex to C_*.

5.5. **The relationship between homology and homotopy.** One important result arising from the Dold-Kan correspondence is the following.

Proposition 5.29. *Let A be a simplicial abelian group. Under the Dold-Kan correspondence, $\pi_*(A) \cong H_*(NA)$.*

Proof. Recall that for a simplicial abelian group A, $\widetilde{Z}_n(A) = \bigcap_{i=0}^n \ker(\partial_i : A_n \to A_{n-1})$, and for a chain complex C, $Z_n(C) = \ker(d_n : C_n \to C_{n-1})$. It is clear from these definitions that $Z_n(NA) = \ker(d_n : NA_n \to NA_{n-1}) = \ker((-1)^n \partial_n : NA_n \to NA_{n-1})$, and so $x \in Z_n(NA) \iff x \in \bigcap_{i=0}^n \ker(\partial_i : A_n \to A_{n-1}) = \widetilde{Z}_n(A)$. Hence, $Z_n(A) = \widetilde{Z}_n(A)$

Moreover, $B_n(NA) = \mathrm{img}(d_{n+1} : N_{n+1}A \to N_nA)$, so $x \in B_n(NA)$ iff $x = d_{n+1}(y)$ for some $y \in N_{n+1}A$. This is the same as saying $x = (-1)^{n+1}\partial_{n+1}y$, and so

$$\partial_i(y) = \begin{cases} 0 & \text{if } i < n \\ 0 & \text{if } i = n \\ (-1)^{n+1}x & \text{if } i = n+1. \end{cases}$$

Therefore, we can see that $x \in B_n(NA) \iff x \sim 0$.

Now, consider the quotient map

$$\xi : \widetilde{Z}_n(A) \to \widetilde{Z}_n(A)/\sim.$$

This map is clearly surjective, as for any class $[z] \in \widetilde{Z}_n(A)/\sim$, we have $\xi(z) = [z]$. Hence, by the first isomorphism theorem we have

$$\widetilde{Z}_n(A)/\ker \xi \cong \widetilde{Z}_n(A)/\sim.$$

Now, $\ker \xi = \{x \in \widetilde{Z}_n(A) : x \sim 0\} = B_n(NA)$. Hence,

$$\pi_n(A) := \widetilde{Z}_n(A)/\sim \cong \widetilde{Z}_n(A)/B_n(NA) = Z_n(NA)/B_n(NA) =: H_n(NA).$$

□

Example 5.30. Let G be an abelian group. Consider the chain complex $G[-k]$ with G at level k, and 0 everywhere else.

$$\ldots \longrightarrow 0 \longrightarrow G \longrightarrow 0 \longrightarrow \ldots$$

Then we call the simplicial abelian group $K(G[-n])$ an *Eilenberg-MacLane space* of type $K(G, n)$. This has the special property that

$$\pi_k\left(K\left(G[-n]\right)\right) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

We calculate this through the homology. Firstly, $N_k\left(K\left(G[-n]\right)\right) = G[-n]$ as every element is in the kernel of each face map (which are all necessarily zero maps). From this, it is clear that $Z_n\left(NKG[-n]\right) = G$ and $B_n\left(NKG[-n]\right) = \{0\}$, so $H_n\left(NKG[-n]\right) = G$. By proposition 5.29, this shows the result for homotopy groups.

Eilenberg-MacLane spaces are important in topology; homology, cohomology and homotopy of spaces can be captured with maps to and from an Eilenberg-MacLane space.

6. A Quillen Equivalence

The notion of an equivalence of categories is somewhat weaker than the notion of an isomorphism of categories: indeed, for an equivalence of categories we only need there to exist functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, such that FG (respectively GF) is naturally isomorphic to the identity on $\mathcal{D}$ (resp. $\mathcal{C}$), whereas an isomorphism of categories requires FG (respectively GF) to be *equal* to the identity on $\mathcal{D}$ (resp. $\mathcal{C}$). With our functors, we clearly do not have an equality, and so we might guess that asking for these categories to be isomorphic is asking for too much. However, there are other, more structured notions of equivalence between categories. One kind of equivalence we might be interested in is one that says two categories have the same homotopical information in some way. One such equivalence is called a *Quillen equivalence*; this is a relationship between two categories $\mathcal{C}$ and $\mathcal{D}$ that induces an equivalence of categories between associated categories called *the homotopy categories* $\mathbf{Ho}(\mathcal{C})$ and $\mathbf{Ho}(\mathcal{D})$. A Quillen equivalence allows us to study the homotopy theory of $\mathcal{C}$ through the homotopy theory of $\mathcal{D}$ and vice versa. This concept was first introduced in 1967 by D. Quillen [Qui67].

In fact, the Dold-Kan correspondence can be upgraded to a Quillen equivalence. This was first proven in [Qui67] and is the content of this chapter. In order to prove this, and indeed for the statement above to have any meaning, we need to put a *model structure* upon these categories. This gives the way in which this equivalence is 'more structured'. We then need to show that our functors K and N preserve this extra structure.

6.1. **Model Categories.** A model category is a context in which we can do abstract homotopy theory in; in fact it was developed by Quillen as an axiomatic framework for homotopy theory [Qui67]. One idea that turns out to be useful to isolate is maps which have *lifting properties* against other maps; this is inspired from homotopy theory in the topological context [Str21, Hat01].

Definition 6.1. Let $i : A \to B$ and $p : X \to Y$ in a category $\mathcal{C}$. The map i is said to have the *left lifting property* against p if for any maps $f : A \to X$ and $g : B \to Y$ such that $g \circ i = p \circ f$, there exists an $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$. In other words, there exists an h making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

Equivalently, p has the *right lifting property* against i.

Definition 6.2. Let X and Y be simplicial sets. A simplicial map $p : X \to Y$ is called a *Kan fibration* if it has the right lifting property against the inclusion of all horns $\Lambda^n_k \hookrightarrow \Delta^n$. More precisely, for any $n \geq 1$ and $0 \leq k \leq n$, and for any maps $f : \Lambda^n_k \to X$ and $g : \Delta^n \to Y$ such that $p \circ f = g \circ i$, there exists a map $h : \Delta^n \to X$ making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

Remark 6.3. Note the condition $n \geq 1$ in this definition. For $n = 0$, we have $\Lambda^0_0 = \Delta^0$, and so this condition would hold for every map.

Example 6.4. Recall the definition of Kan complexes from definition 3.34. This can now be rephrased by saying that X is a Kan complex if and only if the map $X \to \{*\}$ is a Kan fibration. We can see this by comparing the commutative diagrams below, and by noting that any map to the one-point simplicial set must be the zero map.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "0"] \\ \Delta^n \arrow[r, "0"'] \arrow[ur, dashed, "\exists h"] & \{*\} \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Definition 6.5. Let $\mathcal{C}$ be a category. A *weak factorisation system* on $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that:

(WF1) Every morphism f in $\mathrm{mor}(\mathcal{C})$ can be factorised as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$.
(WF2) $\mathcal{L}$ is the class of morphisms in $\mathcal{C}$ that have the left lifting property against every morphism in $\mathcal{R}$.

(WF3) $\mathcal{R}$ is the class of morphisms in $\mathcal{C}$ that have the right lifting property against every morphism in $\mathcal{L}$.

Definition 6.6. Let $\mathcal{C}$ be a category. A *model structure* on $\mathcal{C}$ is a choice of three distinguished classes of morphisms in $\mathcal{C}$:

- A class called *Cofibrations* $\mathrm{Cof} \subset \mathrm{mor}(\mathcal{C})$.
- A class called *fibrations* $\mathrm{Fib} \subset \mathrm{mor}(\mathcal{C})$.
- A class called *weak equivalences* $\mathcal{W} \subset \mathrm{mor}(\mathcal{C})$.

These must satisfy the axioms:

(M1) Any isomorphism f in $\mathrm{mor}(\mathcal{C})$ is contained in $\mathcal{W}$.
(M2) The *2-out-of-3 property*: given a composable pair of morphisms $f : X \to Y$, $g : Y \to Z$, if any two of of f, g and gf are in $\mathcal{W}$, then so is the third.
(M3) Both $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are weak factorisation systems on $\mathcal{C}$.

The morphisms in $\mathrm{Fib} \cap \mathcal{W}$ are called *trivial fibrations*. Similarly, the morphisms in $\mathrm{Cof} \cap \mathcal{W}$ are called *trivial cofibrations*.

This definition differs from the definition given in [Hov99] and has less conditions; it has been shown since, for example in [Rie09], that these conditions given here imply the extra ones in [Hov99]. As we will not need them for the theory we develop, we shall leave these out. We can make one more simplification:

Lemma 6.7. *Let $\mathcal{C}$ be a category with a model structure. The class of weak equivalences and fibrations fully determines the class of cofibrations.*

Proof. Given $\mathcal{W}$ and Fib, we can define a class of morphisms to be the morphisms that have the left lifting property against every trivial fibration. By (M3), these are precisely the cofibrations. □

Definition 6.8. A *model category* is a category $\mathcal{M}$ which is complete and cocomplete equipped with a model structure.

Example 6.9. Let $\mathcal{C}$ be any complete and cocomplete category. There is always a model structure called the *trivial model structure on $\mathcal{C}$*, in which we let the weak equivalences be just the isomorphisms, and let the cofibrations and fibrations be all the morphisms in the category. In this case, (M1) is clearly satisfied, (M2) follows from some basic properties of isomorphisms. For (M3) we show that (isomorphisms, maps) is a weak factorisation system on $\mathcal{C}$. Let $p : X \to Y$ be a morphism in $\mathcal{C}$. Then $p = \mathbb{1}_X p$, so (WF1) is satisfied. Moreover, for diagrams of the form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "i^{-1}f"] & Y, \end{tikzcd}$$

the map $i^{-1}f : B \to X$ exists, giving the left and right lifting properties required in (WF2) and (WF3). The proof that (maps, isomorphisms) is a weak factorisation system is similar. We can always consider $\mathcal{C}$ a model category with this model structure.

Definition 6.10. Let $\mathcal{M}$ be a model category, and denote the initial object by $\mathbf{0}$ and the terminal object by $\mathbf{1}$. An object $X \in \mathcal{M}$ is said to be a *fibrant* if the unique map $X \to \mathbf{1}$ is a fibration. Similarly, we say that X is *cofibrant* if the unique map $\mathbf{0} \to X$ is a cofibration.

We can define the notion of homotopy in a model category as follows. There are many equivalent formulations of this construction, for example in [Hov99, Rie14]; the following is an adaptation of [ZG67].

Definition 6.11. A finite *zig-zag* in a category $\mathcal{C}$ is a string of morphisms in $\mathcal{C}$ of the form

$$C_0 \longleftarrow C_1 \longrightarrow C_2 \longleftarrow \dots \longrightarrow C_{n-1} \longleftarrow C_n$$

The idea is that we would like to invert weak equivalences and make them isomorphisms. We do this by imposing an equivalence relation on finite zig-zags.

Definition 6.12. Let $\mathcal{M}$ be a model category, and consider the collection of finite zig-zags in $\mathcal{M}$ in which the backwards morphisms must be from $\mathcal{W}$. We define the following relations.

- Adjacent morphisms in the same direction may be composed.
- Adjacent pairs $M_{k-1} \xleftarrow{w} M_k \xrightarrow{v} M_{k+1}$ and $M_{k-1} \xrightarrow{w} M_k \xleftarrow{v} M_{k+1}$ for $w, v \in \mathcal{W}$ may be removed.
- Identities pointing forwards or backwards can be removed.

This forms an equivalence relation. Morphisms which are connected by a finite zig-zag are called homotopical.

Definition 6.13. Let $\mathcal{M}$ be a model category. We define $\mathrm{Ho}(\mathcal{M})$ for the category whose objects are the objects in $\mathcal{M}$ which are both fibrant and cofibrant, and whose morphisms are classes of maps which are homotopical to one another. By [Hov99], this is a well-defined category.

Example 6.14. If a complete and cocomplete category $\mathcal{C}$ is equipped with the trivial model structure, then we have $\mathrm{Ho}(\mathcal{C}) \cong \mathcal{C}$, as all weak equivalences are already invertible, and all objects in $\mathcal{C}$ are both fibrant and cofibrant (since all morphisms are both fibrations and cofibrations).

Example 6.15. When we work out the details in the category **Top**, this recovers our usual notion of the topological homotopy category which is given, for example, in [Str21]. This result is proven in [Hov99].

In order to extend the Dold-Kan correspondence to a Quillen equivalence, we must specify model structures on the categories $Ch^+(\mathbf{Ab})$ and **sAb**. The proofs that the following give model structure are long and require a lot of extra theory to do with model categories; in the interest of space constraints these proofs are left out of this project and are referenced instead.

6.1.1. *The Projective Model Structure on* $Ch^+(\mathbf{Ab})$. There are many model structures on $Ch^+(\mathbf{Ab})$; the most commonly used are the injective and projective model structures. We will use the latter to form our Quillen equivalence.

Proposition 6.16. *The following gives a model structure on* $Ch^+(\mathbf{Ab})$*: a chain map* $u_* : C_* \to C'_*$ *is:*

- *A weak equivalence if* u_* *induces an isomorphism in homology, i.e.*
$$H_k(u) : H_k(C_*) \to H_k(C'_*)$$
is an isomorphism of groups for every $k \geq 0$.
- *A fibration if* $u_k : C_k \to C'_k$ *is an epimorphism in each positive degree.*
- *A cofibration if* $u_k : C_k \to C'_k$ *is a monomorphism whose cokernel is a projective abelian group for all* k.

This is called the projective model structure *on* $Ch^+(\mathbf{Ab})$.

This was originally proven to be a model structure in [Qui67], but the result is proved using more modern machinery in ([GS07], theorem 1.5).

Remark 6.17. We note that we have not defined the notion of a monomorphism whose cokernel is a projective abelian group in this project. However, by lemma 6.7, we know that these are must be the morphisms with the left lifting property against the trivial fibrations— and indeed this is more or less the definition of them. In a longer project, it would be interesting to look at this further, but as we do not need this concept we reference ([Wei95], 2.2).

6.1.2. *The Classic Model Structure on* **sAb**. It has been shown that there are an infinite number of model structures on **sSet** [Bek10]. This is perhaps surprising when we consider that there are only nine model structures on **Set** [Bal21], and only one on **CAT** [SP12], which has functors giving an equivalence of categories as the weak equivalences. The model structure we will be interested in is called the *classical model structure on* **sAb**. This is proven to be a model structure in ([GJ99], Theorem III.2.8).

Proposition 6.18. *The following gives a model structure* **sAb***: a simplicial map $f : A \to B$ is:*

- *A weak equivalence if f induces an isomorphism in homotopy groups i.e.*
$$\pi_k(f) : \pi_k(A) \to \pi_k(B)$$
is an isomorphism for every $k \geq 0$.
- *A fibration if $f : A \to B$ is a Kan fibration when considered as a map between simplicial sets.*
- *A cofibration if $f : A \to B$ is a monomorphism of simplicial sets i.e. an injection in every degree.*

This is called the classical model structure *on* **sAb**.

6.2. **Adjoint Pairs.** In this section we focus on the notion of an *adjoint pair* of functors. This can be thought of as a weak form of an equivalence of categories; indeed, every equivalence of categories induces an adjoint pair but the converse is not true.

Definition 6.19. Let $\mathcal{C}, \mathcal{D}$ be categories. Suppose we have functors:
$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$
such that for all A in $\mathcal{C}$ and B in $\mathcal{D}$ we have a natural bijection of sets
$$\mathrm{Hom}_{\mathcal{D}}(LA, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, RB).$$
Then we say L is left adjoint to R (equivalently R is right adjoint to L). In this case, we say (L, R) is an *adjoint pair.*

Example 6.20. The naming of adjoint pairs was inspired by the concept of adjoints in an inner product space. Recall the definition: let A be a linear operator on an inner product space V with inner product $\langle -, - \rangle$. Then the *adjoint operator* A^* is the unique linear map such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$, for all $x, y \in V$. This definition bares resemblance to the definition of adjoint pair.

Example 6.21. Let $U : \mathbf{Ab} \to \mathbf{Set}$ be the forgetful functor, and $\mathbb{Z}$ be the free functor. Then
$$\mathbb{Z} : \mathbf{Set} \rightleftarrows \mathbf{Ab} : U$$
form an adjoint pair, $(\mathbb{Z}, U)$. To see this, first consider a function $U(A) \to X$. There is a unique way to linearly extend this to a abelian group homomorphism $A \to \mathbb{Z}\{X\}$. Conversely, given an abelian group homomorphism $A \to \mathbb{Z}\{X\}$, it is enough to specify what this map does to the generators of $\mathbb{Z}\{X\}$ which is precisely X. This gives a function $U(A) \to X$, which completes the bijection required.

Example 6.22. $(|-|, S_*)$ is an adjoint pair. A proof of this can be found in ([GJ99], proposition 2.2, page 7).

Proposition 6.23. (K, N) *is an adjoint pair, i.e.*

$$\mathrm{Hom}_{\mathbf{sAb}}(KC, A) \cong \mathrm{Hom}_{Ch^+(\mathbf{Ab})}(C, NA)$$

is a bijection of sets.

Proof. Recall that we have natural isomorphisms

$$\Psi^A : KNA \overset{\sim}{\rightleftarrows} A : \left(\Psi^A\right)^{-1}, \quad \Phi^C : NKC \overset{\sim}{\rightleftarrows} C : \left(\Phi^C\right)^{-1}$$

which we constructed in corollaries 5.24 and 5.26. We form maps

$$\Theta : \mathbf{sAb}(KC, A) \rightleftarrows Ch^+(\mathbf{Ab})(C, NA) : \Lambda,$$

and show that these are mutually inverse.

For $f : KC \to A$, we define $\Theta(f) : C \to NA$ by $\Theta(f) = Nf \circ \left(\Phi^C\right)^{-1}$:

$$C \xrightarrow{\left(\Phi^C\right)^{-1}} NKC \xrightarrow{Nf} NA.$$

For $g : C \to NA$, we define $\Lambda(g) : KC \to A$ by $\Lambda(g) = \Psi^A \circ Kg \circ K\Phi^C \circ \left(\Psi^{KC}\right)^{-1}$:

$$KC \xrightarrow{\left(\Psi^{KC}\right)^{-1}} KNKC \xrightarrow{K\Phi^C} KC \xrightarrow{Kg} KNA \xrightarrow{\Psi^A} A.$$

Now,

$$\begin{aligned}
(\Lambda \circ \Theta)(f) &= \Lambda\left(Nf \circ \left(\Phi^C\right)^{-1}\right) \\
&= \Psi^A \circ K\left(Nf \circ \left(\Phi^C\right)^{-1}\right) \circ K\Phi^C \circ \left(\Psi^{KC}\right)^{-1} \\
&= \Psi^A \circ KNf \circ K\left(\left(\Phi^C\right)^{-1}\right) \circ K\Phi^C \circ \left(\Psi^{KC}\right)^{-1} && \text{by functoriality,} \\
&= \Psi^A \circ KNf \circ \left(\Psi^{KC}\right)^{-1}.
\end{aligned}$$

We note that by naturality of Ψ, the diagram

$$\begin{array}{ccc}
KNKC & \xrightarrow{\Psi^{KC}} & KC \\
\Big\downarrow{\scriptstyle KNf} & & \Big\downarrow{\scriptstyle f} \\
KNA & \xrightarrow{\Psi^A} & A,
\end{array}$$

commutes, so $\Psi^A \circ KNf = f \circ \Psi^{KC}$. Hence

$$\begin{aligned}
(\Lambda \circ \Theta)(f) &= \Psi^A \circ KNf \circ \left(\Psi^{KC}\right)^{-1} \\
&= f \circ \Psi^{KC} \circ \left(\Psi^{KC}\right)^{-1} \\
&= f,
\end{aligned}$$

and so $\Lambda \circ \Theta = \mathbb{1}_{\mathbf{sAb}(KC,A)}$.

On the other hand,

$$\begin{aligned}(\Theta\circ\Lambda)(g) &= \Theta\left(\Psi^A\circ Kg\circ K\Phi^C\circ\left(\Psi^{KC}\right)^{-1}\right)\\ &= N\left(\Psi^A\circ Kg\circ K\Phi^C\circ\left(\Psi^{KC}\right)^{-1}\right)\circ\left(\Phi^C\right)^{-1}\\ &= N\left(\Psi^A\right)\circ NKg\circ NK\Phi^C\circ N\left(\left(\Psi^{KC}\right)^{-1}\right)\circ\left(\Phi^C\right)^{-1} \qquad \text{by functoriality.}\end{aligned}$$

Now, thinking level-wise with, we note that $N_n\left(\Psi^A\right) = \Psi^A|_{N_n(A)} = \mathbb{1}_{NKNA_n}$. Similarly, $N_n\left(\left(\Psi^{KC}\right)^{-1}\right) = \mathbb{1}_{N_n(A)}$. Therefore,

$$(\Theta\circ\Lambda)(g) = NKg\circ NK\Phi^C\circ\left(\Phi^C\right)^{-1}$$

Moreover, in lemma 5.23 we showed that $NKu = u$ for all chain maps u, so

$$\begin{aligned}(\Theta\circ\Lambda)(g) &= g\circ\Phi^C\circ\left(\Phi^C\right)^{-1}\\ &= g\end{aligned}$$

and so $\Theta\circ\Lambda = \mathbb{1}_{Ch^+(\mathbf{Ab})(C,NA)}$. Hence

$$\mathrm{Hom}_{\mathbf{sAb}}(KC, A) \cong \mathrm{Hom}_{Ch^+(\mathbf{Ab})}(C, NA),$$

and (K, N) is an adjoint pair. □

Remark 6.24. In this case, since we are working with an equivalence of categories, it is true that both (K, N) and (N, K) are adjoint pairs. We have chosen to do it this way as the calculations turn out to be simpler.

6.3. **Quillen Adjunctions.** We now look at why the notion of adjunction is important.

Definition 6.25. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and let (L, R) be a pair of adjoint functors.

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

(L, R) is said to be a *Quillen adjunction* if R preserves fibrations and trivial fibrations.

Remark 6.26. The usual definition of a Quillen adjunction, as in [Hov99] has a list of equivalent conditions to the one in this definition. Equivalently, (L, R) is a Quillen adjunction if:

- L preserves cofibrations and trivial cofibrations;
- L preserves cofibrations and R preserves fibrations;
- L preserves trivial cofibrations and R preserves trivial fibrations.

As these are equivalent, it suffices to prove just one of these. We have chosen to look at fibrations and trivial fibrations, largely because the functor N is easier to work with than the functor K.

Remark 6.27. Quillen adjunctions turn out to be a suitable notion of morphism between model categories. We would like to form a category of model categories; however, we cannot due to set theoretic issues [Hov99]. We instead form what is known as a 2-category[1] of model categories, which has the extra structure of natural transformations being maps between the Quillen adjunctions.

Given a Quillen adjunction $(L, R) : \mathcal{C} \to \mathcal{D}$, we can form associated functors in the homotopy category.

[1] A category with a class of objects, a class of morphisms and a class of morphisms between morphisms

Theorem 6.28. *Let $(L, R) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. Then there exists functors called the* total derived functors *which give an equivalence of categories between* $\mathrm{Ho}(\mathcal{C})$ *and* $\mathrm{Ho}(\mathcal{D})$.

Proof. For a proof and more details, the reader is referred to ([DS95], theorem 9.7). $\square$

6.4. **Quillen Equivalence.**

Definition 6.29. A Quillen adjunction $(L, R) : \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if for all cofibrant objects X in $\mathcal{C}$ and fibrant objects Y in $\mathcal{D}$, a map $f : LX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\phi(f) : X \to RY$ is a weak equivalence in $\mathcal{C}$, where ϕ is the isomorphism $\mathrm{Hom}_{\mathcal{C}}(LX, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, RY)$.

Example 6.30. The adjoint pair $(| - |, S_*)$ can be extended to a Quillen adjunction, and in fact gives a Quillen equivalence between **Top** and **sSet**. Therefore, there is an equivalence of categories between Ho(**Top**) and Ho(**sSet**). This is proven in ([Hov99], theorem 3.6.7 together with theorem 2.4.23), and gives a rigorous meaning to the statement that simplicial sets are useful to understand the homotopy of topological spaces.

Remark 6.31. Since (K, N) is an adjoint pair which gives an equivalence of categories, a Quillen equivalence between $Ch^+(\mathbf{Ab})$ and **sAb** is easy to prove: as soon as we have a model structure on one side, we could declare the model structure on the other side to be the images of fibrations and weak equivalences under N. This would necessarily provide a Quillen equivalence as we have an inverse functor, showing that these things must be preserved up to natural isomorphism. This is sometimes called the *transferred model structure*, and this is how the Quillen equivalence is proven in [Qui67] and [GJ99]. In light of this, it is not the fact that there is a Quillen equivalence between these two categories that is the particularly interesting bit; rather, it is the fact that the classical model structure on simplicial abelian groups and the projective model structure on non-negatively graded chain complexes correspond in this way; a Kan fibration in **sAb** is homotopically equivalent to the notion of surjection in every positive degree in $Ch^+(\mathbf{Ab})$.

6.5. **The Dold-Kan Quillen Equivalence.** In this section we prove the Dold-Kan correspondence extends to a Quillen equivalence. This was originally due to Quillen in [Qui67], without much proof. A more fleshed out proof appears in [GJ99]. To prove this, we prove that N preserves weak equivalences and fibrations; this is a slightly stricter condition that we require, but certainly does show that N preserves fibrations and trivial fibrations and therefore (K, N) is a Quillen adjunction.

Proposition 6.32. *Let $f : A \to B$ be a weak equivalence in* **sAb**, *i.e. the induced map $\pi_*(f) : \pi_*(A) \to \pi_*(B)$ is an isomorphism. Then $Nf : NA \to NB$ is a weak equivalence in $Ch^+(\mathbf{Ab})$, i.e. $H_*(Nf) : H_*(NA) \to H_*(NB)$ is an isomorphism.*

Proof. This follows from proposition 5.29, which states that $H_*(NA) \cong \pi_*(A)$. By thinking about π_*, H_* and N as functors, this proposition says that π_* is naturally isomorphic to H_*N. Let $f : A \to B$ be a weak equivalence in **sAb**. Then, by functoriality we have the following commutative diagram.

$$\begin{tikzcd}
\pi_*(A) \arrow[r, "\pi_*(f)"] \arrow[d, equal, "\wr"] & \pi_*(B) \arrow[d, equal, "\wr"] \\
H_*(NA) \arrow[r, "H_*(Nf)"] & H_*(NB)
\end{tikzcd}$$

Now, since $H_*(NA) \cong \pi_*(A) \cong \pi_*(B) \cong H_*(NB)$ we must have that $H_*(Nf)$ is an isomorphism of homology groups, as required. $\square$

So, N preserves weak equivalences. Furthermore, we have:

Proposition 6.33. *Let $f : A \to B$ is a Kan fibration. Then $Nf : NA \to NB$ is a surjection in every positive degree.*

Proof. Let $f : A \to B$ be a Kan fibration. Let $n \geq 1$ and consider $\beta \in NB_n$, so $\partial_i(\beta) = 0$ for $0 \leq i \leq n-1$. By proposition 3.17, this is the same as a simplicial map $\tilde{\beta} : \Delta^n \to B$ with $\tilde{\beta}|_{\Lambda^n_n} = 0$. This follows from the fact that $\Lambda^n_n = \bigcup_{i=0}^{n-1} \delta_i(\Delta^n)$ and as $\tilde{\beta}$ is a simplicial map, it commutes with face maps, so we have $\tilde{\beta} \circ \delta_i = \partial_i \circ \tilde{\beta} = \partial_i(\beta) = 0$. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "0"] \arrow[d, hook] & A \arrow[d, "f"] \\
\Delta^n \arrow[r, "\tilde{\beta}"'] \arrow[ur, dashed, "\exists \tilde{x}"] & B
\end{tikzcd}$$

As f is a Kan fibration, there exists a lift $\tilde{x}$ as indicated. This has the property $f \circ \tilde{x} = \tilde{\beta}$, or equivalently $f(x) = \beta$. Also $\tilde{x}|_{\Lambda^n_n} = 0$ which corresponds to $x \in NA_n$. Since this was true for arbitrary $\beta \in NB_n$, it follows that $Nf_n : NA_n \to NB_n$ is surjective for all $n \geq 1$. ☐

Therefore, N preserves fibrations.

Corollary 6.34. *(K, N) is a Quillen adjunction.*

Proof. This follows from the definition of Quillen adjunctions; by proposition 6.32 and proposition 6.33, the functor N preserves fibrations and trivial fibrations. ☐

We now have all the ingredients we need to prove our key result.

Theorem 6.35 (The Dold-Kan Quillen equivalence)**.** *There is a Quillen equivalence between the projective model structure on $Ch^+(\mathbf{Ab})$ and the classical model structure on* **sAb***, given by (K, N).*

Proof. Let C be a cofibrant object in $Ch^+(\mathbf{Ab})$, A be a fibrant object in **sAb**, and $f : KC \to A$ be a weak equivalence in **sAb**. We show that $\Theta(f) : C \to NA$, is a weak equivalence in $Ch^+(\mathbf{Ab})$. Recall from proposition 6.23, that we have an isomorphism $\left(\Phi^C\right)^{-1}$ in $Ch^+(\mathbf{Ab})$. By (M1), this is a weak equivalence. By proposition 6.32, $N(f)$ is also a weak equivalence. Hence by the 2-out-of-3 property (M2), $\Theta(f) = N(f) \circ \left(\Phi^C\right)^{-1}$ is a weak equivalence.

Hence (K, N) is a Quillen equivalence. ☐

7. Concluding Remarks and Further Reading

In this project, we have proven that there is a Quillen equivalence given by:

$$Ch^+(\mathbf{Ab}) \underset{N}{\overset{K}{\rightleftarrows}} \mathbf{sAb}.$$

This allows us to think of chain complexes of abelian groups as the same objects as simplicial abelian groups homotopically, and as a result we can calculate homotopy groups using homology groups, which are often significantly easier to calculate. Along the way, we have seen motivation for where these abstract ideas came from and the problems that they were originally intended to solve. One such example has been looking at the singular simplicial set of a topological space. We have described a translation between topology and homological algebra:

$$\mathbf{Top} \underset{|-|}{\overset{S_*}{\rightleftarrows}} \mathbf{sSet} \underset{U}{\overset{\mathbb{Z}}{\rightleftarrows}} \mathbf{sAb} \underset{N}{\overset{K}{\leftrightarrows}} Ch^+(\mathbf{Ab}),$$

where $(|-|, S_*)$ and (K, N) give Quillen equivalences, and $(\mathbb{Z}, U)$ is an adjoint pair and therefore preserves some properties like limits and colimits [Rie17]. A longer project could look more closely at this translation and look at examples of each it in action.

There are many generalisations of the Dold-Kan correspondence. Throughout this project, we have used the category **Ab** as the basis for our tools. In fact, the desirable properties of this category can be abstractified into what is known as an *abelian category*, which gives many more uses to homological algebra, as shown in ([Wei95], 1.3). Examples of abelian categories include **Ab** and the category of right R-modules for a ring R, amongst others. For an abelian category $\mathcal{A}$, we can generalise the Dold-Kan correspondence to a Quillen equivalence between the categories $Ch^+(\mathcal{A})$ and $\mathbf{s}\mathcal{A}$ [Wei95, GJ99]. The proof is not too dissimilar to the one presented here; by working with R-modules we can argue in much the same way using elements, and then we can invoke *the Freyd-Mitchell Embedding theorem* (1964), which allows us to think of abelian categories as R-modules [Fre64].

There are also more structured extensions of Dold-Kan. For example in 2003, Stefan Schwede and Brooke Shipley proved a monoidal version of the Dold-Kan Quillen equivalence to show that simplicial rings, modules and algebras are Quillen equivalent to differential graded rings, modules and algebras respectively [SS03].

An extended project could attempt to prove that there is a Quillen equivalence between the model structure on $Ch^+(\mathcal{A})$ given by Christensen and Hovey in [CH02] and the effective model structure on $\mathbf{s}\mathcal{A}$ [GHSS21]. This would generalise the work in this project so that when $\mathcal{A} = \mathbf{Ab}$, we would recover exactly the statement of theorem 6.35.

To an algebraic topologist, the Dold-Kan correspondence is an essential tool. This is seen by its continuous use in current research papers such as [GHSS21, SC19, Wal22].

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