Internal Categories, Internal Groupoids, and Models of Type Theory

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Abstract

In this thesis, we examine the logic and properties of 2-categories of internal categories and groupoids. There are two main approaches to studying the logic of a 1-category: firstly, by comparing it to various axiomatic set theories; secondly, by relating it to various type theories, including versions of Martin-Löf type theory with the Uniqueness of Identity Proofs principle. This thesis develops these ideas further, in the setting of 2-categories of internal categories and groupoids.

Building upon work of John Bourke which gives purely 2-categorical axioms that ensure that a 2-category is equivalent to a 2-category of internal categories, we give elementary descriptions of those 2-categories which are equivalent to a 2-category of categories internal to a 1-category of logical interest such as an arithmetic Π-pretopos, an elementary topos, a model of Lawvere's elementary theory of the category of sets and a model of Palmgren's constructive elementary theory of the category of sets. In doing so, we isolate important 2-categorical properties of logical interest.

We prove that for each of the logical 1-categories of interest as considered above, the 2-category of categories internal to it has finite 2-colimits, generalising a result by Johnstone and Wraith. Furthermore, we provide an explicit construction for these colimits, allowing for direct calculations. The existence of such colimits is important for both logical considerations and category-theoretic considerations.

In the setting of $(2, 1)$-categories, we show that these axioms give internal groupoids instead of internal categories. We provide a list of first order $(2, 1)$-categorical axioms such a $(2, 1)$-category satisfying them is equivalent to a $(2, 1)$-category of groupoids internal to an arithmetic Π-pretopos; we say that a $(2, 1)$-category satisfying these axioms models *the elementary theory of the* $(2, 1)$*-category of small abstract groupoids*. We show that any model of the elementary theory of the $(2, 1)$-category of small abstract groupoids gives a model of intensional Martin-Löf type theory that does not satisfy the Uniqueness of Identity Proofs principle, generalising the 1-dimensional setting.

We then move on to incorporating a notion of size into our axiomatisation, in order to capture the behaviour of the $(2, 1)$-category of large groupoids in von Neumann-Bernays-Gödel set theory. In the 1-dimensional setting, an axiomatisation of the category of classes in von Neumann-Bernays-Gödel set theory is given by Joyal and Moerdijk's notion of a class category. In a class category, there is a notion of small objects, and the full subcategory of small objects is an arithmetic Π-pretopos and so forms a model of Martin-Löf type theory that satisfies the Uniqueness of Identity Proofs principle. Building on this, we develop the theory of class $(2, 1)$-categories. We give a notion of small objects which allows us to prove that the full sub-$(2, 1)$-category of small objects forms a model of the elementary theory of the $(2, 1)$-category of small abstract groupoids, and hence forms a model of intuitionistic Martin-Löf type theory.

Declaration of originality

The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapter 3 is based on the paper "Colimits of internal categories" jointly authored by Calum Hughes and Adrian Miranda [HM26], accepted for publication by the Bulletin of the Belgian Mathematical Society— Simon Stevin (BBMS). The contribution of the authors is equally distributed. Chapter 4 is based on the paper "The elementary theory of the 2-category of small categories" jointly authored by Calum Hughes and Adrian Miranda, published in Theory and Applications of Categories, special edition in memory of Bill Lawvere [HM25]. The contribution of the authors is equally distributed. Some of the statements in Section 2.2 are also based on [HM25], having been moved to this section for cohesion of the thesis. Chapter 6 is based on the solely authored paper "The algebraic internal groupoid model of Martin-Löf type theory" [Hug25], which is under review.

Copyright statement

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Chapter 1

Introduction

1.1 Context and Motivation

Investigations into finding a good foundation for mathematics have been made since eminent logicians such as Zermelo, Cantor and Russell noticed that, if approached naively, set theory contained contradictions; this idea was made evident by Russell's paradox in 1901 [Rus20]. Three common frameworks for formalising the foundation of mathematics are given by:

1. Set theory, which axiomatises sets together with a membership relation ($\emptyset \in 1$, $3 \in \mathbb{N}$, etc.)
2. Category theory, which axiomatises objects, morphisms between objects and the composition of these morphisms ($\emptyset \to 1, \mathbb{N} \to \mathbb{Z}, \sin : \mathbb{R} \to \mathbb{R}$, etc.)
3. Martin-Löf type theory, which axiomatises types ($\mathbb{N}, \mathtt{Bool}$ etc.) together with terms ($3 : \mathbb{N}, \top : \mathtt{Bool}$, etc.)

In set theory, two sets are considered the same if and only if they have the same elements— this is the Axiom of Extensionality. In category theory, two objects are considered the same if and only if the have the same incoming morphisms from all other objects— this is a corollary of the Yoneda lemma. In Set-level type theory, we postulate the Uniqueness of Identity Proofs principle (UIP)— or equivalently axiom K [Str93]— which ensures that there is at most one way in which two terms can be identified. Given two topological spaces, if equality on points implies that they are equal, they must have been equipped with the discrete topology and are therefore 0-dimensional spaces. Therefore, we consider the three frameworks above to be 0-*dimensional foundations*. The 0-dimensional setting and the relationships between these frameworks is well-understood.

Within the framework of set theory, the most commonly accepted foundation is Zermelo-Fraenkel set theory with choice (ZFC) [Zer08; Fra22]. This is a non-constructive theory— meaning that it proves the Law of the Excluded Middle— and it cannot be finitely axiomatised due to the Axiom Schema of Replacement [Kun84, Corollary IV 7.7]. A finitely axiomatisable version of ZFC without the Axiom schema of Replacement but with a bounded form of the Axiom of Separation is given by Bounded Zermelo-Fraenkel set theory with choice (BZFC) [Mac86]. Other well-established variants are constructive versions which do not include the Axiom of Choice and cannot prove the Law of the Excluded Middle; these are given by Aczel's Constructive Zermelo-Fraenkel (CZF) [Acz78] and Intuitionistic Zermelo-Fraenkel (IZF) [Fri73; Myh06].

Working in a suitable metatheory, we can form the category of sets. For each of the set theories mentioned above, this category is an arithmetic Π-pretopos— that is a locally cartesian closed, exact and extensive category with natural numbers object [AFW06]. Lawvere's categorical approach to foundations, first given in his *Elementary Theory of the Cate-*

gory of Sets (ETCS) [Law64] approaches foundations from the opposite direction; it takes sets, functions and composition of functions (i.e. a category) as the primitive notions of the foundation rather than sets and the membership relation. The usual axioms of set theory are replaced by axioms which we require this category to satisfy; in particular, these axioms give an arithmetic Π-pretopos [Col71; Mit73]. Lawvere proved that in an appropriate set theory which contains ZFC, models of ETCS are equivalent to the category of sets in ZFC. On the other hand, any arithmetic Π-pretopos has an "internal set theory" for working with objects of the category in almost the same way that one would work with sets in ZFC— this was developed for elementary toposes by [Osi74; Mit73; Bén73], and generalised to the setting of arithmetic Π-pretoposes later. It was shown in [Shu19, Corollary 9.5] that the internal set theory of ETCS has the same logical strength as BZFC. Other important instances of arithmetic Π-pretoposes are given by models of Palmgren's *constructive* elementary theory of the category of sets (CETCS) [Pal12], which gives a characterisation of the categorical properties of the category of sets in CZF [Acz78] and the effective topos [Hyl82], which gives a model of set theory in which every function on the natural numbers is computable. It should be noted that the internal language of an arithmetic Π-pretopos is unable to satisfy the Axiom Schema of Replacement— indeed it is given by a finite axiomatisation. We therefore see arithmetic Π-pretoposes as models of generalised bounded set theories.

Given an arithmetic Π-pretopos, it was shown by Streicher that we also have an associated internal set-level type theory [Str93]. On the other hand, from a type theory, we can form its syntactic category and obtain a locally cartesian closed category [See84]; if this type theory has a type of natural numbers, well-behaved quotients and well-behaved sums this actually forms an arithmetic Π-pretopos [Mai10]. Therefore, we see that generalised models of set theory correspond to Martin-Löf type theories which satisfy UIP. We therefore have a correspondence as displayed below.

$$\begin{tikzcd} \text{Set Theories} \arrow[r, shift left, "\text{cat. of sets}"] & \text{Arithmetic }\Pi\text{-Pretopos} \arrow[l, shift left, "\text{internal set theory}"] \arrow[r, shift right, "\text{internal type theory}"'] & \text{Type Theories with UIP} \arrow[l, shift right, "\text{syntactic category}"'] \end{tikzcd}$$

Fig. 1.1. Correspondence between 0-dimensional foundations

The Axiom Schema of Replacement states that the image of each definable mapping is a set. Thus, there are definable objects that are in some sense "too large" to be in BZFC. For example, the Axiom of Replacement in particular asserts that for any set I and any indexed family $\{X_i\}_{i\in I}$, the disjoint union

$$\coprod_{i\in I} X_i$$

is a set. For $I = \mathbb{N}$, and $X_n := \mathcal{P}^n(\mathbb{N})$, this coproduct cannot be formulated in BZFC [Lei14]. In order to add the Axiom of Replacement to BZFC, we must add an infinite number of axioms saying that these large definable objects are in fact sets. One of the aims of developing von Neumann-Bernays-Gödel set theory (NBG) [Neu28; Ber76; Göd40] was to solve this by incorporating large objects as part of the theory. In this theory, the primitive objects are called classes, and those classes which are elements of other classes are called sets. Replacement becomes either a theorem or a single axiom, being handled by the theory's ability to talk about classes as well as sets. It therefore gives a finitely axiomatisable, conservative extension of ZFC. A good summary of NBG is given in [Men09, §4].

Again, in a suitable metatheory, we can form the category of classes in NBG. In a similar vein to Lawvere's ETCS, the categorical properties of the category of classes were axiomatised by Joyal and Moerdijk's notion of a class category [JM95] which is a category together with a distinguished family of 'small maps' satisfying some properties. Variations on these axioms were subsequently given to capture constructive class theories [MP02; Gam05; Sim05; Ber06] and intuitionistic class theories [Gam02; But03; For04; AF05]. In each of these axiomatisations, we can give a definition of 'small objects'— which correspond to the sets in the class theory— and form the full subcategory of small objects; for each of the approaches mentioned, the category of small objects forms an arithmetic Π-pretopos.

$$\begin{tikzcd}
\text{Set theories} \arrow[r, shift left, "\text{cat. of sets}"] \arrow[d, hook] & \text{arithmetic } \Pi\text{-pretoposes} \arrow[l, shift left, "\text{internal set theory}"] \arrow[d, hook] \\
\text{Class theories} \arrow[r, shift left, "\text{cat. of classes}"] & \text{Class categories} \arrow[l, shift left, "\text{internal class theory}"]
\end{tikzcd}$$

Fig. 1.2. The objects of a class category are to be thought of as generalised *classes* whilst the small objects are to be thought of as generalised sets.

1.2 Thesis Objectives

It is the objective of this thesis to categorify the story presented above from 0-dimensional foundations to 1-dimensional foundations. There are various motivations for wanting this, coming from different research areas such as type theory, category theory, topos theory and homotopy theory.

Firstly, in [HS98], Hofmann and Streicher provided a mathematical model of a type theory in which UIP was not satisfied. In this model, types are interpreted as *groupoids*— that is: as collections of objects equipped with higher-dimensional structure in the form of paths between those objects. Terms are interpreted as objects of the corresponding groupoids. Two terms are considered to be equal if there is a path between them, which in particular shows that UIP is not satisfied as there can be more than one path between two points in a groupoid. This model illuminated the perspective that in some sense type theory without UIP is a higher dimensional version of set theory— when UIP is not included as an axiom, the types are no longer guaranteed to be "set-like" but are instead "space-like". This perspective was taken seriously by Awodey and Warren [AW09] and Voevodsky [Voe06] whose ideas paved the way for homotopy type theory (HoTT) and the Univalent Foundations program [Uni13], in which types are thought of as topological spaces (or ∞-groupoids)— indeed, Voevodsky proved that the $(\infty, 1)$-category of ∞-groupoids forms a model of homotopy type theory [KL21]. Homotopy type theory and its subsequent variations have been successful in providing foundations for mathematics in which it is more convenient to work with higher-dimensional structures, such as ∞-groupoids, since the objects of that foundation are natively higher-dimensional. In contrast, to define an ∞-groupoid in 0-dimensional foundations, we have to define an infinite amount of data and coherence conditions and then keep track of this data when working with them. As such, theorems about higher structures are easier to formalise in homotopy type theory than in set theory— for example, the Yoneda lemma for ∞-categories [RV17; Mar21]. This has led to wide-reaching applications from ∞-category theory [RS17; Cis+25] to algebraic geometry [Koc06; Ble17; CCH24]. As such, it has become increasingly interesting to explore non-set-level foundations of mathematics. The 1-dimensional approach has types which are "groupoid-like", and give a 1-truncated HoTT.

Another motivation for examining 1-dimensional foundations comes from Lawvere, who worked on a first order axiomatisation of the category of categories in [Law66]. This approach was unsuccessful due to an error, noted by [Isb67], in the so-called "category description theorem" [Law66, p. 15] which attempted to deduce from his theory that categories are built out of discrete objects as a 1-dimensional quotient; [BP75] then constructed a model of Lawvere's theory which showed that the category of discrete objects did not form a model of Lawvere's ETCS, which was a central result in [Law66]. Whilst suggestions to remedy this approach were given in [Isb67; BP75], Lawvere desired further study of a first-order axiomatisation of the category of categories [Law]. Perhaps one of the limitations of Lawvere's approach was in his consideration of the 1-category of categories, which overlooks the additional data of natural transformations between functors which form an important part of category theory; indeed a kind of "category description theorem" states that categories are a certain kind of genuinely 2-dimensional colimit of discrete objects. As such it makes sense to consider a first order axiomatisation of the 2-category of categories rather than just the 1-category of categories. However, we maintain that in the context of 1-dimensional foundations of mathematics, it is also more natural to study the $(2,1)$-category of groupoids rather than the 2-category of categories for two reasons: firstly, in an attempt to categorically model Martin-Löf type theories without UIP in the same vein as Hofmann and Streicher; secondly, it is groupoids that are the direct higher dimensional version of sets, whilst categories are the higher dimensional version of partially ordered sets (posets)— this was noted by Voevodsky in [Voe14]. We have the following table:

	0-dimensions	1-dimensions	...	∞-dimensions
Undirected	Sets	Groupoids		Topological spaces
Directed	Posets	Categories		$(\infty, 1)$-categories

In [Men71], a Lawvere-style elementary axiomatisation of the 1-category of posets is given; it is significantly more complicated than Lawvere's ETCS and does not have foundational motivations. Similarly, axiomatising the 2-category of categories is more complicated than axiomatising the $(2,1)$-category of groupoids and also has less uses from a foundational point of view. Indeed, groupoids model MLTT whereas categories model directed type theory, which is notably more complicated than plain MLTT— see, for example [LH11]. Given an elementary axiomatisation of the $(2,1)$-category of groupoids, we could define a category to be a groupoid together with an ordering as structure— much like a poset is a set together with an ordering. Notably, this definition is more category-theoretic than the standard one. It avoids 'evil' formulations by remaining invariant under equivalence; in contrast, the traditional definition allows for equivalent categories with non-isomorphic sets of objects.

A third motivation for studying 1-dimensional foundations comes from topos theory. Since every elementary topos is an arithmetic Π-pretopos, it has a corresponding internal set theory and internal set-level type theory. It is therefore reasonable to suggest that an elementary $(2,1)$-topos should have both an associated internal groupoid theory and an internal groupoid-level type theory. A proposed categorification of the notion of an elementary topos to two dimensions was given in [Web07], which crucially introduced the notion of a discrete opfibration classifier (although the primordial ideas were already present in Lawvere's work [Law66]). In the 2-category of large categories **CAT**, the important discrete opfibration classifier is derived from the category **Set**, which we call the classifying object. In the 1-dimensional elementary topos **Set**, the important classifying object is $\{\emptyset, \mathbf{1}\}$ in which $\emptyset$ represents falsity and $\mathbf{1}$ represents truth; in the 2-dimensional setting, the category **Set** can be thought of as generalised truth values; $X \in \mathbf{Set}$ is a statement of truth with $x \in X$ a distinct witness of that truth. Note that $\emptyset$ is still the only representation of falsity. This interpretation is consistent with the philosophy of intensional MLTT as a foundational logic; it is *proof relevant* meaning that it not only matters whether something is true, but also in which way that truth is proven. In [Web07], it is shown that in a cartesian closed 2-category with finite limits, a duality involution and a discrete opfibration classifier, there is a notion of internal Yoneda embedding, introduced by [SW78], which is a crucial aspect of category theory. Another approach to axiomatising the 2-category of categories was given by the notion of an elementary cosmos [Str80], which consists of a 2-category $\mathcal{K}$ together with a way to associate a presheaf object $\mathbf{Psh}(\mathbb{X})$ to each object $\mathbb{X} \in \mathcal{K}$; it can be shown that an elementary cosmos also has a Yoneda embedding. However, both approaches seem to lack several other desired properties. For example, unlike in the 1-categorical theory, the axioms given do not seem to be strong enough to imply any sort of exactness properties that are key in both category theory and logic.

Moreover, there is a fairly subtle conceptual issue with these ideas which does not appear in the 1-dimensional case. We must be clear about which 2-category it is that we are hoping to axiomatise. If we are hoping to axiomatise the 2-category of small categories, then we cannot have **Set** as the classifying object since it is not a small category. Instead, it has a classifier determined by the category of finite sets, yet this is somehow unnatural to consider in the theory: rarely in category theory do we refer to finite categories and in proof relevant mathematics it is unimportant to consider statements which are true with finite numbers of proofs. On the other hand, if we are hoping to axiomatise the 2-category of large categories **CAT** (or similarly the 2-category of locally small categories), we must first be clear what a non-small category is; we must work in a metatheory which support the notion of the category of sets. Two common solutions are either to work in ZFC with the addition of two Grothendieck universes, or to work in an enlarged version of NBG set theory which we denote NBG+ and explain formally later. Since in this thesis we are concerned with elementary axioms for 2-categories, which are in particular finite axiomatisations, we will work in NBG+. In this metatheory, **Set** is a classifying object in **CAT**, yet we run into issues with cartesian closure: recall that presheaves on a locally small category may not be locally small; similarly, presheaves on a non-locally small category may be too large to be in **CAT**. Therefore, it is necessary to include a notion of size when trying to axiomatise the 2-category of categories. These same issues with cartesian closure appear when looking at the category of classes in NBG+. This issue is solved categorically by Joyal and Moerdijk's class categories. As explained above, on the logical side this also adds in a categorical formulation of the Axiom of Replacement into the internal set theory of the category. Accordingly, we propose that a categorification of class categories is more appropriate in the axiomatisation of large categories and groupoids. Recently, another proposed definition of an elementary 2-topos was given in [Hel24] which does not include cartesian closure, and which proves that the classifying object S is an internal 1-topos. The work presented there does not try to give a model of intensional MLTT, and it does not seem that it is possible from the axioms given.

Finally, another motivation for exploring this work comes from homotopy theory. Voevodsky proved that the $(\infty, 1)$-category of ∞-groupoids forms a model of Univalent Foundations using the Kan-Quillen model structure on simplicial sets, using Quillen's abstract homotopy theory to deal with the problem of higher coherences— the model structure presents the 1-category of Kan complexes as an $(\infty, 1)$-category [KL21]. Our model of (1-dimensional) MLTT also comes from the so-called natural model structure on the 1-category of groupoids, which presents the 1-category of groupoids as a $(2, 1)$-category. There are a wealth of other proofs that model categories give models of type theories [AK11; Shu15; GK17; LS20; Awo26] and also a range of ways that homotopical structures can model type theories [Awo18; Ber18; KL18; Awo26]. Voevodsky's proof that ∞-groupoids model homotopy type theory was non-constructive. The model of homotopy type theory given by cubical sets [Coh+15; Awo+24; Awo26] gave a constructive model, showing that homotopy type theory is consistent relative to constructive set theory; the problem as to whether Voevodsky's original model is constructive remains an open problem.

Notably for our work, [GL23] provides a way in which an *algebraic* weak factorisation system can model type theory. Algebraic weak factorisation systems were defined in [GT06] and used in the definition of algebraic model categories in [Rie11]. They are designed to be more constructive and proof-relevant than their non-algebraic counterparts, since extra algebraic data is carried around and is therefore automatically coherent in calculations. Our approach in this work follows this philosophy, allowing for an explicit description of the model of type theory. This makes certain computations more tractable which makes it easier to see the specifics of the type theories of a particular model. Whilst algebraic homotopy theory provides a more constructive framework for modelling type theories, it is frequently overlooked. This work aims to demonstrate the practical advantages of the algebraic approach through its implementation.

To summarise, our first objective is to give $(2, 1)$-categorical axioms which gives an first order axiomatisation of the theory of groupoids in a generalised set theory, which we call the *elementary theory of the* $(2, 1)$*-category of small abstract groupoids* (ETCSAG) in the sense that it fits in Figure 1.3 below, which is a categorified version of Figure 1.1.

$$\begin{tikzcd}
\text{Groupoid theories} \arrow[r, shift left, "{(2,1)\text{-cat. of groupoids}}"] & \text{ETCSAG} \arrow[l, shift left, "\text{internal groupoid theory}"] \arrow[r, shift right, "\text{internal type theory}"'] & \text{1-truncated HoTT} \arrow[l, shift right, dashed, "{\text{syntactic } (2,1)\text{-category}}"']
\end{tikzcd}$$

Fig. 1.3. Correspondence between 1-dimensional frameworks

In this figure, the syntactic $(2, 1)$-category structure is the one described in [Spa24, Section 3.2], although note that we do not explicitly use it in this thesis— this is left for future work.

It should be possible to compare between the 0-dimensional and 1-dimensional approaches. Given the $(2, 1)$-category of groupoids, we can recover the category of sets (up to equivalence) by taking the discrete groupoids. Similarly, given the category of sets, we can define the $(2, 1)$-category of (small) groupoids by taking groupoids internal to sets. Since we can define the notion of discrete object in any arbitrary $(2, 1)$-category, and define the notion of a $(2, 1)$-category of groupoids internal to any 1-category with pullbacks, we would like our ETCSAG to be a part of a $(2, 1)$-equivalence between the very large 2-categories of categories displayed below:

$$\begin{tikzcd}
\text{Arithmetic } \Pi\text{-pretoposes} \arrow[r, bend right, "\mathbf{Gpd}(-)"'] & \text{ETCSAG} \arrow[l, bend right, "\mathbf{Disc}(-)"']
\end{tikzcd}$$

Fig. 1.4. Comparing between 1-categorical approach and the $(2, 1)$-categorical approach

Our result will build upon the work of [Bou10], which gave a 2-equivalence between the large 2-categories of 1-categories with pullbacks and 2-categories satisfying some elementary 2-categorical axioms. We can compose figures 1.1, 1.3 and 1.4 to get translations between the frameworks in Figure 1.5.

$$\begin{tikzcd}
\text{Set Theories} \arrow[r, shift left, "\text{cat. of sets}"] & \text{Arithmetic } \Pi\text{-Pretoposes} \arrow[l, shift left, "\text{internal set theory}"] \arrow[r, shift right, "\text{internal type theory}"'] \arrow[d, shift left, "\mathbf{Gpd}(-)"] & \text{0-truncated HoTT} \arrow[l, shift right, "\text{syntactic category}"'] \\
\text{Groupoid theories} \arrow[r, shift left, "{(2,1)\text{-cat. of groupoids}}"] & \text{ETCSAG} \arrow[l, shift left, "\text{internal groupoid theory}"] \arrow[r, shift right, "\text{internal type theory}"'] \arrow[u, shift left, "\mathbf{Disc}(-)"] & \text{1-truncated HoTT} \arrow[l, shift right, "{\text{syntactic } (2,1)\text{-category}}"']
\end{tikzcd}$$

Fig. 1.5. Correspondence between 0-dimensional and 1-dimensional frameworks

In the 0-dimensional setting, by adding extra axioms to an arithmetic Π-pretopos, we can isolate categorical models of specific set theories— this is exhibited by ETCS and CETCS which are categorical models of BZFC and some bounded form of CZF respectively. In this spirit, our second objective is to give 2-categorical axioms which gives a first order axiomatisation of the 2-category of categories in BZFC and CZF respectively; this also should give $(2, 1)$-categorical axioms which gives a first order axiomatisation of the $(2, 1)$-category of groupoids in BZFC and CZF respectively.

The approaches explained above provide elementary theories for the 2-category (resp. $(2,1)$-category) of *small* categories (resp. groupoids). Our third objective is to extend this to provide an elementary theory for the $(2,1)$-category of *large* groupoids. Following the 1-dimensional case, we give a categorification of the notion of a class category, which we call a *class* $(2,1)$*-category*. As in Figure 1.2, this should have a notion of small objects for which the full subcategory of small objects recovers our ETCSAG.

$$\begin{tikzcd}
\text{Small groupoid theories} \arrow[r, shift left, "\text{(2,1)-cat. of small groupoids}"] \arrow[d, hook] & \text{ETCSAG} \arrow[l, shift left, "\text{internal groupoid theory}"] \arrow[d, hook] \\
\text{Large groupoid theories} \arrow[r, shift left, "\text{(2,1)-cat. of large groupoids}"] & \text{Class } (2,1)\text{-categories} \arrow[l, shift left, "\text{internal groupoid theory}"]
\end{tikzcd}$$

Fig. 1.6. The objects of a class $(2,1)$-category are to be thought of as generalised large groupoids, and the small objects are to be thought of as generalised groupoids

By the second objective of this thesis, we should be able to add extra axioms to the axioms for a class $(2,1)$-category such that the small objects form a model of small groupoids in BZFC; since class $(2,1)$-categories satisfy an internal version of the axiom of replacement, this will give a theory in which the small objects are a model of ZFC.

1.2.1 Notes on strictness

All of the results that we present in this thesis are strict in nature i.e. **Cat**-enriched (or **Gpd**-enriched) as opposed to the bicategorical ones i.e. weakly **Cat**-enriched (or weakly **Gpd**-enriched). Whilst from the higher-categorical perspective, it is more natural to study bicategories in order to incorporate examples from topology and mathematical physics, there is little motivation to study bicategories from our approach.

Firstly, from the perspective of modelling type theory, a strict approach seems to be more convenient. In particular, the substitution mechanism in type theory is rather strict; indeed substitution is given by pullback and if this was a bipullback rather than a strict pullback, it would be unique only up to equivalence; this would then lead to a weak substitution rule.

Moreover, strict 2-categories are simply less complicated to work with than bicategories; hence there are results that are foundational for this work that are proven in the case of strict 2-categories but do not yet have bicategorical analogues. As such, progress in the higher-categorical dimension of this work is only currently possible in the strict 2-categorical case.

The approach taken by the Australian school of Category Theory is to prove strictification results between the weak world and the strict world, so that we can work with the ease of the strict world without loss of generality of the weak world— for example: every bicategory is equivalent to a strict 2-category [Bén67]. We hope that there is a bicategorical version of the theory we present in this thesis, alongside a result which says that it can be suitably strictified to match with our theory. We leave this for future work.

1.3 Outline and Main results

In this section, we give an outline of the thesis and flag up one main result from each of its chapters. In many of the chapters, we focus on 2-categories instead of $(2,1)$-categories in the interest of proving more general results that are of interest to a wider audience; however all of our results work for $(2,1)$-categories too with minimal adjustments.

Chapter 2 gives necessary background information and fixes notation, as well as recalling results which form the basis for which our theory is built upon. In particular, we simply state the definitions of BO-exactness and SO-exactness for a 2-category, given in [BG14] and discuss notions of two dimensional extensionality that do not appear in the literature.

In Chapter 3, we prove a result in pure internal category theory which is necessary for our work but that we believe is of independent interest, which is stated as follows.

Theorem 3.5.2 (2-colimits in $\mathbf{Cat}(\mathcal{E})$)**.** Let $\mathcal{E}$ be an extensive category with pullbacks and pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. Then the 2-category $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits.

This result is important for our purposes as colimits are an integral part of category theory and moreover it will be used to give our model of intensional MLTT. This is a generalisation of [JW78, Corollary 6.10] which states the same result for elementary toposes with a natural numbers objects. Our result is an improvement for several reasons. Firstly, it is more general allowing for our results to apply to many more examples, including arithmetic Π-pretoposes. Secondly, our proof fixes some gaps in the reasoning given in their paper. Thirdly, the proof we give not only proves the existence of coequalisers but also gives a constructive recipe to calculate them, which is of practical use and also paves the way for potential generalisations of this construction to settings other than internal categories.

In our general setting, we additionally show that $\mathbf{Cat}(\mathcal{E})$ is extensive, has pullbacks, and codescent coequalisers are stable under pullback along discrete Conduché fibrations. Moreover, we give converse results to this, (Theorem 3.6.8) showing a correspondence between these 2-dimensional properties and the assumed 1-dimensional properties.

In Chapter 4, we give an elementary description of 2-categories of the form $\mathbf{Cat}\,(\mathcal{E})$ where $\mathcal{E}$ is a category modelling ETCS. We call this the *elementary theory of the* 2*-category of small categories* (ET2CSC) (Definition 4.8.1). This extends Bourke's characterisation of 2-categories of the form $\mathcal{K} \simeq \mathbf{Cat}\,(\mathcal{E})$ where $\mathcal{E}$ has pullbacks [Bou10] to take account for the extra properties in ETCS. We explain Bourke's result and its relation to two dimensional exactness and two dimensional exact completions in Section 2.4. Our main result is the following:

Theorem 4.8.2 (Characterisation of 2-categories of categories internal to ETCS)**.**

(i) Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ models the elementary theory of the category of sets if and only if $\mathbf{Cat}(\mathcal{E})$ models the elementary theory of the 2-category of small categories, and in this case $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$.

(ii) Conversely, let $\mathcal{K}$ be a 2-category. Then $\mathcal{K}$ models the elementary theory of the 2-category of small categories if and only if $\mathbf{Disc}\,(\mathcal{K})$ models the elementary theory of the category of sets, and in this case $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$.

Important two-dimensional concepts which we introduce include 2-well-pointedness (Definition 4.4.11), full-subobject classifiers (Definition 4.6.1), and the Categorified Axiom of Choice (Definition 4.7.12). Along the way, we show how generating families (resp. orthogonal factorisation systems) on $\mathcal{E}$ give rise to generating families (resp. orthogonal factorisation systems) on $\mathbf{Cat}(\mathcal{E})_1$, results which we believe are of independent interest (Corollary 4.4.8, Proposition 4.7.7).

In the $(2,1)$-setting, this gives an elementary description of the $(2,1)$-category of groupoids internal to the elementary theory of the category of sets. In Theorem 4.8.13, we show that there is a Morita biequivalence between the 2-category **ETCS**, whose objects are categories modelling ETCS and **ET2CSC**, whose objects are 2-categories modelling ET2CSC.

$$\mathbf{ETCS} \underset{\mathbf{Cat}(-)}{\overset{\mathbf{Disc}(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\sim}}}} \mathbf{ET2CSC}$$

Since ETCS is shown to have the same logical power as BZFC, it follows the elementary theory of the $(2,1)$-category of small groupoids also has the same logical power as BZFC.

In Chapter 5, we develop further the ideas introduced in Chapter 4 and give a 2-dimensional version of Palmgren's constructive elementary theory of the category of sets (CETCS). We give a list of elementary 2-categorical axioms which we call the *constructive elementary theory of the* 2*-category of small categories* (CET2CSC) (Definition 5.4.1). We have an analogue of Theorem 4.8.2 in this case, stated as below.

Theorem 5.4.2 (Constructive version of Theorem 4.8.2)**.**

(i) Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ models the constructive elementary theory of the category of sets if and only if $\mathbf{Cat}(\mathcal{E})$ models the constructive elementary theory of the 2-category of small categories, and in this case $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$.

(ii) Conversely, let $\mathcal{K}$ be a 2-category. Then $\mathcal{K}$ models the constructive elementary theory of the 2-category of small categories if and only if $\mathbf{Disc}\,(\mathcal{K})$ models the constructive elementary theory of the category of sets, and in this case $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$.

We introduce a categorified notion of projective objects (Definition 5.3.11) which differs from notions currently in the literature through their association with acute morphisms rather than codescent morphisms. From this, we extract a categorified version of the Presentation Axiom. We arrive at a framework for constructive and predicative category theory which captures the 2-categorical properties of categories in Constructive Zermelo-Fraenkel set theory. We explore aspects of this theory, for example we have a decidable full-subobject classifier, which we define in Definition 5.4.12, which means that in the internal logic of such a 2-category, we can speak about full subcategories of internal categories which are decidable. We give a definition of decidably fully faithful morphism abstractly in any 2-category (Definition 5.4.10), allowing us to give a purely 2-dimensional definition of the Law of the Excluded Middle (Definition 5.4.13).

Along the way, we produce a characterisation of 2-categories of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ for $\mathbf{Disc}(\mathcal{K})$ an arithmetic Π-pretopos, a result which allows us to translate between 0-dimensional foundations and 1-dimensional foundations (cf. Diagram 1.4). This gives us our desired ETCSAG (see Figure 1.3). This result requires us to prove a correspondence between exactness of $\mathcal{E}$ and SO-exactness of $\mathbf{Cat}(\mathcal{E})$, a result which we believe is of independent interest (Proposition 5.3.5).

Up until this point, we have explored different 2-categorical axioms such that we can deduce that $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ for $\mathbf{Disc}(\mathcal{K})$ a 1-category with desirable logical properties. In Chapter 6 we restrict our attention to $(2,1)$-categories and show that these logical properties allow us to form a model of intensional MLTT. The main result is the following:

Theorem 6.5.9 (Internal groupoids model type theory)**.** Let $\mathcal{E}$ be a locally cartesian closed locos with coequalisers. Then $\mathbf{Gpd}(\mathcal{E})$ forms a model of intensional Martin-Löf type theory.

The proof of this theorem goes through a proof that the 1-category $\mathbf{Cat}(\mathcal{E})$ forms an algebraic model structure (Theorem 6.3.15) with one of its associated algebraic weak factorisation systems a type theoretic algebraic weak factorisation system in the sense of [GL23].

Our general setting allows us to give many different examples, including $\mathbf{Gpd}(\mathcal{E})$ in which $\mathcal{E}$ is an arithmetic Π-pretopos, in particular implying the case for $\mathcal{E}$ a model of ETCS or CETCS. Therefore, this accomplishes another of the goals of this thesis; we have shown that ETCSAG provides a model of intensional MLTT, generalising the fact that arithmetic Π-pretoposes provide a model of set-level MLTT.

In Chapter 7, we move on to adding in a notion of size and adding in the Axiom of Replacement into our theory. We introduce the notion of a class $(2,1)$-category (Definition 7.4.4) which is a categorified version of a class category [JM95]. In such a $(2,1)$-category $\mathcal{K}$, we can give a definition of small object and form the full subcategory of small objects $\mathcal{K}_\sigma$. Our main theorem is the following:

Theorem 7.5.29. Let $\mathcal{K}$ be a class $(2,1)$-category. Then $\mathcal{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ and $\mathbf{Disc}(\mathcal{K}_\sigma)$ is an arithmetic Π-pretopos. As such, $\mathcal{K}_\sigma$ forms a model of intensional MLTT.

Class $(2,1)$-categories also satisfy an internal form of the Axiom of Replacement. We bring together all the results of the previous chapters to give a definition of the *elementary theory of the* $(2,1)$*-category of groupoids* (Definition 7.6.4), which has the additional property of the internal language of $\mathbf{Disc}(\mathcal{K}_\sigma)$ being a model of ZFC (Theorem 7.6.5). Our abstract definition of class $(2,1)$-category also captures many examples, including large groupoids, prestacks and a certain kind of stacks. We show that $(2,1)$-categories of semi-strict stacks prove all but one of the axioms we provide.

The conclusion in Chapter 8 outlines some directions for future work based off of the results in this thesis.

Chapter 2

Preliminaries

In this chapter we establish the notation, terminology and conventions used in this thesis, and catalogue concepts from internal category theory that will be crucial for our proofs. We also describe some less standard result in two dimensional category theory that we will use throughout.

Remark 2.0.1. We assume some familiarity with standard category theory [Mac71], including some topos theory [Joh02a; Joh02b; MM94], enriched category theory [Kel82] and in particular two-dimensional category theory [Lac10; JY21]. We briefly remind the reader of common 2-categorical notions and conventions that are used in this thesis. A 2-category is a **Cat**-enriched category, whilst a $(2,1)$-category is a **Gpd**-enriched category; hence a 2-category has a hom-category between any two objects, and a $(2,1)$-category has a hom-groupoid between any two objects— this is equivalent to being a 2-category in which every 2-cell is invertible. We will refer to **Cat**-enriched (co)limits as 2-*(co)limits*. We say that a 2-category $\mathcal{K}$ has *powers* and *copowers* by the category $\mathbf{2} := \{ \bullet \longrightarrow \bullet \}$ (elsewhere also called cotensors and tensors by $\mathbf{2}$ respectively) if for any $\mathbb{X} \in \mathcal{K}$, there exists an object $\mathbb{X}^{\mathbf{2}} \in \mathcal{K}$, respectively $\mathbf{2} \odot \mathbb{X} \in \mathcal{K}$ such that for any $\mathbb{A}, \mathbb{B} \in \mathcal{K}$, we have the following isomorphism of categories:

$$\mathcal{K}(\mathbf{2} \odot \mathbb{A}, \mathbb{B}) \cong \mathbf{Cat}(\mathbf{2}, \mathcal{K}(\mathbb{A}, \mathbb{B})) \cong \mathcal{K}(\mathbb{A}, \mathbb{B}^{\mathbf{2}}).$$

We also consider powers by the category $\mathcal{I} := \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$, defined similarly.

A notion of *finiteness* for weights for 2-(colimits) is described in [Str76], and all 2-(co)limits that we will consider are finite in this sense. We will call an adjoint equivalence in the 2-category $\mathcal{V}$-**Cat** for $(\mathcal{V}, \otimes, I) = (\mathbf{Cat}, \times, \mathbf{1})$ a 2-*equivalence*. A functor (resp. 2-functor) will be said to *preserve* some structure if it does so up to isomorphism.

Definition 2.0.2. Let $\mathcal{K}$ be a 2-category. We call an object $X \in \mathcal{K}$ *discrete* if for any $\mathbb{A} \in \mathcal{K}$, the hom-category $\mathcal{K}(\mathbb{A}, X)$ is isomorphic to a set i.e. is in the essential image of the functor $\mathbf{disc} : \mathbf{Set} \to \mathbf{Cat}$.

Below, we give an alternative, equivalent definition of discrete objects in a 2-category which will be useful in our work. The proof is straightforward by unwinding the definition.

Lemma 2.0.3. *Let $X \in \mathcal{K}$. Then X is discrete if and only if any 2-cell*

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & X \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

must have the property that $f = g$ and $\alpha = 1_f$.

We write **Disc**$(\mathcal{K})$ for the full sub-2-category of $\mathcal{K}$ consisting of discrete objects. Note that **Disc**$(\mathcal{K})$ is a locally discrete 2-category, so we can equally think of it as a 1-category without loss of generality. Frequently in this thesis, we will write **Disc**$(\mathcal{K})$ to mean the underlying 1-category.

Notation 2.0.4. In this thesis we will use the following conventions for font.

- Calligraphic font $\mathcal{E}$, $\mathcal{C}$, $\mathcal{K}$ will be used for categories, with the letter $\mathcal{K}$ typically being reserved for a 2-category.
- Ordinary mathematical font will be used for the contents of a 1-category. These will typically be capitals X, Y, Z when they are objects, and lower case f, g, h when they are morphisms.
- Blackboard bold $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ will be used for general objects in a 2-category. Discrete objects will be denoted by ordinary mathematical font A, B, C to emphasise that they are equivalent to objects in a 1-category.
- Greek letters will typically be used for 2-cells i.e. $\alpha : f \Rightarrow g$, $\epsilon : LR \Rightarrow 1$.
- When we need to be even more careful in distinguishing data in **Cat**$(\mathcal{E})$ from data in $\mathcal{E}$, the former will be either underlined or overlined. As an example, in Definition 2.2.7 we distinguish between the 2-cell $\overline{\alpha} : f \Rightarrow g$ in **Cat**$(\mathcal{E})$, and its components assigner, which is a morphism $\alpha : A_0 \to B_1$ in $\mathcal{E}$.
- Given $f : A \to B$ and $g : C \to B$ in a category, the pullback will be denoted either $A \times_B C$ or $A_f{\times_B}_g C$, when we wish to be clear about which morphisms we are pulling back.
- We denote identity morphisms by 1 or $1_C : C \to C$ or diagrammatically by $C = C$.
- We denote terminal objects by **1**, or occasionally $\underline{\mathbf{1}}$ for a 2-category, when we need to distinguish between the one dimensional terminal object and the two dimensional terminal object. When we denote the terminal object in **Set**, we sometimes denote this $\{*\}$ to emphasise that it is a set with a unique element $*$. Similarly, we denote initial objects by **0** or $\underline{\mathbf{0}}$. Again, for the initial object in **Set** we sometimes denote this by $\emptyset$.

2.1 Notions of fibration

We begin by establishing some special types of morphisms in a 2-category, which are characterised by the property of being able to lift data.

We begin with a definition which tells us that a morphism has discrete fibres.

Definition 2.1.1.

- In **Cat** a *discrete fibration* is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that if we have a morphism in $\mathbb{Y}$ of the form $\phi : y \to Fx$, then there exists a unique $\psi : x' \to x$ in $\mathbb{X}$ such that $F(\psi) = \phi$.

- In **Cat** a *discrete opfibration* is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that if we have a morphism in $\mathbb{Y}$ of the form $\phi : Fx \to y$, then there exists a unique $\psi : x \to x'$ in $\mathbb{X}$ such that $F(\psi) = \phi$.

- For a 2-category $\mathcal{K}$, a functor $F : \mathbb{X} \to \mathbb{Y}$ is called a *discrete (op)fibration* if, for all $\mathbb{A} \in \mathcal{K}$, the functor

$$\mathcal{K}(\mathbb{A}, F) : \mathcal{K}(\mathbb{A}, \mathbb{X}) \to \mathcal{K}(\mathbb{A}, \mathbb{Y})$$

is a discrete (op)fibration in **Cat**.

Remark 2.1.2. The above definition is representable, however we can unwind this definition to give the notion of discrete opfibration without reference to **Cat**. Let $F : \mathbb{X} \to \mathbb{Y}$ be a discrete opfibration in an arbitrary 2-category $\mathcal{K}$. Then, for any diagram:

$$\begin{tikzcd}
 & \mathbb{X} \arrow[d, "F"] \\
\mathbb{A} \arrow[ur, bend left, "x"] \arrow[r, bend left, "Fx"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbb{Y}
\end{tikzcd}$$

there exists a unique 2-cell $\psi : x \to x'$ as indicated below such that $y = Fx'$ and $F.\psi = \phi$.

$$\begin{tikzcd}
 & \mathbb{X} \arrow[d, "F"] \\
\mathbb{A} \arrow[ur, bend left=50, "x"] \arrow[ur, dashed, bend right, "x'"'] \arrow[ur, phantom, "\exists!\psi"] \arrow[r, bend right, "y"'] & \mathbb{Y}
\end{tikzcd}$$

In **Cat**, by taking $\mathbb{A} = \mathbf{1}$, we recover the definition given in Definition 2.1.1 part (1). Conversely, given Definition 2.1.1 part (1), it is not too hard to show that this is enough to lift natural transformations in **Cat** by the definition of natural transformations.

In **Cat**, this precisely says that the fibres above every point are discrete. In fact, we have the following, which relates discrete opfibrations to discrete objects.

Lemma 2.1.3. *If $\mathcal{K}$ has a terminal object $\mathbf{1}$, then $X \in \mathcal{K}$ is a discrete object if and only if the unique map $! : X \to \mathbf{1}$ is a discrete opfibration.*

Proof. As the definition of discrete object and discrete opfibration are both representable, it is enough to check this in **Cat**. Let $\mathbb{X} \to \mathbf{1}$ be a discrete opfibration. It is clear that for any $x \in \mathbb{X}$, $1_x : x \to x$ in $\mathbb{X}$ is a lifting of $1_* :!(x) \to *$ in $\mathbf{1}$.

But then since any $p : x \to x'$ is also necessarily a lifting of $1_* :!(x) \to *$ in **1**, then by the uniqueness of lifts, it follows that $p = 1_x$, and so $\mathbb{X}$ is discrete. The converse is easy to see. $\square$

A general definition of fibration (otherwise known as Grothendieck fibration) drops the need of uniqueness of the lift. However, we will not need this; instead we drop the need for uniqueness of a lift, but add the restriction that we can only lift isomorphisms in $\mathcal{K}$ to obtain the notion of an *isofibration*. Note that in a $(2,1)$-category, since all 2 cells are isomorphisms, the notions of isofibration and Grothendieck fibration coincide.

Definition 2.1.4.

- In **Cat** an *isofibration* is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that if we have an isomorphism in $\mathbb{Y}$ of the form $\phi : y \cong Fx$, then there exists an isomorphism $\psi : x' \cong x$ such that $F(\psi) = \phi$.

- For a 2-category $\mathcal{K}$, a functor $F : \mathbb{X} \to \mathbb{Y}$ is called an *isofibration* if, for all $\mathbb{A} \in \mathcal{K}$, the functor

$$\mathcal{K}(\mathbb{A}, F) : \mathcal{K}(\mathbb{A}, \mathbb{X}) \to \mathcal{K}(\mathbb{A}, \mathbb{Y})$$

is an isofibration in **Cat**.

We can recover an inherent description of isofibration in a 2-category $\mathcal{K}$ without reference to **Cat** by replacing every instance of 2-cell in Remark 2.1.2 with an *invertible* 2-cell, and dropping the requirement for uniqueness. For our purposes, we will take an algebraic perspective on isofibrations; an isofibration is a map $F : \mathbb{X} \to \mathbb{Y}$ in a 2-category together with a *choice* of lifts for each invertible 2-cell $\alpha : G \Rightarrow H : \mathbb{A} \to \mathbb{Y}$— these are sometimes called *cloven* isofibrations. Note that this is not needed in the case of discrete (op)fibrations since lifts are unique. Further explorations of isofibrations are given in Chapter 6.

The third and final notion of fibration to be utilised in this thesis is that of a 2-sided discrete fibration.

Definition 2.1.5. [LR20] In **Cat**, a 2-sided discrete fibration is a span $(p, q) : \mathbb{E} \to \mathbb{B} \times \mathbb{A}$ such that:

- For any $\phi : qe \to a$ in $\mathbb{A}$, there is a unique lift $\psi : e \to e'$ in $\mathbb{E}$ with $q(\psi) = \phi$ and $pe' = pe$.

- For any $\xi : b \to pe'$ in $\mathbb{B}$, there is a unique lift $\zeta : e \to e'$ in $\mathbb{E}$ such that $p(\zeta) = \xi$ and $qe = qe'$.

- For any $\theta : e \to e'$ in $\mathbb{E}$, the codomain of the unique lift of $q(\theta) : qe \to qe'$ equals the domain of the unique lift of $p(\theta) : pe \to pe'$ and their composite equals θ— that is we obtain $\psi : e \to e''$ and $\zeta : e'' \to e'$ such that $\zeta \circ \psi = \theta$.

This allows us to give the definition of 2-sided discrete fibration in any 2-category $\mathcal{K}$.

Definition 2.1.6. [LR20] For a 2-category $\mathcal{K}$, a 2-sided discrete fibration is a span $(p, q) : \mathbb{E} \to \mathbb{B} \times \mathbb{A}$ such that for all $\mathbb{X} \in \mathcal{K}$, the span in **Cat** $(\mathcal{K}(\mathbb{X}, p), \mathcal{K}(\mathbb{X}, q)) : \mathcal{K}(\mathbb{X}, \mathbb{E}) \to \mathcal{K}(\mathbb{X}, \mathbb{B}) \times \mathcal{K}(\mathbb{X}, \mathbb{A})$ is a 2-sided discrete fibration in **Cat**.

By Definition 2.1.5 parts (1) and (2), it is clear that a 2-sided discrete fibration $(p, q) : \mathbb{E} \to \mathbb{B} \times \mathbb{A}$ consists of a discrete fibration $p : \mathbb{E} \to \mathbb{A}$ and a discrete opfibration $q : \mathbb{E} \to \mathbb{A}$, and part (3) of the definition says that these must interact well with each other. In **Cat** (and in any 2-category with the structure of a duality involution), a 2-sided discrete fibration $(p, q) : \mathbb{E} \to \mathbb{B} \times \mathbb{A}$ is equivalently a discrete opfibration $(p, q) : \mathbb{E} \to \mathbb{B}^{\mathrm{op}} \times \mathbb{A}$ [Web07].

2.2 Internal categories and the 2-category $\mathbf{Cat}(\mathcal{E})$

Internal categories were formally introduced by Grothendieck in [Gro60], but their structure was already implicit in [Ehr59] and further early applications to differential geometry appeared in the subsequent [Ehr63]. See chapter 8 of [Bor94] for a modern textbook account of internal category theory, and B2 of [Joh02b] for its relation to topos theory.

Let Δ denote the skeleton of the 'simplex category', whose objects are non-empty finite ordered sets and morphisms are order preserving functions. Identify each object in Δ with its representative ordered set $[n] := \{0, 1, 2..., n\}$. For $k \leq n$, let $\delta^n_k : [n] \to [n+1]$ denote the unique monotonic function whose image does not contain $k \in [n+1]$ and let $\sigma^n_k : [n+1] \to [n]$ denote the unique monotonic function mapping two elements to k and one element to every other possible output. Let $\Delta_{\leq 3}$ denote the full-subcategory of Δ on the objects $[n]$ for $0 \leq n \leq 3$.

Definition 2.2.1. A category *internal* to a locally small category $\mathcal{E}$ is a diagram in $\mathcal{E}$ as displayed below left, which sends the pushout squares in $\Delta_{\leq 3}$ displayed below right to pullback squares in $\mathcal{E}$.

$$\Delta^{\mathrm{op}}_{\leq 3} \xrightarrow{\quad \mathbb{C} \quad} \mathcal{E} \qquad\qquad \begin{tikzcd} n+2 \arrow[dr, phantom, "\lrcorner" very near start] & n+1 \arrow[l, "\delta_2^{n+1}"'] \\ n+1 \arrow[u, "\delta_0^{n+1}"] & n \arrow[l, "\delta_1^n"] \arrow[u, "\delta_0^n"'] \end{tikzcd}$$

Remark 2.2.2. We unpack this definition, and establish notation and terminology which we will use in this thesis. A category internal to $\mathcal{E}$, denoted $\mathbb{C} := (C_0, C_1, d_0, d_1, i, m)$ is given by the datum of a diagram in $\mathcal{E}$ as displayed below.

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

The objects $C_0, C_1 \in \mathcal{E}$ are called the *object of objects* and *object of arrows* respectively, and the morphisms d_1, d_0, i, m are called *source*, *target*, *identity assigner* and *composition*. The *object of composable n-tuples C_n* for $n \in \{2, 3\}$ are pullbacks as depicted below.

$$\begin{tikzcd} C_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] \arrow[dr, phantom, "\lrcorner" very near start] & C_2 \arrow[d, "\pi_1"] \\ C_2 \arrow[r, "\pi_0"'] & C_1 \end{tikzcd}$$

These data are subject to axioms asserting the commutativity of the diagrams displayed below.

- Sources and targets for identities and composites:

$$\begin{tikzcd} C_0 \arrow[r, "i"] \arrow[dr, "1_{C_0}"'] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \quad \begin{tikzcd} C_0 \arrow[r, "i"] \arrow[dr, "1_{C_0}"'] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd} \quad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_0"'] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \quad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

- The associativity and left and right unit laws for composition:

$$\begin{tikzcd} C_3 \arrow[r, "m_0"] \arrow[d, "m_1"'] & C_2 \arrow[d, "m"] \\ C_2 \arrow[r, "m"'] & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "i_0"] \arrow[dr, "1_{C_1}"'] & C_2 \arrow[d, "m"] & C_1 \arrow[l, "i_1"'] \arrow[dl, "1_{C_1}"] \\ & C_1 & \end{tikzcd}$$

Where the morphisms $m_0 := (m\pi_{3,0}, \pi_1\pi_{3,1})$, $m_1 := (\pi_0\pi_{3,0}, m\pi_{3,1})$, $i_0 := (id_0, 1_{C_1})$ and $i_1 := (1_{C_0}, id_1)$ are induced by the universal property of C_2 as a pullback. For example, the equation required for m_0 to be well-defined is witnessed by the following calculation.

$$d_1.m.\pi_{3,0} = d_1.\pi_1.\pi_{3,0} = d_1.\pi_0.\pi_{3,1} = d_0.\pi_1.\pi_{3,1}$$

These conditions correspond to the simplicial identities which must be preserved by functoriality of $\mathbb{C} : \Delta^{\mathrm{op}}_{\leq 3} \to \mathcal{E}$.

Definition 2.2.3. Let $\mathcal{E}$ be a category with pullbacks and let $\mathbb{A}, \mathbb{B} : \Delta^{\mathrm{op}}_{\leq 3} \to \mathcal{E}$ be categories internal to $\mathcal{E}$. An *internal functor* from $\mathbb{A}$ to $\mathbb{B}$ is a natural transformation as depicted below.

$$\begin{tikzcd} \Delta^{\mathrm{op}}_{\leq 3} \arrow[r, bend left, "\mathbb{A}"] \arrow[r, bend right, "\mathbb{B}"'] \arrow[r, phantom, "f\Downarrow"] & \mathcal{E} \end{tikzcd}$$

Remark 2.2.4. Internal functors can also be defined explicitly as given by a *component on objects* $f_0 : A_0 \to B_0$ and a *component on arrows* $f_1 : A_1 \to B_1$ in $\mathcal{E}$ which satisfy the commutativity of the diagrams shown in 2.2.4. Here the morphism $f_2 := (f_1\pi_0, f_1\pi_1)$, is induced by the universal property of B_2, as witnessed by the following calculation

$$d_1.f_1.\pi_0 = f_0.d_1.\pi_0 = f_0.d_0.\pi_1 = d_0.f_1.\pi_1$$

The component $f_3 : A_3 \to B_3$ is uniquely determined from this information by the universal property of B_3 in a similar way. The diagrams below express f's respect for sources, targets, identities, and composition, and they all correspond to naturality conditions for $f : \mathbb{A} \to \mathbb{B}$.

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "d_0^{\mathbb{A}}"'] & B_1 \arrow[d, "d_0^{\mathbb{B}}"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \quad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "d_1^{\mathbb{A}}"'] & B_1 \arrow[d, "d_1^{\mathbb{B}}"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \quad \begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "i^{\mathbb{A}}"'] & B_0 \arrow[d, "i^{\mathbb{B}}"] \\ A_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd} \quad \begin{tikzcd} A_2 \arrow[r, "f_2"] \arrow[d, "m^{\mathbb{A}}"'] & B_2 \arrow[d, "m^{\mathbb{B}}"] \\ A_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd}$$

The morphism f_2 is thought of as taking a composable pair in $\mathbb{A}$ and returning the composable pair given by its image under f. Given $(x, y) : X \to A_2$, the morphism f_2 composes with (x, y) to give (f_1x, f_1y), and so the equation $f_1m\,(x, y) = m\,(f_1x, f_1y)$ follows by respect for composition.

Remark 2.2.5. It is evident from their definition that internal categories and internal functors form a category, in fact a full subcategory of $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$. We write this category as $\mathbf{Cat}\,(\mathcal{E})_1$, using the subscript '1' to distinguish it from the 2-category $\mathbf{Cat}(\mathcal{E})$ which we will recall in Proposition 2.2.8. In particular, $\mathbf{Cat}\,(\mathcal{E})_1$ is small (resp. locally small) if $\mathcal{E}$ is small (resp. locally small), since $\Delta^{\mathrm{op}}_{\leq 3}$ is certainly small. The inclusion functor $N : \mathbf{Cat}\,(\mathcal{E})_1 \hookrightarrow [\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$, which sends an internal category to its underlying truncated simplicial object in $\mathcal{E}$, is called the *nerve*.

Proposition 2.2.6. *Consider the functors $(-)_0, (-)_1 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$, which send an internal category to its object of objects and object of arrows respectively.*

1. $(-)_1 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ *is faithful.*

2. $(-)_0$ *and* $(-)_1$ *preserve and jointly reflect limits.*

Proof. For part (1), let $f, g : \mathbb{A} \to \mathbb{B}$ be internal functors in $\mathcal{E}$ such that $f_1 = g_1$. We need to show that $f = g$. Since $f_1 = g_1$, in particular $f_1 i^{\mathbb{A}} = g_1 i^{\mathbb{A}}$. Since f and g both preserve identities, this is equivalent to saying that $i^{\mathbb{B}} f_0 = i^{\mathbb{B}} g_0$. But by sources (or targets) for identities in $\mathbb{B}$, we may compose these equal morphisms in $\mathcal{E}$ with the source (or target) map of $\mathbb{B}$ to see that $f_0 = g_0$. For part (2), it is standard that the family of functors $(-)_n : [\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}] \to \mathcal{E}$ for $n \leq 3$ preserve and jointly reflect limits, and that limits in $\mathbf{Cat}(\mathcal{E})_1$ are computed in $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$. But since the outputs for $n \in \{0, 1\}$ are enough to determine the rest of an internal category structure, it follows that $(-)_0$ and $(-)_1$ also jointly reflect limits. $\square$

We now review how $\mathbf{Cat}(\mathcal{E})_1$ can be upgraded to a 2-category by incorporating the internal natural transformations of Definition 2.2.7, to follow.

Definition 2.2.7. Given internal functors $(f_0, f_1), (g_0, g_1) : \mathbb{A} \to \mathbb{B}$, an *internal natural transformation*

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\overline{\alpha}\Downarrow"] & \mathbb{B} \end{tikzcd}$$

is a morphism $\alpha : A_0 \to B_1$ called the *component assigner*, making the following diagrams in $\mathcal{E}$ commute.

- Assignation of components: the commutative diagrams displayed below left and below centre commutes.
- Internal naturality: the square displayed below right commutes, where the morphisms $\alpha_0 := (\alpha d_1, g_1) : A_1 \to B_2$ and $\alpha_1 := (f_1, \alpha d_0) : A_1 \to B_2$ are induced by the universal property of B_2.

$$\begin{tikzcd} A_0 \arrow[r, "\alpha"] \arrow[dr, "f_0"'] & B_1 \arrow[d, "d_1"] \\ & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "\alpha"] \arrow[dr, "g_0"'] & B_1 \arrow[d, "d_0"] \\ & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"'] & B_2 \arrow[d, "m"] \\ B_2 \arrow[r, "m"'] & B_1 \end{tikzcd}$$

Internal natural transformations correspond to simplicial homotopies $\{\alpha_{0,\ldots,n} : A_n \to B_{n+1}\}_{n \in \mathbb{N}}$ [GJ09], but are once again determined by significantly less data than in the setting of general simplicial objects due to the universal property of pullbacks in $\mathcal{E}$.

Proposition 2.2.8 (Proposition 8.1.4 of [Bor94], Section 1.4 of [Mir18])**.** *Let $\mathcal{E}$ be a category with pullbacks. Categories, functors and natural transformations internal to $\mathcal{E}$ form a 2-category $\mathbf{Cat}(\mathcal{E})$ whose underlying category is $\mathbf{Cat}(\mathcal{E})_1$, identity 2-cells $\overline{1_f}$ have component assigners given by if_0, vertical composite of 2-cells below left has component assigner given by the morphism in $\mathcal{E}$ depicted below right.*

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\overline{\alpha}"] \arrow[r, phantom, shift right=3, "\Downarrow\overline{\beta}"] & \mathbb{B} \end{tikzcd} \qquad\qquad A_0 \xrightarrow{(\alpha,\beta)} B_2 \xrightarrow{m} B_1$$

The left whiskering and right whiskering pictured below are defined as the composites in $\mathcal{E}$ given by βf_0 and $g_1\alpha$ respectively, and the horizontal composition of 2-cells is defined via whiskering and vertical composition in the usual way as described in Proposition II 3.1 of [Mac71].

$$\begin{tikzcd} \mathbb{A} \arrow[r, "f"] & \mathbb{B} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\overline{\beta}"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] & \mathbb{B} \arrow[r, "g"] & \mathbb{C} \end{tikzcd}$$

If $\mathcal{E}$ is small (resp. locally small), then **Cat** $(\mathcal{E})$ *is small (resp. has small hom-categories).*

Further background on properties of the 2-category **Cat**$(\mathcal{E})$ will be reviewed in Remark 4.2.1.

Fully-faithfulness for internal functors is recalled in Definition 2.2.9, to follow. Unlike in the enriched setting, this is equivalent to the representably defined notion of fully-faithfulness for morphisms in **Cat**$(\mathcal{E})$.

Definition 2.2.9. Let $\mathcal{E}$ be a category with products. An internal functor $f : \mathbb{A} \to \mathbb{B}$ is called

- *faithful* if the morphism into the pullback induced by the following commutative square is a monomorphism.
- *fully faithful* if the induced morphism into the pullback is an isomorphism.

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "(d_0,d_1)"'] & B_1 \arrow[d, "(d_0,d_1)"] \\ A_0 \times A_0 \arrow[r, "f_0 \times f_0"'] & B_0 \times B_0 \end{tikzcd}$$

Remark 2.2.10. An internal functor (f_0, f_1) is a monomorphism in **Cat**$(\mathcal{E})$ if and only if it is faithful and f_0 is a monomorphism. In Section 4.6 we will relate subobject classifiers in $\mathcal{E}$ to classifiers for morphisms in **Cat**$(\mathcal{E})$ which are both fully faithful and monomorphisms; these notions being definable representably in **Cat**$(\mathcal{E})$. In Subsection 4.7.2, we will exhibit such morphisms in **Cat**$(\mathcal{E})$ as the right class $\mathcal{R}'$ of an orthogonal factorisation system, giving an internal version of the analysis in Section 5.2 of [BG14]. The left class $\mathcal{L}'$ of this factorisation system will consist of internal functors $f : \mathbb{A} \to \mathbb{B}$ for which $f_0 : A_0 \to B_0$ are epimorphisms in $\mathcal{E}$. This will allow us to detect them via the 2-category structure of **Cat**$(\mathcal{E})$, despite the fact that representables $\mathcal{E}(X, -) : \mathcal{E} \to$ **Set** typically fail to preserve or jointly reflect epimorphisms. The class $\mathcal{L}'$ features in our categorification of the axiom of choice, in Definition 4.7.12.

Definition 2.2.11. Let $\mathcal{E}$ be a category with pullbacks. A *groupoid internal to* $\mathcal{E}$ is an internal category $(C_0, C_1, d_0, d_1, i, m)$ together with $(-)^{-1} : C_1 \to C_1$ satisfying the following equations.

$$\begin{tikzcd} C_1 \arrow[r, "(-)^{-1}"] \arrow[dr, "d_0"'] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "(-)^{-1}"] \arrow[dr, "d_1"'] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r, "(1_{C_1},1_{C_1})"] \arrow[d, "d_1"'] & X_1{}_{d_0}\times_{C_0}{}_{d_0}X_1 \arrow[r, "1\times_{C_0}(-)^{-1}"] & X_1{}_{d_0}\times_{C_0}{}_{d_1}X_1 \arrow[d, "m"] \\ C_0 \arrow[rr, "i"'] & & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "(1_{C_1},1_{C_1})"] \arrow[d, "d_0"'] & X_1{}_{d_1}\times_{C_0}{}_{d_1}X_1 \arrow[r, "(-)^{-1}\times_{C_0}1"] & X_1{}_{d_0}\times_{C_0}{}_{d_1}X_1 \arrow[d, "m"] \\ C_0 \arrow[rr, "i"'] & & C_1 \end{tikzcd}$$

Define **Gpd**$(\mathcal{E})$ to be the full sub-2-category of **Cat**$(\mathcal{E})$ whose objects are the internal groupoids.

These equations state that $(-)^{-1}$ internally inverts morphisms in $\mathbb{C}$. For $\mathcal{E} = $ **Set**, the first two equations say that given a category $\mathbb{C}$ and a morphism $f : c \to c'$, there is a morphism $(-)^{-1} \circ f : c' \to c$; the second two equations encode the fact that this is a two-sided inverse to f.

Note that given $\overline{\alpha} : f \Rightarrow g : \mathbb{X} \to \mathbb{Y}$ corresponding to $\alpha : X_0 \to Y_1$ is invertible with inverse given by

$$\alpha^{-1} := X_0 \xrightarrow{\alpha} Y_1 \xrightarrow{(-)^{-1}} Y_1.$$

It is a standard calculation to see that this forms an internal natural transformation $\overline{\alpha}^{-1} : g \Rightarrow f : \mathbb{X} \to \mathbb{Y}$ and by the process of composing 2-cells described in Proposition 2.2.8 that this is actually an inverse for $\overline{\alpha}$. Therefore, **Gpd**$(\mathcal{E})$ is a $(2,1)$-category.

2.3 Adjunctions between $\mathcal{E}$ and Cat$(\mathcal{E})_1$

We review some adjunctions between **Cat**$(\mathcal{E})_1$ to $\mathcal{E}$. These adjunctions will be invaluable in our proofs that various universal properties in one of these categories imply analogous properties in the other.

Remark 2.3.1. The functor $(-)_0 : $ **Cat**$(\mathcal{E})_1 \to \mathcal{E}$ has a left adjoint **disc** $: \mathcal{E} \to$ **Cat** $(\mathcal{E})_1$. This sends $X \in \mathcal{E}$ to the internal category **disc**$(X) : \Delta^{\text{op}}_{\leq 3} \to \mathcal{E}$ which is constant at X. The components of the unit of this adjunction on $X \in \mathcal{E}$ are all given by identities, and as such **disc** $: \mathcal{E} \to$ **Cat**$(\mathcal{E})_1$ is fully faithful. Meanwhile, the components of the counit on an internal category $\mathbb{A}$ are given by the internal functor whose component on objects is 1_{A_0} and component on arrows is $i : A_0 \to A_1$. It is easy to see that the naturality square for the counit on an internal functor f is a pullback precisely if f reflects identities, in the sense that the square $f_1 i = i f_0$ is a pullback. It is also easy to see that **disc** preserves finite limits, even when it does not have the left adjoint that will be described in Remark 2.3.4.

Remark 2.3.2. When $\mathcal{E}$ has products, $(-)_0 : $ **Cat**$(\mathcal{E})_1 \to \mathcal{E}$ also has a right adjoint, which we call **indisc** $: \mathcal{E} \to$ **Cat**$(\mathcal{E})_1$. This sends X to the internal category defined by $\{n \mapsto X^n\}$, with n-simplices given by the n-fold product for $n \in \Delta^{\text{op}}_{\leq 3}$. When $\mathcal{E} = $ **Set**, this is the groupoid with set of objects is X and a unique morphism between any two objects. The counit of $(-)_0 \dashv$ **indisc** is the identity, and as such **indisc** $: \mathcal{E} \to$ **Cat**$(\mathcal{E})_1$ is fully faithful. Meanwhile the unit has its component on an internal category $\mathbb{A}$ given by the internal functor $\eta_{\mathbb{A}} : \mathbb{A} \to$ **indisc**$(\mathbb{A})$ which is given by the identity on objects, and the morphism $(d_0, d_1) : A_1 \to A_0 \times A_0$ between objects of arrows.

We call internal categories of the form **indisc**(X) for some $X \in \mathcal{E}$ *indiscrete*. Note that there are other names for this in the literature: *chaotic, codiscrete, coarse* and *Brandt*.

Remark 2.3.3. The counit of **disc** $\dashv (-)_0$ and the unit of $(-)_0 \dashv$ **indisc** both have components which are internal functors given by isomorphisms (indeed, identities) between objects of objects. Isomorphism on objects internal functors

$f : \mathbb{A} \to \mathbb{B}$ play a special role in the 2-category $\mathbf{Cat}(\mathcal{E})$. They are strongly left orthogonal to fully faithful internal functors, in the sense of Definition 2.3.3 of [Bou10]. Indeed, they form the left class of an orthogonal factorisation system in $\mathbf{Cat}(\mathcal{E})_1$, for which the right class are the fully faithfuls. This factorisation is constructed via certain 2-categorical limits and colimits in $\mathbf{Cat}(\mathcal{E})$, which we will describe in more detail in Remark 4.2.1.

Remark 2.3.4. Assume $\mathcal{E}$ has coequalisers of reflexive pairs. Then **disc** has a left adjoint $\Pi_0 : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$ which sends every internal category $\mathbb{A}$ to the codomain of the coequaliser q_A of its source and target, and every internal functor $(f_0, f_1) : \mathbb{A} \to \mathbb{B}$ to the morphism shown below, which is induced by the universal property of $\Pi_0(\mathbb{A})$, given the serial commutativity of the square on the left.

$$
\begin{tikzcd}
A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f_1"'] & A_0 \arrow[r, "q_{\mathbb{A}}"] \arrow[d, "f_0"'] & \Pi_0(\mathbb{A}) \arrow[d, dashed, "\Pi_0(f)"] \\
B_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B_0 \arrow[r, "q_{\mathbb{B}}"'] & \Pi_0(\mathbb{B})
\end{tikzcd}
$$

Since **disc** $: \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ is fully faithful, the component of the counit on an object $X \in \mathcal{E}$ can again be chosen to be the identity. Meanwhile, the component of the unit $q : 1_{\mathbf{Cat}(\mathcal{E})_1} \Rightarrow \mathbf{disc} \circ \Pi_0$ on an internal category $\mathbb{A}$ is given on objects by the coequaliser $q_{\mathbb{A}}$ above, and on arrows by the subsequent composite from A_1 to $\Pi_0(\mathbb{A})$. The triangle identities can be shown using the universal properties of the coequalisers.

For the proof of Theorem 4.4.13 we will need the more nuanced observation that there is also a natural bijection

$$\mathcal{E}(\Pi_0(\mathbb{A}), B) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbb{A}, \mathbf{disc}(B)),$$

defined whenever the coequaliser of the source and target morphisms for $\mathbb{A}$ exists in $\mathcal{E}$. It is straightforward to see that this also holds, via a similar argument to the one sketched above.

2.4 Two dimensional regularity and exactness

In this section, we review some notions of 2-dimensional regularity and exactness, given in [BG14]. We begin by recalling the 1-dimensional notions, so that we can draw parallels.

Definition 2.4.1. Let $\mathcal{E}$ be a category with pullbacks. A *regular epimorphism* is a morphism which is the coequaliser of some parallel pair.

Note that in general, a regular epimorphism is an epimorphism, but not every epimorphism needs to be a regular epimorphism.

Definition 2.4.2. Let $\mathcal{E}$ be a category with pullbacks. An *internal equivalence relation* is an $\mathbb{X} \in \mathbf{Gpd}(\mathcal{E})$ such that $(d_1, d_0) : X_1 \to X_0 \times X_0$ is a monomorphism.

Remark 2.4.3. For $\mathcal{E} = \mathbf{Set}$ and a set X, this recovers the definition of equivalence relation. Given for $x, x' \in X$ we define an equivalence relation R on X by xRx' if there is a morphism $f : x \to x'$. In this case, the identity assigner of the groupoid $\mathrm{id}_x : x \to x$ corresponds to reflexivity (xRx), the fact that morphisms are invertible in a groupoid corresponds to the symmetry property of the equivalence relation ($xRx' \iff x'Rx$) and the composition operation corresponds to transitivity of the relation (xRx' and $x'Rx'' \implies xRx''$). We note that the condition that (d_1, d_0) is a monomorphism means that there cannot be two morphisms $f, g : x \to x'$ and so there is at most one witness of the relation between x and x'.

Given $f : X \to Y$ in $\mathcal{E}$, we can form the kernel pair of f:

$$\begin{tikzcd}
X\,{}_f{\times}_Y{}_f\,X \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

We use the notation for the pullback below:

$$\begin{tikzcd}
X\,{}_f{\times}_Y{}_f\,X\,{}_f{\times}_Y{}_f\,X \arrow[r, "p"] \arrow[d, "q"'] & X\,{}_f{\times}_Y{}_f\,X \arrow[d, "d_1"] \\
X\,{}_f{\times}_Y{}_f\,X \arrow[r, "d_0"'] & X.
\end{tikzcd}$$

We form a map $m : X\,{}_f{\times}_Y{}_f\,X\,{}_f{\times}_Y{}_f\,X \to X\,{}_f{\times}_Y{}_f\,X$ by the universal property of the pullback given the commutativity of the outside of the following diagram:

$$\begin{tikzcd}
X\,{}_f{\times}_Y{}_f\,X\,{}_f{\times}_Y{}_f\,X \arrow[r, "p"] \arrow[d, "q"'] \arrow[drr, dashed, "m"] & X\,{}_f{\times}_Y{}_f\,X \arrow[drr, "d_1"] & & \\
X\,{}_f{\times}_Y{}_f\,X \arrow[drr, "d_0"'] & & X\,{}_f{\times}_Y{}_f\,X \arrow[r, "d_0"] \arrow[d, "d_1"'] & X \arrow[d, "f"] \\
& & X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Similarly, we form a map $i : X \to X\,{}_f{\times}_Y{}_f\,X$ by the universal property of the pullback:

$$\begin{tikzcd}
X \arrow[dr, dashed, "i"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& X\,{}_f{\times}_Y{}_f\,X \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

This data forms an equivalence relation:

Definition 2.4.4. Let $\mathcal{E}$ be a 2-category and let $f : X \to Y$ be a morphism in $\mathcal{E}$. The *kernel* of f, denoted $K(f)$ is the internal equivalence relation given by

$$\begin{tikzcd}
X\,{}_f{\times}_Y{}_f\,X \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & X\,{}_f{\times}_Y{}_f\,X \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l, "i" description]
\end{tikzcd}$$

We note that the colimit of this diagram is the same as the coequaliser of the pair $d_1, d_0 : X_f \times_{Y_f} X \to X$. Hence, we may refer to the kernel of f as just this data, and the colimit of the kernel gives a regular epimorphism.

We write $QK(f)$ for the colimit of the kernel of f, when it exists.

Definition 2.4.5. Let $\mathcal{E}$ be a 2-category and let $f : X \to Y$ be a regular epimorphism. We say that f is *effective* if f is the coequaliser of its kernel, i.e. $f : X \to Y = X \to QK(f)$. We say that an internal equivalence relation is *effective* if it is the kernel of its coequaliser.

Definition 2.4.6. A category $\mathcal{E}$ is called *regular* if:

(R1) It is finitely complete.

(R2) For any morphism $f : X \to Y$, the kernel $K(f)$ admits a coequaliser.

(R3) The pullback of a regular epimorphism along any morphism is again a regular epimorphism.

It is called exact if internal equivalence relations are effective.

Example 2.4.7. The categories **Set**, **Ab**, and $\mathcal{E}$ for any topos are exact and therefore also regular. The category of stone spaces is regular but not exact. Neither the 1-category of categories nor the category of partially ordered sets is regular.

In one dimensions, the stipulation that an internal equivalence relation has the property that $(d_1, d_0) : X_1 \to X_0 \times X_0$ is a monomorphism says that there is at most one way in which two elements can be related. In two dimensions, we want there to be a set of ways in which two elements can be related— this is the perspective given by proof-relevant mathematics— hence we replace the notion of monomorphism by that of 2-sided discrete fibration (Definition 2.1.6). For higher dimensional internal equivalence relations in a 2-category, we also drop the symmetry condition, giving a higher dimensional version of a preorder. This gives the following notion, first due to Bourn-Pennon [BP78] and studied further by Bourke [Bou10]. Firstly, let $\Delta^-_{\leq 2}$ denote the full subcategory of Δ on the objects $[n]$ for $0 \leq n \leq 2$ with the extra condition that $\Delta^-_{\leq 2}([2],[1]) = \emptyset$; let $W : \Delta^-_{\leq 2} \to \mathbf{Cat}$ denote the evident inclusion.

Definition 2.4.8. A *codescent object* is a W-weighted 2-colimit of a diagram $\mathbb{X} : (\Delta^-_{\leq 2})^{\mathrm{op}} \to \mathcal{K}$. We denote the 2-colimit of $\mathbb{X}$ by $Q\mathbb{X}$; the associated morphism $\mathbb{X}_0 \to Q\mathbb{X}$ is called a *codescent morphism*.

Definition 2.4.9. Let $\mathcal{K}$ be a 2-category. A *catead* $\mathbb{X}$ is a category internal to $\mathcal{K}$

$$\begin{tikzcd} \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & \mathbb{X}_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{X}_0 \arrow[l, "i" description] \end{tikzcd}$$

such that the span of the source and target map $X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_0$ forms a 2-sided discrete fibration.

By definition, a catead in $\mathcal{K}$ can be thought of as a functor $\mathbb{X} : (\Delta^-_{\leq 2})^{\mathrm{op}} \to \mathcal{K}$; hence it makes sense to ask if it has codescent object.

Example 2.4.10. In $\mathbf{Cat}(\mathcal{E})$, codescent morphisms are internal functors which are isomorphism-on-objects [Bou10]. Given $\mathbb{X} = (X_0, X_1, d_1, d_0, i, m) \in \mathbf{Cat}(\mathcal{E})$, we can consider X_0 and X_1 as discrete internal categories; it follows easily from discreteness that $(d_1, d_0) : X_1 \to X_0 \times X_0$ forms a 2-sided discrete opfibration and so we can form the catead:

$$
\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 . \arrow[l, "i" description]
\end{tikzcd}
$$

The codescent object is $\mathbb{X}$ itself and the codescent morphism is the inclusion of X_0 into $\mathbb{X}$ that is equal-on-objects.

Due to this, internal categories are precisely the proof-relevant directed internal equivalence relations for a 1-category $\mathcal{E}$. This is made precise in [BG14], which shows that the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ is a kind of exact completion of $\mathcal{E}$ under these equivalence relations. We discuss this further below.

In a 2-category $\mathcal{K}$, given $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{A} \to \mathbb{Y}$, we denote the comma object by

$$
\begin{tikzcd}
f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathbb{X} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow"] \\
\mathbb{A} \arrow[r, "g"'] & \mathbb{Y}.
\end{tikzcd}
$$

We write $f \downarrow f \downarrow f$ for the pullback below:

$$
\begin{tikzcd}
f \downarrow f \downarrow f \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & f \downarrow f \arrow[d, "d_1"] \\
f \downarrow f \arrow[r, "d_0"'] & \mathbb{X}.
\end{tikzcd}
$$

We can define a map $m : f \downarrow f \downarrow f \to f \downarrow f$ by the universal property of the comma object:

$$
\begin{tikzcd}
f \downarrow f \downarrow f \arrow[r, "p"] \arrow[d, "q"'] \arrow[drr, dashed, "m"] & f \downarrow f \arrow[drr, "d_1"] & & \\
f \downarrow f \arrow[drr, "d_0"'] & & f \downarrow f \arrow[r, "d_0"] \arrow[d, "d_1"] & \mathbb{X} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow"] \\
& & \mathbb{X} \arrow[r, "f"'] & \mathbb{Y}.
\end{tikzcd}
$$

Similarly, we define a map $i : \mathbb{X} \to f \downarrow f$ by the universal property of the comma object as below.

$$
\begin{tikzcd}
\mathbb{X} \arrow[dr, dashed, "i"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& f \downarrow f \arrow[r, "d_0"] \arrow[d, "d_1"] & \mathbb{X} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow"] \\
& \mathbb{X} \arrow[r, "f"] & \mathbb{Y}.
\end{tikzcd}
$$

Definition 2.4.11. [Str04] Let $\mathcal{K}$ be a 2-category and let $f : \mathbb{X} \to \mathbb{Y}$ be a morphism in $\mathcal{K}$. The *higher kernel* of f, denoted $\mathbf{K}(f)$ is the catead given by

$$
\begin{tikzcd}
f \downarrow f \downarrow f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f \downarrow f \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{X} \arrow[l, "i" description]
\end{tikzcd}
$$

We write $Q\mathbf{K}(f)$ for the codescent object of the higher kernel of f, when it exists.

Definition 2.4.12. Let $\mathcal{K}$ be a 2-category and let $f : \mathbb{X} \to \mathbb{Y}$ be a codescent morphism. We say that f is *effective* if f is the codescent morphism of its higher kernel, i.e. $f : \mathbb{X} \to \mathbb{Y} = \mathbb{X} \to Q\mathbf{K}(f)$. We say that a catead is *effective* if it is the higher kernel of its codescent object.

More precisely, this means that a catead

$$\begin{tikzcd} \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & \mathbb{X}_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{X}_0 \arrow[l, "i" description] \end{tikzcd}$$

with codescent morphism $f : \mathbb{X}_0 \to Q\mathbb{X}$ is effective if

$$\begin{tikzcd} \mathbb{X}_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathbb{X}_0 \arrow[d, "f"] \arrow[dl, phantom, "\swArrow"] \\ \mathbb{X}_0 \arrow[r, "f"'] & Q\mathbb{X} \end{tikzcd}$$

exhibits the comma object $f \downarrow f$.

We arrive at a definition of exactness for 2-categories.

Definition 2.4.13. [BG14, Theorem 43, Theorem 45] A 2-category is called *BO-regular* when

(BO1) It is finitely complete.

(BO2) Codescent objects of cateads exist.

(BO3) Codescent morphisms are stable under pullback.

(BO4) Codescent morphisms are effective.

(BO5) If $f : \mathbb{A} \to \mathbb{B}$ is a codescent morphism, so is $\delta_f : \mathbb{A} \to \mathbb{A} \times_{\mathbb{B}} \mathbb{A}$.

It is called *BO-exact* if cateads are effective.

Example 2.4.14. The 2-category **Cat** is BO-exact (and hence BO-regular), as is $\mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ any 1-category with pullbacks.

In contrast, the category **Set** when thought of as a locally discrete 2-category is not BO-exact nor BO-regular; in fact it satisfies (BO1)-(BO4) but not (BO5). In this case, codescent morphisms are exactly given by regular epimorphisms, and so satisfy (BO1)-(BO4) by the exactness properties of **Set**. The diagonal of a regular epimorphism is a regular epimorphism if and only if it is an isomorphism and so (BO5) can be thought of as a condition that ensures that we have 2-dimensional behaviour.

Remark 2.4.15. We note that the "BO" in "BO-regular/ BO-exact" stands for "bijective-on-objects"; the paper [BG14] constructs a general framework for regularity and exactness in a $\mathcal{V}$-enriched category, and the notion of regularity or exactness corresponds to an orthogonal factorisation system on $\mathcal{V}$. The BO-regular/exact definition corresponds to the (bijective on objects, fully faithful) factorisation system on **Cat**.

Later, we also give the definition of "SO-regularity/ exactness"— this corresponds instead to the (surjective on objects, injective on objects and fully faithful) factorisation system on **Cat**.

Remark 2.4.16. We might also want to relax the condition that the 2-category be finitely complete to include more examples, such as the 2-category of pseudofunctors, pseudonatural transformations and modifications on functor 2-category. This does not have all finite limits, but does have pullbacks along isofibrations, terminal object and arrow objects. These conditions are enough to form the definition given above, and were suggested by Bourke; in [Hel24], this condition is known as having PITA limits. For the purposes of this thesis, we stick with the condition of working with finitely complete 2-categories.

Proposition 2.4.17. *Let $\mathcal{K}$ be a BO-regular 2-category. Then the classes (codescent morphism, fully faithful) forms an orthogonal factorisation system on $\mathcal{K}$.*
Consequently, codescent morphisms in a BO-regular category are closed under pushouts, coproducts (transfinite) compositions and all isomorphisms are codescent morphisms.

Proof. See [BG14, Proposition 4] and note that this is an example of a kernel-quotient factorisation system studied in that paper, which converges immediately since $\mathcal{K}$ is BO-regular.

More precisely, one can factorise any arrow as a codescent morphism followed by a fully faithful functor by taking it's higher kernel and forming the codescent object of this; the second factor is necessarily a fully faithful functor. □

In [BG14], it is proven that $\mathbf{Cat}(\mathcal{E})$ is always BO-exact— in fact it is the BO-exact completion of the 1-category $\mathcal{E}$ [BG14, Corollary 61]. An important theorem [CV98, Theorem 16] recognises that a 1-category with some (co)limits $\mathcal{E}$ is an exact completion if and only if it has enough projectives— in this case, it is the exact completion of its projective objects. The theorem below, due to [Bou10, Theorem 4.18] states a 2-dimensional version of this result. It roughly says that a 2-category is the BO-exact completion of a 1-category if and only if it has enough BO-projectives and discrete objects are BO-projective. We first recall the definition of BO-projectivity.

Definition 2.4.18. [Bou10, Definition 4.13] Let $\mathcal{K}$ be a 2-category. An object $\mathbb{X} \in \mathcal{K}$ is called *BO-projective* if the functor $\mathcal{K}(\mathbb{X}, -) : \mathcal{K} \to \mathbf{Cat}$ preserves codescent morphisms.

Proposition 2.4.19. *[Bou10, Theorem 4.18] If $\mathcal{E}$ is a category with pullbacks then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ satisfies the conditions listed below, then there is a 2-equivalence*

$$\mathcal{K} \simeq \mathbf{Cat}\,(\mathcal{E})$$

where $\mathcal{E} := \mathbf{Disc}\,(\mathcal{K})$.

1. *$\mathcal{K}$ has pullbacks and powers by $\mathbf{2}$.*

2. *$\mathcal{K}$ has codescent objects of cateads.*

3. *Codescent morphisms and cateads are effective in $\mathcal{K}$.*

4. *Discrete objects in $\mathcal{K}$ are BO-projective.*

5. *For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*

The second notion of exactness we will explore in this thesis is the notion of SO-exactness. We describe this below. Let $\hat{\Delta}^{-}_{\leq 2}$ be the category $\Delta^{-}_{\leq 2}$ appended with an extra object $[1]'$ and singular morphism $j : [1] \to [1]'$. Let $W' : \hat{\Delta}^{-}_{\leq 2} \to$ **Cat** be the evident inclusion.

Definition 2.4.20. An *SO-quotient* is a W'-weighted 2-colimit of a diagram $\mathbb{X} : (\hat{\Delta}^{-}_{\leq 2})^{\mathrm{op}} \to \mathcal{K}$. We denote the 2-colimit by $Q\mathbb{X}$ and the associated morphism $\mathbb{X}_0 \to Q\mathbb{X}$ is called an *SO-regular epimorphism*.

We note that in a 2-category with codescent objects and coidentifiers, an SO-quotient can be calculated by first calculating the codescent object of the associated catead, then taking a coidentifier to impose the extra compatibility with j.

Definition 2.4.21. [BG14, Proposition 27] Let $\mathcal{K}$ be a 2-category. An SO-congruence is a diagram in $\mathcal{K}$ as below

$$\begin{tikzcd} & \mathbb{X}_{2'} \arrow[d, "j"] & \\ \mathbb{X}_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & \mathbb{X}_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{X}_0 \arrow[l, "i" description] \end{tikzcd} \tag{2.1}$$

such that

- $(\mathbb{X}_0, \mathbb{X}_1, d_1, d_0, i, m)$ is a catead.
- $(d_1 j, d_0 j) : \mathbb{X}_{2'} \to \mathbb{X}_0$ is an internal equivalence relation in $\mathcal{U}(\mathcal{K})$ in which $\mathcal{U}(\mathcal{K})$ is the underlying 1-category of $\mathcal{K}$.
- The map j is a full monomorphism.
- The graph homomorphism

$$\begin{tikzcd} \mathbb{X}_{2'} \arrow[r, "j"] \arrow[d, shift right=2, "d_1 j"'] \arrow[d, shift left=2, "d_0 j"] & \mathbb{X}_1 \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] \\ \mathbb{X}_0 \arrow[r, equal] & \mathbb{X}_0 \end{tikzcd}$$

  is an internal functor.

By definition, an SO-congruence in $\mathcal{K}$ can be thought of as a functor $\mathbb{X} : (\hat{\Delta}^{-}_{\leq 2})^{\mathrm{op}} \to \mathcal{K}$; hence it makes sense to ask if an SO-congruence has an SO-quotient.

Remark 2.4.22. The definition of full monomorphism used in the above is given in Definition 4.6.1, but these are defined representably; $f : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$ is a full monomorphism if for all $\mathbb{A} \in \mathcal{K}$ the functor $\mathcal{K}(A, f)$ is a full monomorphism in **Cat**. Note that this is equivalent to saying fully faithful and injective on objects.

These are the diagrams that we wish to have well-behaved quotients of. For any $f : \mathbb{X} \to \mathbb{Y}$, we can construct an SO-congruence as below:

$$\begin{tikzcd}
 & \mathbb{X}{}_f\times_{\mathbb{Y}}{}_f\mathbb{X} \arrow[d, "j"] & \\
f\downarrow f\downarrow f \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & f\downarrow f \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{X} \arrow[l, "i" description]
\end{tikzcd}$$

We call this the SO-kernel of f and denote the associated diagram by $\mathbf{Ker}(F)$. We write $Q'\mathbf{Ker}(F)$ for the 2-colimit of this diagram, if it exists.

Definition 2.4.23. Let $\mathcal{K}$ be a 2-category and let $f : \mathbb{X} \to \mathbb{Y}$ be an SO-regular epimorphism. We say that f is *effective* if f is the 2-colimit of its SO-kernel i.e. $f : \mathbb{X} \to \mathbb{Y} = \mathbb{X} \to Q\mathbf{Ker}(f)$. We say that an SO-congruence is *effective* if it is the SO-kernel of its 2-colimit.

Definition 2.4.24. [BG14, Theorem 36, Theorem 38] A 2-category is called *SO-regular* if

(SO1) It is finitely complete.

(SO2) For any morphism $F : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$, the associated SO-congruence $\mathbf{Ker}(F)$ admits a 2-colimit.

(SO3) The pullback of any SO-regular epimorphism is an SO-regular epimorphism.

(SO4) SO-regular epimorphisms are effective.

An SO-regular category is called *SO-exact* if SO-congruences are effective.

Example 2.4.25. The 2-category **Cat** is SO-exact (resp. SO-regular) and so is $\mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ an exact (resp. regular) 1-category. Any SO-exact 2-category is BO-exact. Hence **Set** is not SO-exact. However, a locally discrete 2-category is SO-regular if and only if it is regular when considered as a 1-category. Hence **Set** *is* SO-regular.

2.5 Two dimensional extensivity

In this section, we review some notions of 2-dimensional extensivity. One version appears in [HM25, Definition 5.1] (which is Definition 4.4.1 in this thesis). This suffices for the purposes there, but in Chapter 7 we need a more genuinely 2-dimensional definition of extensivity that interacts well with isofibrational slicing (See Remark 7.7.12). This notion is due to [Shub]. There, it notes that in general these definitions do not seem to agree.

In this thesis, we use Definition 4.4.1 only in the case we are talking about 2-categories of internal categories; we show that for $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, the differing notions of extensivity do actually agree.

We start by recalling the 1-dimensional case.

Definition 2.5.1. [CLW93, Definition 2.1] A 1-category with pullbacks $\mathcal{E}$ is called *extensive* if it has finite coproducts and the functor

$$+ : \mathcal{E}/X \times \mathcal{E}/Y \to \mathcal{E}/X + Y$$

is an equivalence of categories.

Example 2.5.2. The categories **Set**, **Cat**, **Top** are all extensive 1-categories. The category **Vect** is not an extensive 1-category.

In [CLW93], a few alternative characterisations are worked out.

Definition 2.5.3. In a 1-category $\mathcal{E}$, a coproduct $A + B$ is called *disjoint* if the following commutative squares are pullbacks:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "\iota_A"] \\ A \arrow[r, "\iota_A"'] & A+B. \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "\iota_B"] \\ B \arrow[r, "\iota_B"'] & A+B. \end{tikzcd} \qquad \begin{tikzcd} \mathbf{0} \arrow[r] \arrow[d] & A \arrow[d, "\iota_A"] \\ B \arrow[r, "\iota_B"'] & A+B. \end{tikzcd} \tag{2.2}$$

A coproduct $A + B$ is called *universal* if given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "g"] & Y \arrow[l] \arrow[d, "h"] \arrow[dl, phantom, "\llcorner"] \\ A \arrow[r, "\iota_A"'] & A+B & B \arrow[l, "\iota_B"] \end{tikzcd}$$

in which each of the individual squares is a pullback as indicated, then $Z \cong X + Y$ and $g \cong f + h$.

Extensivity can be captured by the property that coproducts are universal.

Lemma 2.5.4. [CLW93, Proposition 2.2] *A* 1*-category with finite products $\mathcal{E}$ is extensive if and only if pullbacks along coproduct injections exist and coproducts are universal.*

Similarly, extensivity can be captured by the property that coproducts are disjoint and stable under pullback.

Lemma 2.5.5. [CLW93, Proposition 2.6, Lemma 2.11] *A* 1*-category with finite products $\mathcal{E}$ is extensive if and only if coproducts are disjoint and stable under pullback.*

Remark 2.5.6. Note that Lemma 2.5.5 implies that coproduct inclusions are monomorphisms— this is precisely what the first two diagrams in Equation (2.2) say.

We now move onto the 2-dimensional case. The following appears in [HM25, Definition 5.1] (which is Definition 4.4.1 in this thesis).

Definition 2.5.7. Call a 2-category with pullbacks $\mathcal{K}$ *extensive* if it has finite coproducts and the 2-functor

$$+ : \mathcal{K}/_{\mathbb{X}} \times \mathcal{K}/_{\mathbb{Y}} \to \mathcal{K}_{\mathbb{X}+\mathbb{Y}}$$

is a 2-equivalence.

The following notions were first given in [Shub].

Definition 2.5.8. Let $\mathcal{K}$ be a 2-category with pullbacks and finite coproducts.

- We call a coproduct $\mathbb{A}+\mathbb{B}$ *disjoint* if we have comma squares:

$$\begin{tikzcd} \mathbb{A}^{\mathbf{2}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathbb{A} \arrow[d, "\iota_{\mathbb{A}}"] \arrow[dl, Rightarrow] \\ \mathbb{A} \arrow[r, "\iota_{\mathbb{A}}"'] & \mathbb{A}+\mathbb{B}. \end{tikzcd} \quad \begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathbb{B} \arrow[d, "\iota_{\mathbb{B}}"] \arrow[dl, Rightarrow] \\ \mathbb{B} \arrow[r, "\iota_{\mathbb{B}}"'] & \mathbb{A}+\mathbb{B}. \end{tikzcd} \quad \begin{tikzcd} \mathbf{0} \arrow[r] \arrow[d] & \mathbb{A} \arrow[d, "\iota_{\mathbb{A}}"] \arrow[dl, Rightarrow] \\ \mathbb{B} \arrow[r, "\iota_{\mathbb{B}}"'] & \mathbb{A}+\mathbb{B}. \end{tikzcd} \quad \begin{tikzcd} \mathbf{0} \arrow[r] \arrow[d] & \mathbb{B} \arrow[d, "\iota_{\mathbb{B}}"] \arrow[dl, Rightarrow] \\ \mathbb{A} \arrow[r, "\iota_{\mathbb{A}}"'] & \mathbb{A}+\mathbb{B}. \end{tikzcd}$$

- We call a coproduct $\mathbb{A}+\mathbb{B}$ *universal* if given a commutative diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d, "g"] & \mathbb{Y} \arrow[l] \arrow[d, "h"] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{A} \arrow[r, "\iota_{\mathbb{A}}"'] & \mathbb{A}+\mathbb{B} & \mathbb{B} \arrow[l, "\iota_{\mathbb{B}}"] \end{tikzcd}$$

  in which each of the individual squares is a pullback as indicated, then $\mathbb{Z} \cong \mathbb{X}+\mathbb{Y}$ and $g \cong f+h$.

- We call $\mathcal{K}$ 2*-extensive* if coproducts are universal and disjoint.

Proposition 2.5.9. *Let $\mathcal{E}$ be a category with pullbacks. The 2-category $\mathbf{Cat}(\mathcal{E})$ is extensive (in the sense of Definition 2.5.7) if and only if it is 2-extensive (in the sense of Definition 2.5.8).*

Proof. By Proposition 4.4.2, $\mathbf{Cat}(\mathcal{E})$ is extensive if and only if $\mathcal{E}$ is extensive. Now, if $\mathbf{Cat}(\mathcal{E})$ is 2-extensive, then it is not hard to see that $\mathcal{E}$ is extensive; coproducts in $\mathcal{E}$ can be considered discrete coproducts in $\mathbf{Cat}(\mathcal{E})$ via the functor $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$, which is a left adjoint and so preserves coproducts. This coproduct is universal in $\mathbf{Cat}(\mathcal{E})$, and this implies the the original coproduct in $\mathcal{E}$ is universal; by Lemma 2.5.4, this implies $\mathcal{E}$ is extensive.

It remains to show that if $\mathcal{E}$ is extensive, then $\mathbf{Cat}(\mathcal{E})$ is 2-extensive. This can be shown using the fact that both coproducts and pullbacks are calculated pointwise, and using the explicit description of a comma object as a pullback. The proof that we have universal coproducts becomes straightforward by pointwise calculation. We explain that the following commutative square:

$$\begin{tikzcd} \mathbb{A}^{\mathbf{2}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathbb{A} \arrow[d, "\iota_{\mathbb{A}}"] \\ \mathbb{A} \arrow[r, "\iota_{\mathbb{A}}"'] & \mathbb{A}+\mathbb{B}. \end{tikzcd}$$

is a comma object; the proof for the other squares in the definition of disjoint coproducts is similar.

The comma object of the cospan $\iota_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}+\mathbb{B} \leftarrow \mathbb{A} : \iota_{\mathbb{A}}$ can be written as the pullback square below.

$$\begin{tikzcd} \iota_{\mathbb{A}} \downarrow \iota_{\mathbb{A}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathbb{A}+\mathbb{B})^{\mathbf{2}} \arrow[d, "(d_1, d_0)"] \\ \mathbb{A} \times \mathbb{A} \arrow[r] & (\mathbb{A}+\mathbb{B}) \times (\mathbb{A}+\mathbb{B}). \end{tikzcd}$$

Since we are in an extensive category, $(\mathbb{A} + \mathbb{B})^{\mathbf{2}} \cong \mathbb{A}^{\mathbf{2}} + \mathbb{B}^{\mathbf{2}}$ since powers are preserved under the 2-equivalence of Definition 2.5.7. Since pullbacks are computed pointwise, this means that on objects we have the following pullback square in $\mathcal{E}$.

$$\begin{tikzcd} (\iota_{\mathbb{A}} \downarrow \iota_{\mathbb{A}})_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 + B_1 \arrow[d, "{(d_1, d_0)}"] \\ A_0 \times A_0 \arrow[r, "\iota_{A_0} \times \iota_{A_0}"'] & (A_0 + B_0) \times (A_0 + B_0). \end{tikzcd} \tag{2.3}$$

Note that here, we have used that $\mathbb{A}_0^{\mathbf{2}} = A_1$; this is proven in Section 4.2.

Now, in an extensive category, $(A_0 + B_0) \times (A_0 + B_0) \cong (A_0 \times A_0) + (B_0 \times B_0) + (A_0 \times B_0) + (B_0 \times A_0)$ and under this isomorphism, $\iota_{A_0} \times \iota_{A_0} \cong \iota_{A_0 \times A_0}$— this is the property of distributivity [CLW93, Proposition 4.5]. By disjointness of coproducts, $(d_1, d_0) : A_1 + B_1 \to (A_0 + B_0) \times (A_0 + B_0)$ factors through $(A_0 \times A_0) + (B_0 \times B_0)$ under this isomorphism, and so by universality of the coproduct we have the property that $A_1 \cong (\iota_{\mathbb{A}} \downarrow \iota_{\mathbb{A}})_0$. The same proof strategy applies on morphisms, proving that $\iota_{\mathbb{A}} \downarrow \iota_{\mathbb{A}} \cong \mathbb{A}^{\mathbf{2}}$, as required.

□

We relate two dimensional extensivity and one dimensional extensivity in the following.

Lemma 2.5.10. *Let $\mathcal{K}$ be a 2-extensive category. Then* **Disc**$(\mathcal{K})$ *is extensive as a* 1*-category.*

Proof. For discrete objects X, 2-cells must be identities and $X^{\mathbf{2}} \cong X$, so the diagrams witnessing disjoint and universal coproducts in the definition of 2-extensivity implies that coproducts are disjoint and universal in **Disc**$(\mathcal{K})$ in the one dimensional sense; hence **Disc**$(\mathcal{K})$ is extensive by Lemma 2.5.4. □

Example 2.5.11. By the above, since **Set** is extensive as a 1-category, it follows that when considered as a locally discrete 2-category, **Set** is extensive in both senses, and that **Cat** $=$ **Cat**(**Set**) is extensive in both senses.

Chapter 3

Colimits of internal categories

3.1 Introduction

3.1.1 Context and Motivation

Whilst coproducts in **Cat** are easily calculated by working levelwise, the coequaliser of a parallel pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ has a more complicated description involving not just equivalence classes of objects and morphisms of $\mathcal{D}$ but also equivalence classes of paths, as described in [BBP99]. Together, these constructions ensure that **Cat** has all finite colimits, which is a foundational result in the field.

The goal of this work is to provide conditions on a category $\mathcal{E}$ such that the 2-category $\mathbf{Cat}(\mathcal{E})$ of internal categories, internal functors, and internal natural transformations has finite 2-colimits. In order to show that a 2-category has finite 2-colimits it suffices to show that it has coproducts, copowers by the free-living arrow in **Cat** (which we denote **2**) and coequalisers [Kel89, §3]. Extensivity of an $\mathcal{E}$ with pullbacks suffices for coproducts and copowers by **2** to exist in $\mathbf{Cat}(\mathcal{E})$, as shown in Lemma 4.4.2 and Theorem 4.4.5 and reviewed in Section 3.2. We show that for a 1-category $\mathcal{E}$, the property of having pullback-stable coequalisers (Definition 3.1.3) gives rise to very special coequalisers in $\mathbf{Cat}(\mathcal{E})$, as treated in Section 3.3. However, the following example illustrates that the property of having pullback-stable coequalisers in $\mathcal{E}$ is insufficient for $\mathbf{Cat}(\mathcal{E})$ to have coequalisers.

Example 3.1.1. Let $\mathcal{E} := \mathbf{FinSet}$, the elementary topos of finite sets. Note that elementary toposes are locally cartesian closed, and so have pullback stable coequalisers in the sense of Definition 3.1.3. Consider the following diagram in $\mathbf{Cat}(\mathcal{E})$, in which the two functors pick out the source and target of the free-living arrow.

$$\mathbf{1} \underset{d^0}{\overset{d^1}{\rightrightarrows}} \mathbf{2} \tag{3.1}$$

If a coequaliser to this diagram existed in $\mathbf{Cat}(\mathbf{FinSet})$, then it would have a single object since the d^1 and d^0 would cause the source and target of **2** to be glued together. Given any finite monoid on **1**, we can send unique arrow of **2** to any of the generators of the monoid; this will coequalise Diagram (3.1) and so induces a unique arrow out of the coequaliser by its universal property. Therefore the coequaliser of Diagram (3.1), if it exists, provides the free monoid on the terminal. Remark D5.3.4 of [Joh02b] explains that in an elementary topos $\mathcal{E}$, the existence of such a free monoid is equivalent to $\mathcal{E}$ having a natural numbers object. The category **FinSet** does not have a natural numbers object; this can be deduced from the fact that the natural numbers object satisfies the axioms for Peano arithmetic [LM05, Theorem 2], which cannot be satisfied by any finite set. Therefore, the coequaliser of this diagram does not exist in $\mathbf{Cat}(\mathbf{FinSet})$. In fact, the coequaliser of this diagram in $\mathbf{Cat}(\mathbf{Set})$ is given by the monoid of natural numbers, which is equivalently the free monoid on the singleton set.

The above example shows that having free monoids is important in the construction of coequalisers in $\mathbf{Cat}(\mathcal{E})$. In the absence of cartesian closedness and a subobject classifier, having a *parametrised list object* on $A \in \mathcal{E}$ implies the existence of the free monoid on A. This follows from [Mai10, Proposition 7.3] by restricting the construction of the free internal category on a free internal graph to one object categories and graphs. Since we want to work in a general setting, we will assume that the free internal category on internal graphs exists in $\mathcal{E}$ on top of preservation of coequalisers under pullback.

On the other hand, if we assume that $\mathcal{E}$ is locally finitely presentable, then the existence of 2-colimits in $\mathbf{Cat}(\mathcal{E})$ is relatively easy to prove.

Proposition 3.1.2. *Let $\mathcal{E}$ be accessible. Then $\mathbf{Cat}(\mathcal{E})$ is accessible as a 1-category. Furthermore, if $\mathcal{E}$ also has finite colimits (so is locally finitely presentable), then $\mathbf{Cat}(\mathcal{E})$ has 2-colimits.*

Proof. Recall that $\mathbf{Cat}(\mathcal{E})$ is of the form $\mathbf{Mod}(\mathcal{S}, \mathcal{E})$, the category of models for a finite limit sketch $\mathcal{S}$ in $\mathcal{E}$. As $\mathcal{E}$ is accessible, we can apply [LT23, Proposition 5.13] and deduce that $\mathbf{Mod}(\mathcal{S}, \mathcal{E})$ is accessible. For $\mathcal{E}$ locally finitely presentable, we instead apply Proposition 1.53 of [JJ94], and conclude that $\mathbf{Cat}(\mathcal{E})_1$ is locally finitely presentable, so has finite colimits, in particular coequalisers. Therefore, $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits. □

We restrict ourselves to the elementary setting of an extensive category $\mathcal{E}$ with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. Our main result is Theorem 3.5.2, which says that $\mathbf{Cat}(\mathcal{E})$ has coequalisers under these assumptions. Finite 2-colimits follow as a consequence. Moreover, the assumed properties of $\mathcal{E}$ imply certain properties of $\mathbf{Cat}(\mathcal{E})$. In Section 3.6, we investigate the necessity of the assumed properties on $\mathcal{E}$ by showing that certain 2-categorical properties of $\mathbf{Cat}(\mathcal{E})$ imply these assumptions on $\mathcal{E}$. This is all brought together in Theorem 3.6.8, which shows an equivalence of extensive categories $\mathcal{E}$ with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint and 2-categories satisfying a list of axioms. It should be noted that this theorem is written in purely 2-categorical terms, without reference to the fact that $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$, so that this gives an elementary characterisation of such 2-categories, using the work of [Bou10].

Examples of categories $\mathcal{E}$ satisfying the necessary condition are given in Section 3.7 and include elementary toposes with natural numbers objects. Therefore this work gives a generalisation of the work in [JW78, Corollary 6.10][1]. In particular, we generalise the result from the setting of elementary toposes with natural numbers objects to a setting which does not need cartesian closedness, regularity, exactness or a subobject classifier. Our proof uses a different method to theirs, from which it is easier to understand the actual construction of coequalisers of internal categories. Moreover, our proof is self-contained and does not rely on a monadic functor theorem, potentially allowing us to generalise to more elementary settings. However, it is also of interest that internal categories are exactly the algebras for the free-forgetful monad on internal graphs, and so we provide a proof using the method of [JW78, Corollary 6.10] as well in Appendix B.

The study of 2-categories of internal categories has been of increasing interest in recent years. [BG14] shows that assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ is a kind of 2-exact completion of the 1-category $\mathcal{E}$. 2-categories of internal categories are

[1] We thank Nathanael Arkor for pointing this reference out to us, as well as very helpful discussions.

also of interest for matters relating to 2-dimensional foundations of mathematics. In Chapter 4 we describe the elementary theory of the 2-category of small categories, which extends Lawvere's elementary theory of the category of sets to the higher dimensional setting. This will be extended in Chapter 7, where we will describe $(2,1)$-categories of large groupoids, which should provide examples of elementary $(2,1)$-toposes. Although many possible definitions of elementary 2-toposes have been given [Web07; Str80; Hel24], it is generally agreed that 2-toposes should have 2-colimits. Hence, it is important to understand 2-categories which have 2-colimits, and our present work establishes this for 2-categories of internal categories under appropriate assumptions on $\mathcal{E}$. Relatedly, our result allows for a proof that the model structure on internal categories described in [EKL05] is cofibrantly generated and algebraic, in Chapter 6.

3.1.2 Structure of the chapter

This work is divided into six sections. Section 3.2 recalls the construction of coproducts and copowers by **2** in $\mathbf{Cat}(\mathcal{E})$, and gives a more detailed outline of our strategy in constructing coequalisers. Section 3.3 constructs coequalisers of parallel pairs of internal functors that agree on objects. This simple case allows us to construct coequifiers in $\mathbf{Cat}(\mathcal{E})$. We prove that in this case coequalisers of parallel pairs of internal functors that agree on objects are stable under pullback along discrete Conduché fibrations. Section 3.4 uses free internal categories on internal graphs to construct coequalisers of pairs of arrows out of a discrete category. Section 3.5 brings together all these parts to prove that $\mathbf{Cat}(\mathcal{E})$ has coequalisers for an arbitrary pair of parallel morphisms. Section 3.6 considers results in the converse direction. We isolate pullback stability of codescent coequalisers (Definition 3.6.11) along discrete Conduché fibrations in $\mathbf{Cat}(\mathcal{E})$ as being important as it is equivalent to pullback stability of coequalisers in $\mathcal{E}$ (Proposition 3.3.7 and Proposition 3.6.7). Theorem 3.6.8 states that the assumptions that $\mathcal{E}$ is an extensive category with pullbacks and pullback-stable coequalisers admitting free internal categories over internal graphs is equivalent to the assumptions that $\mathbf{Cat}(\mathcal{E})$ is extensive, has finite 2-colimits, pullbacks and that coequalisers that agree on objects are stable under pullback along discrete Conduché fibrations. Theorem 3.6.12 uses the work of [Bou10] to state this in elementary terms for a general 2-category $\mathcal{K}$ without reference to the fact that $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$. We conclude in Section 3.7 with examples of when our results can be applied. This involves reproving a theorem of Maietti about constructing a left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ in a more general setting, though the proof is the same. We also show, due to an argument by Peter LeFanu Lumsdaine, that list-arithmetic pretoposes have pullback-stable coequalisers.

3.1.3 Notational and terminology

We begin by establishing some notation and terminology. For a parallel pair of arrows $f, g : X \to Y$ and arrow $h : Y \to Z$, we say h *coequalises* f and g for the situation in which $hg = hf$. We say that h is *the coequaliser* of f and g in the universal such case.

Important to our proof is that coequalisers are stable under pullback.

Definition 3.1.3. Let $\mathcal{E}$ be a category with pullbacks. We say that $\mathcal{E}$ has *pullback-stable coequalisers* if it has coequalisers and for any morphism $f : X \to Y$ in $\mathcal{E}$ the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ preserves coequalisers.

More explicitly, this means that if we have $f : X \to Y$ and a coequaliser diagram:

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \xrightarrow{q} C \tag{3.2}$$

in $\mathcal{E}/Y$, then the diagram :

$$f^*(A) \underset{f^*(h)}{\overset{f^*(g)}{\rightrightarrows}} f^*(B) \xrightarrow{f^*(q)} f^*(C) \tag{3.3}$$

is a coequaliser diagram in $\mathcal{E}/X$ (and hence also in $\mathcal{E}$).

Recall that a *regular epimorphism* is a morphism that is the coequaliser of some diagram, and a *regular category* is a category with finite limits, coequalisers of kernel pairs and the property that the pullback of a regular epimorphism is a regular epimorphism.

If $\mathcal{E}$ has pullback stable coequalisers, then regular epimorphisms are stable under pullback. Therefore, a category with pullback stable coequalisers and equalisers is a regular category, since we are assuming that we have pullbacks and coequalisers.

On the other hand, if $\mathcal{E}$ is a regular category with coequalisers, this does not guarantee that we have pullback-stable coequaliser. Given a coequaliser diagram as in Equation (3.2), $q : B \to C$ is a regular epimorphism, so by regularity of $\mathcal{E}$ it follows that $f^*(q) : f^*(B) \to f^*(C)$ is a regular epimorphism, and so it is the coequaliser of some diagram:

$$E \underset{l}{\overset{k}{\rightrightarrows}} f^*(B) \xrightarrow{f^*(q)} f^*(C).$$

However, there is nothing to guarantee that $E = f^*(A)$ and $k = f^*(g), l = f^*(h)$, and so this does not imply pullback stability of coequalisers in the sense of Definition 3.1.3. In our explicit description of coequalisers, it is important that we have this extra control over the pullback of coequalisers.

3.2 Constructing finite 2-colimits of internal categories via simpler colimits

Recall (for example from [Kel89, §3]) that finite 2-colimits can be constructed using finite coproducts, coequalisers of parallel pairs, and copowers by **2**. We briefly review the construction of finite coproducts and copowers by **2** in the 2-category $\mathbf{Cat}(\mathcal{E})$ under the assumption that $\mathcal{E}$ is extensive and has pullbacks. We then outline the construction of coequalisers of parallel pairs in $\mathbf{Cat}(\mathcal{E})$, which we will develop over the subsequent sections.

First, we describe an internal free-living arrow in $\mathbf{Cat}(\mathcal{E})$, which we denote $\mathbf{2}_{\mathcal{E}}$. For any object $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, the cartesian product $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$ will have the universal property of the copower of $\mathbb{A}$ by **2**. The internal category $\mathbf{2}_{\mathcal{E}}$ can be concretely described as a truncated simplicial object, with n-simplices given by the $(n+2)$-fold coproduct of the terminal object $\mathbf{1} \in \mathcal{E}$; see Example 2.3.2 of [Mir18] for further details. Abstractly, it is the image of **2** under $\mathbf{Cat}(F) : \mathbf{Cat}(\mathbf{FinSet}) \to \mathbf{Cat}(\mathcal{E})$, where $F : \mathbf{FinSet} \to \mathcal{E}$ is the unique coproduct and terminal object preserving functor, which is described in Definition 4.4.4. We note that with the additional assumption of cartesian closedness, Proposition 3.2.1 (2) is Theorem 4.4.5 (2), but this proof is more general as we only assume lextensivity.

Proposition 3.2.1. *Let $\mathcal{E}$ be an extensive category with pullbacks. Then* **Cat**$(\mathcal{E})$ *has*

(i) extensive coproducts which are created by N : **Cat**$(\mathcal{E})_1 \to [\Delta^{op}_{\leq 3}, \mathcal{E}]$.

(ii) copowers by **2***, which for an internal category $\mathbb{A}$ are given by* $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$.

Proof. For part (1), the coproduct of a pair of internal categories $\mathbb{A}$ and $\mathbb{B}$ is given levelwise by $n \mapsto \mathbb{A}_n + \mathbb{B}_n$, as proven in Lemma 4.4.2. For part (2), the internal functor $\mathbf{2}_{\mathcal{E}} \times \mathbb{A} \to \mathbb{B}$ corresponding to an internal natural transformation $\overline{\alpha} : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ is given via the description of $\mathbf{2}_{\mathcal{E}}$ by two morphisms $(f_0, g_0) : A_0 + A_0 \to B_0$ and $(f_1, m.\alpha, g_1) : A_1 + A_1 + A_1 \to B_1$ in $\mathcal{E}$. Further details can be found in [Mir18].

□

In light of Proposition 3.2.1, to show that **Cat**$(\mathcal{E})$ has finite 2-colimits it suffices to show that the 2-category **Cat**$(\mathcal{E})$ has coequalisers of parallel pairs. Moreover, since **Cat**$(\mathcal{E})$ has powers by **2**, it suffices to show that the underlying category **Cat**$(\mathcal{E})_1$ has coequalisers of parallel pairs.

A naive attempt at constructing a coequaliser of a pair of internal functors would be to do this levelwise. We have already seen in Example 3.1.1 that this does not work even internal to **Set** since pairs of morphisms may become newly composable once a coequaliser is also taken at the level of objects. In Example 3.1.1, the single non-identity morphism of the free living arrow becomes composable with itself after gluing together its source and target; this new composite is not created by coequalising on morphisms, and so one must take the free category on the graph obtained by coequalising on objects and then morphisms.

Our construction of coequalisers of arbitrary parallel pairs of internal functors $F, G : \mathbb{A} \to \mathbb{B}$ decomposes into the following two steps.

(i) First restrict F and G along $\varepsilon_{\mathbb{A}} : \mathbf{disc}(A_0) \to \mathbb{A}$ and form the coequaliser $K : \mathbb{B} \to \mathbb{D}$ of the parallel pair $F \cdot \varepsilon_{\mathbb{A}}$ and $G \cdot \varepsilon_{\mathbb{A}}$.

$$\begin{tikzcd} \mathbf{disc}(A_0) \arrow[r, bend left, "F\cdot\varepsilon_{\mathbb{A}}"] \arrow[r, bend right, "G\cdot\varepsilon_{\mathbb{A}}"'] & \mathbb{B} \arrow[r, "K"] & \mathbb{D}. \end{tikzcd}$$

In Proposition 3.4.6 we show that if $\mathcal{E}$ is a category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U}$: **Cat**$(\mathcal{E}) \to$ **Gph**$(\mathcal{E})$ has a left adjoint, then coequalisers of parallel pairs of internal functors out of discrete categories exist in **Cat**$(\mathcal{E})$.

(ii) Next, form the coequaliser $P : \mathbb{D} \to \mathbb{C}$ of the parallel pair of internal functors KF and KG.

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "K\cdot F"] \arrow[r, bend right, "K\cdot G"'] & \mathbb{D} \arrow[r, "P"] & \mathbb{C}. \end{tikzcd}$$

Note that since K coequalises $F.\varepsilon_{\mathbb{A}}$ and $G.\varepsilon_{\mathbb{A}}$, the functors KF and KG agree on objects. In Proposition 3.3.1 we show that if $\mathcal{E}$ has pullback-stable coequalisers then **Cat**$(\mathcal{E})$ has coequalisers of parallel pairs of internal functors that agree on objects.

Finally, in Section 3.5 we show that for abstract reasons these steps combine in such a way that $Q := PK : \mathbb{B} \to \mathbb{C}$ is the coequaliser of the original parallel pair $F, G : \mathbb{A} \to \mathbb{B}$. We prove Proposition 3.4.6, as required for step (1) above, using the following two auxiliary constructions.

(i) The construction of free categories on graphs. We use their universal property.

(ii) The construction of coequifiers of parallel pairs of internal natural transformations. We show in Corollary 3.3.3 that when $\mathcal{E}$ has pullback-stable coequalisers then **Cat**$(\mathcal{E})$ has coequifiers of arbitrary pairs of internal natural transformations.

In step (1) above, we first forget about any morphisms in $\mathbb{A}$ and instead generate the coequaliser on objects and consider the graph $\mathcal{G}$ which has equivalence classes of objects in $\mathbb{B}$ as objects and morphisms in $\mathbb{B}$ as edges. The free category on this graph gives us a category whose morphisms are strings of morphisms in $\mathbb{B}$ that become composable once we coequalise on objects. We require an internal functor $\mathbb{B} \to \mathbb{F}(\mathcal{G})$, but the construction so far only guarantees us a morphism of their underlying graphs. The final two coequifiers extend this to a morphism of graphs which respects identities and composition.

Step (2) then considers the morphisms of $\mathbb{A}$, and takes the coequaliser just on morphisms. This requires only exactness properties in $\mathcal{E}$.

Remark 3.2.2. It is interesting to compare this construction with the method used in §4 of [BBP99] in the context of **Cat**. Let $F, G : \mathcal{A} \to \mathcal{B}$. The construction of a coequaliser in [BBP99] first constructs a relation ${}_F{=}_G$ on $\mathcal{B}$ generated by F and G defined on objects by $a \,{}_F{=}_G\, a \in \mathcal{A}_0$ iff $F(a) = G(a)$ and on morphisms by $f \,{}_F{=}_G\, f$ iff $F(f) = G(f)$. It then constructs the *generalised congruence* ${}_F{\simeq}_G$ generated by this relation, which closes this relation on morphisms under some axioms. It then quotients $\mathcal{B}$ by this generalised congruence, and the result is the coequaliser. In contrast, Step (1) of our construction constructs a category in which the *generalised* congruence on $\mathcal{B}$ is simply an ordinary congruence (in the standard sense of [Mac71], for example) on this new category. In other words, the category constructed by Step (1) is the setting in which the generalised congruence is defined. In internal category theory, one must be very careful to state precisely where things are defined. Step (2) takes the usual quotient of a category by a congruence.

We do not, however, attempt to define the notion of a generalised congruence on an internal category.

3.3 Coequalisers of arrows that agree on objects

Throughout this section, $\mathcal{E}$ will be assumed to be a category with pullbacks and pullback-stable coequalisers. The goal of this section is to show that under these assumptions, the 2-category **Cat**$(\mathcal{E})$ has coequalisers of pairs of internal functors $F, G : \mathbb{A} \to \mathbb{B}$ which agree on objects in the sense that the morphisms $F_0, G_0 : A_0 \to B_0$ are equal in $\mathcal{E}$. As a corollary, we find that **Cat**$(\mathcal{E})$ also has coequifiers under these assumptions. Finally, we show that these coequalisers are stable under pullback along discrete Conduché fibrations.

Proposition 3.3.1. *Let $\mathcal{E}$ be a category with pullbacks and pullback-stable coequalisers. Any pair $F, G : \mathbb{A} \to \mathbb{B}$ of internal functors that agree on objects has a coequaliser in* **Cat**$(\mathcal{E})$.

Proof. Consider the limit L in $\mathcal{E}$ of the following diagram:

$$\begin{tikzcd}
 & & L \arrow[dll, dashed, "\pi_0"'] \arrow[d, dashed, "\pi_1"] \arrow[drr, dashed, "\pi_2"] & & \\
B_1 \arrow[dr, "d_0"] & & A_1 \arrow[dl, "F_0 d_1"] \arrow[dr, "F_0 d_0"] & & B_1 \arrow[dl, "d_1"] \\
 & B_0 & & B_0 &
\end{tikzcd}$$

and define $\widetilde{F} := (\pi_0, F_1\pi_1, \pi_2), \widetilde{G} := (\pi_0, G_1\pi_1, \pi_2) : L \to B_3$. We define $Q_1 : B_1 \to C_1$ as the coequaliser of the following parallel pair of arrows in $\mathcal{E}$:

$$\begin{tikzcd}
L \arrow[r, shift left, "\widetilde{F}"] \arrow[r, shift right, "\widetilde{G}"'] & B_3 \arrow[r, "m^2"] & B_1.
\end{tikzcd}$$

We show that (B_0, C_1) turns out to be the objects of objects and morphisms for an internal category which has the universal property of the desired coequaliser. We define source and target $d_0, d_1 : C_1 \to B_0$ using the universal property of the coequaliser given the commutativity of the following diagram for $(i, j) \in \{(0, 2), (1, 0)\}$.

$$\begin{tikzcd}
L \arrow[r, "\widetilde{F}"] \arrow[d, "\widetilde{G}"'] \arrow[dr, "\pi_j"] & B_3 \arrow[r, "m^2"] \arrow[d, "\pi_j"] & B_1 \arrow[dd, "d_i"] \\
B_3 \arrow[r, "\pi_j"] \arrow[d, "m^2"'] & B_1 \arrow[dr, "d_i"] & \\
B_1 \arrow[rr, "d_i"'] & & B_0
\end{tikzcd}$$

We define $i : B_0 \to C_1$ as the composite

$$B_0 \xrightarrow{i} B_1 \xrightarrow{Q_1} C_1.$$

Next, define C_2 as the pullback of $d_0, d_1 : C_1 \to B_0$ and define $Q_2 : B_2 \to C_2$ to be induced by the universal property of the pullback, given the morphisms $Q_1 \cdot \pi_0$ and $Q_1 \cdot \pi_1$. Since coequalisers are stable under pullback in $\mathcal{E}$, it follows that the following is a coequaliser diagram.

$$\begin{tikzcd}
L \times_{B_0} L \arrow[r, shift left, "m^2 \cdot \widetilde{F} \times_{B_0} m^2 \cdot \widetilde{F}"] \arrow[r, shift right, "m^2 \cdot \widetilde{G} \times m^2 \cdot \widetilde{G}"'] & B_2 \arrow[r, "Q_2"] & C_2.
\end{tikzcd}$$

By the same reasoning, the following is also a coequaliser diagram.

$$\begin{tikzcd}
B_3 \times_{B_0} L \arrow[r, shift left, "B_3 \times_{B_0} m^2 \cdot \widetilde{F}"] \arrow[r, shift right, "B_3 \times m^2 \cdot \widetilde{G}"'] & B_2 \arrow[r, "B_3 \times_{B_0} Q_1"] & B_3 \times_{B_0} C_1.
\end{tikzcd}$$

In the following, the morphism $Q_1.m^2 : B_3 \times_{B_0} B_1 \to C_1$ coequalises the pair $B_3 \times_{B_0} m^2 \cdot \widetilde{F}, B_3 \times_{B_0} m^2 \cdot \widetilde{G}$ by definition of the maps involved and associativity of composition in $\mathbb{B}$. This induces an arrow $u : B_3 \times_{B_0} C_1 \to C_1$ by the universal property of the coequaliser.

$$\begin{tikzcd}
B_3 \times_{B_0} L \arrow[r, shift left, "B_3\times_{B_0}\widetilde{F}"] \arrow[r, shift right, "B_3\times_{B_0}\widetilde{G}"'] \arrow[d, "m^3\times_{B_0}A_1\times_{B_0}B_1"'] & B_3 \times_{B_0} B_3 \arrow[r, "B_3\times_{B_0}m^2"] \arrow[d, "m^3"] & B_3 \times_{B_0} B_1 \arrow[d, "m^2"] & \\
L \arrow[r, shift left, "\widetilde{F}"] \arrow[r, shift right, "\widetilde{G}"'] & B_3 \arrow[r, "m^2"] & B_1 \arrow[r, "Q_1"] & C_1
\end{tikzcd}$$

By definition of the coequalisers involved and associativity, the following diagram commutes.

$$\begin{tikzcd}
L \times_{B_0} L \arrow[r, "\widetilde{F}\times_{B_0}L"] \arrow[d, "\widetilde{G}\times_{B_0}L"'] & B_3 \times_{B_0} L \arrow[rr, "B_3\times_{B_0}\widetilde{F}"] \arrow[d, "B_3\times_{B_0}\widetilde{G}"] & & B_3 \times_{B_0} B_3 \arrow[r, "m^4"] \arrow[d, "B_3\times_{B_0}m^2"'] & B_2 \arrow[d, "m"] \\
B_3 \times_{B_0} L \arrow[r, "B_3\times_{B_0}\widetilde{F}"] \arrow[ddd, "B_3\times_{B_0}\widetilde{G}"'] & B_3 \times_{B_0} B_3 \arrow[dr, "B_3\times m^2"] & & B_3 \times_{B_0} B_1 \arrow[r, "m^3"] \arrow[dd, "B_3\times_{B_0}Q_1"] & B_1 \arrow[ddd, "Q_1"] \\
& & B_3 \times_{B_0} B_1 \arrow[dr, "B_3\times_{B_0}Q_1"] & & \\
& B_3 \times_{B_0} B_1 \arrow[rr, "B_3\times_{B_0}Q_1"] \arrow[dr, "m^3"] & & B_3 \times_{B_0} C_1 \arrow[dr, "u"] & \\
B_3 \times_{B_0} B_3 \arrow[ur, "B_3\times_{B_0}m^2"] \arrow[r, "m^2\times_{B_0}m^2"'] & B_2 \arrow[r, "m"'] & B_1 \arrow[rr, "Q_1"'] & & C_1
\end{tikzcd}$$

Therefore, we obtain a unique arrow $m : C_2 \to C_1$ such that $mQ_2 = Q_1 m$.

We claim that $\mathbb{C} := (B_0, C_1, d_0, d_1, i, m)$ forms an internal category. The laws specifying the source and target of identity morphisms are satisfied as shown below:

$$\begin{tikzcd}
B_0 \arrow[r, "i"] \arrow[drr, "B_0"'] & B_1 \arrow[r, "Q_1"] \arrow[dr, "d_i"] & C_1 \arrow[d, "d_i"] \\
& & B_0
\end{tikzcd} \qquad i \in \{0, 1\}.$$

To show that the laws specifying the source and target of composite morphisms are satisfied, we appeal to the universal property of Q_2 as the coequaliser of F_2 and G_2. We show that, for $i \in \{0, 1\}$, the maps $Q_2 d_i m, Q_2 d_i \pi_i : B_2 \to B_0$ are equal in the diagram below. Both maps clearly coequalise $F_2, G_2 : A_2 \to B_2$. By uniqueness aspect of the universal property, it follows that $d_i m = d_i \pi_i$.

$$\begin{tikzcd}
B_2 \arrow[r, "Q_2"] \arrow[dr, "m"] \arrow[ddr, "\pi_i"'] \arrow[ddd, "Q_2"'] & C_2 \arrow[r, "m"] & C_1 \arrow[ddd, "d_i"] \\
& B_1 \arrow[ur, "Q_1"] \arrow[ddr, "d_i"] & \\
& B_1 \arrow[d, "Q_1"'] \arrow[dr, "d_i"] & \\
C_2 \arrow[r, "\pi_i"'] & C_1 \arrow[r, "d_i"'] & C_2
\end{tikzcd}$$

The other axioms follow similarly; for example, the left unit law follows from the fact that by the assumption that coequalisers are closed under pullbacks, the following diagram is a coequaliser diagram:

$$B_0 \times_{B_0} L \underset{B_0 \times_{B_0} m^2 \cdot \widetilde{G}}{\overset{B_0 \times_{B_0} m^2 \cdot \widetilde{F}}{\rightrightarrows}} B_0 \times_{B_0} B_1 \xrightarrow{B_0 \times_{B_0} Q_1} B_0 \times_{B_0} C_1$$

and so we can check the left unit law by showing that the maps

$$m \cdot (i \times_{B_0} C_1) \cdot (B_0 \times_{B_0} Q_1), \pi_1 \cdot (B_0 \times_{B_0} Q_1) : B_0 \times_{B_0} B_1 \to C_1$$

are equal, and since both maps clearly coequalise the diagram above, by uniqueness of the universal property, it follows that $m \cdot (i \times_{B_0} C_1) = \pi_1$.

The right unit law and associativity of composition follows using the same method; the details for associativity can be found in appendix A.

This shows that $\mathbb{C}$ is an internal category.

By definition of $d_0, d_1 : C_1 \to B_0$, $i : B_0 \to C_1$ and $m : C_2 \to C_1$, it also follows that $Q := (1_{B_0}, Q_1)$ is well-defined as an internal functor $\mathbb{B} \to \mathbb{C}$. We now show that it has the universal property of the coequaliser of F and G.

Given

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, "Q"] \arrow[dr, "R"'] & \mathbb{C} \\ & & \mathbb{D} \end{tikzcd}$$

where $RF = RG$, we define $K_1 : C_1 \to D_1$ by the universal property of C_1 as a coequaliser given the following equalities.

$$\begin{aligned} R_1 \cdot m^2 \cdot (B_1 \times_{B_0} F_1 \times_{B_0} B_1) &= m^2 \cdot R_3 \cdot (B_1 \times_{B_0} F_1 \times_{B_0} B_1) \\ &= m^2(R \times_{D_0} F_1 \cdot R \times_{D_0} R) \\ &= m^2(R \times_{D_0} G_1 \cdot R \times_{D_0} R) \\ &= m^2 \cdot R_3 \cdot (B_1 \times_{B_0} G_1 \times_{B_0} B_1) \\ &= R_1 \cdot m^2 \cdot (B_1 \times_{B_0} G_1 \times_{B_0} B_1). \end{aligned}$$

The following diagrams show that $K := (R_0, K_1)$ assembles into a functor $\mathbb{C} \to \mathbb{D}$. The diagrams make use of the universal property of Q_1 and Q_2 as coequalisers. Uniqueness of $K : \mathbb{C} \to \mathbb{D}$ follows from uniqueness of K_1.

$$
\begin{tikzcd}
B_0 \arrow[rr, "R_0"] \arrow[dr, "i"] \arrow[dd, "i"'] & & D_0 \arrow[dd, "i"] \\
& B_1 \arrow[dl, "Q_1"] \arrow[dr, "R_1"] & \\
C_1 \arrow[rr, "K_1"'] & & D_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "Q_1"] \arrow[d, "Q_1"'] \arrow[dr, "d_i"] \arrow[rr, bend right, "R_1"] & C_1 \arrow[r, "K_1"] & D_1 \arrow[d, "d_i"] \\
C_1 \arrow[r, "d_i"'] & B_0 \arrow[r, "R_0"'] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[r, "Q_2"] \arrow[rr, bend right, "R_2"] \arrow[dr, "m"'] \arrow[dd, "Q_2"'] & C_2 \arrow[r, "K_2"] & D_2 \arrow[dd, "m"] \\
& B_1 \arrow[d, "Q_1"] \arrow[dr, "R_1"] & \\
C_2 \arrow[r, "m"] & C_1 \arrow[r, "K_1"] & D_1
\end{tikzcd}
$$

□

We will use the following result to show that coequifiers exist in $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ satisfies the assumptions of Proposition 3.3.1 and is moreover extensive.

Lemma 3.3.2. *Let $\mathcal{K}$ be a 2-category with powers by $\mathbf{2}$. Then the equifier of a parallel pair of 2-cells $\overline{\alpha}, \overline{\beta} : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ exists if and only if the equaliser of the corresponding morphisms $\hat{\alpha}, \hat{\beta} : \mathbb{A} \to \mathbb{B}^{\mathbf{2}}$ exists. In this case, the limits agree.*

Proof. We can check this representably in **Cat**. Recall that an equaliser of $\hat{\alpha}, \hat{\beta} : \mathcal{A} \to \mathcal{B}^{\mathbf{2}}$ in **Cat** is given by the full subcategory of those $a \in \mathcal{A}$ such that $\hat{\alpha}(a) = \hat{\beta}(a)$. Similarly, recall that the equifier of $\overline{\alpha}, \overline{\beta} : f \Rightarrow g$ in **Cat** is given by the full subcategory of $a \in \mathcal{A}$ such that $\overline{\alpha}_a = \overline{\beta}_a$. By definition, $\hat{\alpha}(a) = \alpha_a$ and $\hat{\beta}(a) = \beta_a$, so these define the same thing. □

The corollary to follow records conditions under which the 2-category $\mathbf{Cat}(\mathcal{E})$ has coequifiers. These will be used in the construction of coequalisers of parallel pairs of internal functors whose domains are discrete in Section 3.4.

Corollary 3.3.3. *Let $\mathcal{E}$ be an extensive category with pullbacks and pullback-stable coequalisers. The 2-category $\mathbf{Cat}(\mathcal{E})$ has coequifiers.*

Proof. Consider the parallel pair of internal natural transformations displayed below left. By Lemma 3.3.2 applied to $\mathcal{K} = \mathbf{Cat}(\mathcal{E})^{\mathrm{op}}$, these correspond to the parallel pair of internal functors displayed below right. Observe that both functors are given on objects by the morphism $(F_0, G_0) : A_0 + A_0 \to B_0$. Hence the result follows from Proposition 3.3.1.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D, above}] & \mathbb{B}
\arrow[from=U, to=D, Rightarrow, shift right=2, "\overline{\alpha}"'] \arrow[from=U, to=D, Rightarrow, shift left=2, "\overline{\beta}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{2}_{\mathcal{E}} \times \mathbb{A} \arrow[r, bend left, "\hat{\alpha}"] \arrow[r, bend right, "\hat{\beta}"'] & \mathbb{B}
\end{tikzcd}
$$

□

Remark 3.3.4. We also note that under the assumptions that $\mathcal{E}$ is an extensive category with pullbacks and pullback-stable coequalisers, $\mathbf{Cat}(\mathcal{E})$ also has cocomma objects which are constructed in a similar way. Given a span of functors $\mathbb{A} \xleftarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$, their cocomma has object of objects given by $B_0 + C_0$ and object of morphisms constructed using limits and coequalisers in $\mathcal{E}$. Specifically, first construct the limit L of the diagram displayed below.

$$
\begin{tikzcd}
B_1 \arrow[dr, "d_0"'] & & A_0 \arrow[dl, "F_0"] \arrow[dr, "G_0"'] & & C_1 \arrow[dl, "d_1"] \\
& B_0 & & C_0 &
\end{tikzcd}
$$

When $\mathcal{E} = \mathbf{Set}$ this limit consists of a morphism f in $\mathbb{B}$, a morphism g in $\mathbb{C}$ and a 'heteromorphism' from the target Z of f to the source Y of g whenever there is an object X in $\mathbb{A}$ satisfying $FX = Z$ and $GX = Y$. This heteromorphism will correspond to the component on X of the natural transformation forming part of the cocomma cocone. To ensure that these heteromorphisms form a natural transformation, we next form the coequaliser of a parallel pair of maps from $b, c : A_1 \to L$. These maps are induced by the universal property of L, given the data displayed below left for b and below right for c.

$$
\begin{tikzcd}
& & A_1 \arrow[r, "d_0"] \arrow[dll, "F_1"'] \arrow[dd, "d_0"] & A_0 \arrow[dr, "G_0"] & \\
& & & & C_0 \arrow[d, "i"] \\
B_1 \arrow[dr, "d_0"'] & & A_0 \arrow[dl, "F_0"] \arrow[dr, "G_0"'] & & C_1 \arrow[dl, "d_1"] \\
& B_0 & & C_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_0 \arrow[dl, "F_0"'] & A_1 \arrow[l, "d_0"'] \arrow[dd, "d_0"] \arrow[ddrr, "G_1"] & & \\
C_0 \arrow[d, "i"'] & & & & \\
B_1 \arrow[dr, "d_0"'] & & A_0 \arrow[dl, "F_0"] \arrow[dr, "G_0"'] & & C_1 \arrow[dl, "d_1"] \\
& B_0 & & C_0 &
\end{tikzcd}
$$

We leave details of the proof that this gives a well-defined internal category which has the universal property of a cocomma to the interested reader. Cocommas in $\mathbf{Cat}(\mathcal{E})$ will not be needed in this thesis.

It is not true that coequalisers in $\mathcal{E}$ being stable under pullbacks implies that all coequalisers in $\mathbf{Cat}(\mathcal{E})$ are stable under pullback; a counterexample is given in the case that $\mathcal{E} = \mathbf{Set}$ in Example 5.3.1. However, from our construction it is not too hard to see that given pullback stability of coequalisers in $\mathcal{E}$, a certain class of coequalisers is stable under pullback along discrete Conduché fibrations. Discrete Conduché fibrations can be defined representably in any 2-category, but we recall an equivalent description for internal categories below.

Definition 3.3.5. Let $\mathcal{E}$ be a category with pullbacks. An internal functor $F : \mathbb{X} \to \mathbb{Y}$ is called a *discrete Conduché fibration* if the square displayed below is a pullback.

$$
\begin{tikzcd}
X_2 \arrow[r, "F_2"] \arrow[d, "m"'] & Y_2 \arrow[d, "m"] \\
X_1 \arrow[r, "F_1"'] & Y_1
\end{tikzcd}
$$

Discrete Conduché fibrations are precisely those internal functors in which we can lift composites in $\mathbb{Y}$ that are in the image of F uniquely to composites in $\mathbb{X}$. The following result will be useful.

Lemma 3.3.6. *Let $F : \mathbb{X} \to \mathbb{Y}$ be a discrete Conduché fibration in* $\mathbf{Cat}(\mathcal{E})$*. Then the following square is a pullback.*

$$
\begin{tikzcd}
X_3 \arrow[r, "F_3"] \arrow[d, "m^2"'] & Y_3 \arrow[d, "m^2"] \\
X_1 \arrow[r, "F_1"'] & Y_1
\end{tikzcd}
$$

Proof. It is enough to prove this representably in **Cat**, in which case it is an easy exercise in lifting a composable triple in $\mathbb{Y}$ that is in the image of F to a composable triple in $\mathbb{X}$ by using the discrete Conduché property twice. $\square$

Proposition 3.3.7. *Let $\mathcal{E}$ be a category with pullbacks and pullback-stable coequalisers. Then coequalisers of parallel pairs of internal functors that agree on objects are stable under pullback along discrete Conduché fibrations.*

Proof. Let $F, G : \mathbb{A} \to \mathbb{B}$ in $\mathbf{Cat}(\mathcal{E})/\mathbb{Y}$ with $F_0 = G_0$ and coequaliser $Q : \mathbb{B} \to \mathbb{C}$ and $f : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}(\mathcal{E})$. We will show that $f^*(Q) : f^*(\mathbb{B}) \to f^*(\mathbb{C})$ is the coequaliser of $f^*(F), f^*(G) : f^*(\mathbb{A}) \to f^*(\mathbb{B})$. First, note that it is clear that $f^*(F)_0 = f^*(G)_0$. Following Proposition 3.3.1, the coequaliser of $f^*(F)$ and $f^*(G)$ is given by the coequaliser of the following diagram

$$f_3^*(L) \underset{f^*\left(\widetilde{G}\right)}{\overset{f^*\left(\widetilde{F}\right)}{\rightrightarrows}} f_3^*(B_3) \xrightarrow{f^*(m^2)} f_1^*(B_1).$$

The discrete Conduché condition allows us to rewrite the above diagram as a pullback over f_1 rather than a mixture of f_1 and f_3; by the pullback lemma and Lemma 3.3.6 the outside of the following diagrams are pullbacks.

$$\begin{tikzcd}
f_3^*(L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L \arrow[d] \\
X_3 \arrow[r, "f_3"] \arrow[d, "m^2"'] \arrow[dr, phantom, "\lrcorner"] & Y_3 \arrow[d, "m^2"] \\
X_1 \arrow[r, "f_1"] & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
f_3^*(B_3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_3 \arrow[d] \\
X_3 \arrow[r, "f_3"] \arrow[d, "m^2"'] \arrow[dr, phantom, "\lrcorner"] & Y_3 \arrow[d, "m^2"] \\
X_1 \arrow[r, "f_1"] & Y_1
\end{tikzcd}$$

Therefore, the coequaliser of $f^*(F)$ and $f^*(G)$ is given by the coequaliser of the following diagram.

$$f_1^*(L) \underset{f^*\left(\widetilde{G}\right)}{\overset{f^*\left(\widetilde{F}\right)}{\rightrightarrows}} f_1^*(B_3) \xrightarrow{f^*(m^2)} f_1^*(B_1).$$

Now, since coequalisers are stable under pullback in $\mathcal{E}$, the coequaliser of this diagram is $f_1^*(Q) : f_1^*(B_1) \to f_1^*(C_1)$, as required.

$\square$

Recall that an exponentiable functor is a functor such that pulling back along it has a right adjoint (Definition 5.2.3). As left adjoints, exponentiable functors therefore preserve colimits, in particular coequalisers. Therefore, the previous proposition is trivial for $\mathcal{E} = \mathbf{Set}$ since in **Cat** discrete Conduché functors are examples of exponentiable functors. In fact, for any elementary topos $\mathcal{E}$, [Joh77, Theorem 2.37] tells us that Conduché fibrations are exponentiable in $\mathbf{Cat}(\mathcal{E})$ and so this result is only notable in the cases in which $\mathcal{E}$ is more general.

In Lemma 3.6.7 we prove the converse of the above result. This in particular tells us that Conduché fibrations are not always exponentiable in $\mathbf{Cat}(\mathcal{E})$; when $\mathcal{E}$ does not have pullback-stable coequalisers, Conduché fibrations in $\mathbf{Cat}(\mathcal{E})$ cannot be exponentiable.

In light of Corollary 3.3.3, Proposition 3.3.7 implies coequifiers in $\mathbf{Cat}(\mathcal{E})$ are stable under pullback along discrete Conduché fibrations.

Corollary 3.3.8. *Let $\mathcal{E}$ be an extensive category with pullbacks and pullback-stable coequalisers. Then the 2-category $\mathbf{Cat}(\mathcal{E})$ has coequifiers which are stable under pullback along discrete Conduché fibrations.*

Proof. Let $f : \mathbb{X} \to \mathbb{Y}$ be a discrete Conduché fibration in $\mathbf{Cat}(\mathcal{E})$ and consider a pair of parallel internal natural transformations $\overline{\alpha}, \overline{\beta} : F \Rightarrow G : \mathbb{A} \to \mathbb{B}$ in $\mathbf{Cat}(\mathcal{E})/\mathbb{Y}$. By Corollary 3.3.3, the coequifier of $\overline{\alpha}, \overline{\beta}$ in $\mathbf{Cat}(\mathcal{E})$ can be expressed as a coequaliser of a pair of parallel internal functors that agree on objects $\hat{\alpha}, \hat{\beta} : \mathbf{2}_{\mathcal{E}} \times \mathbb{A} \to \mathbb{B}$. This is in $\mathbf{Cat}(\mathcal{E})/\mathbb{Y}$ via the projection $\mathbf{2}_{\mathcal{E}} \times \mathbb{A} \to \mathbb{A}$. By Proposition 3.3.7, this coequaliser is stable under pullback along f. Moreover, notice that $f^*(\mathbf{2}_{\mathcal{E}} \times \mathbb{A}) \cong \mathbf{2}_{\mathcal{E}} \times f^*(\mathbb{A})$ and so in pulling back, we obtain the coequaliser of the pair $f^*(\hat{\alpha}), f^*(\hat{\beta}) : \mathbf{2}_{\mathcal{E}} \times f^*(\mathbb{A}) \to f^*(\mathbb{B})$ in $\mathbf{Cat}(\mathcal{E})/\mathbb{X}$. Again, by Corollary 3.3.3, this corresponds exactly to the coequifier of the pair of parallel internal natural transformations $f^*(\overline{\alpha}), f^*(\overline{\beta}) : f^*(F) \Rightarrow f^*(G) : f^*(\mathbb{A}) \to f^*(\mathbb{B})$. ☐

3.4 Coequalisers of pairs of arrows out of a discrete category

Throughout this section, we assume that $\mathcal{E}$ is a category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint denoted $\mathbb{F} : \mathbf{Gph}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})_1$. In Section 3.7, we give examples of categories satisfying these conditions; these include elementary toposes with natural numbers objects and list-arithmetic pretoposes. The goal of this Section is to prove that $\mathbf{Cat}(\mathcal{E})$ has coequalisers of pairs of arrows $F, G : A_0 \to \mathbb{B}$ where A_0 is a discrete category. Our proof uses the universal property of the free category on a graph, which we state explicitly in Corollary 3.4.1, to follow. Our proof also uses the fact that if we have pullback-stable coequalisers in $\mathcal{E}$, then $\mathbf{Cat}(\mathcal{E})$ has coequifiers of any parallel pair of natural transformations, as recorded in Corollary 3.3.3.

Corollary 3.4.1. *Let A_0 be a discrete category internal to $\mathcal{E}$ and let $F, G : A_0 \to \mathbb{B}$ be a parallel pair of internal functors. Form the coequaliser $k_0 : B_0 \to C_0$ of the parallel pair $F_0, G_0 : A_0 \to B_0$ in $\mathcal{E}$. Consider the graph $\mathcal{G} := (B_1, C_0, k_0 \cdot d_0, k_0 \cdot d_1)$ internal to $\mathcal{E}$. There is a category $\mathbb{F}(\mathcal{G})$ and a morphism of graphs $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ with the property that for any internal category $\mathbb{H}$ and morphism of graphs $h : \mathcal{G} \to \mathcal{U}(\mathbb{H})$ there is a unique internal functor $h' : \mathbb{F}(\mathcal{G}) \to \mathbb{H}$ satisfying $\mathcal{U}(h') \cdot \eta_{\mathcal{G}} = h$.*

Proof. The morphism of graphs $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is the component of the unit for the adjunction $\mathbb{F} \dashv \mathcal{U}$ at the graph $\mathcal{G}$. The property stated for $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is precisely the universal property of the unit. ☐

Lemma 3.4.2. *There is a morphism of graphs $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ defined on vertices by the coequaliser $k_0 : B_0 \to C_0$ of F_0 and G_0, and on edges by the edge-assignment $(\eta_{\mathcal{G}})_1 : \mathcal{G}_1 = B_1 \to \mathbb{F}(\mathcal{G})_1$.*

Proof. Since $\eta_{\mathcal{G}} : \mathcal{G} \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is a morphism of graphs, we see that for $i \in \{0, 1\}$, the equation displayed below holds.

$$d_i^{\mathbb{F}(\mathcal{G})} \cdot (\eta_{\mathcal{G}})_1 = (\eta_{\mathcal{G}})_0 \cdot d_i^{\mathcal{G}} = k_0 \cdot d_i^{\mathbb{B}} \tag{3.4}$$

This is because $k_1 \cdot d_1^{\mathbb{B}} : B_1 \to C_0$ is the source of $\mathcal{G}$ and $k_0 \cdot d_1^{\mathbb{B}} : B_1 \to C_0$ is the target of $\mathcal{G}$ and $(\eta_G)_0 = 1_{C_0}$ (as proven in Lemma B.0.4). This says precisely that $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ is well-defined as a morphism of graphs. ☐

The morphism of graphs $k : \mathcal{U}(\mathbb{B}) \to \mathcal{U}\mathbb{F}(\mathcal{G})$ of Lemma 3.4.2 will typically not be compatible with identity or composition structure. This is rectified by constructing a coequifier ensuring each of these conditions is satisfied.

Lemma 3.4.3. *There is a parallel pair of natural transformations $\overline{\alpha}, \overline{\beta} : k_0 \Rightarrow k_0 : \mathbf{disc}(B_0) \to \mathbb{F}(\mathcal{G})$ as displayed below left, whose component assigning morphisms $\alpha, \beta : B_0 \to \mathbb{F}(\mathcal{G})_1$ are given by $(\eta_{\mathcal{G}})_1 \cdot i^{\mathbb{B}}$ and $i^{\mathbb{F}(\mathcal{G})} \cdot k_0$ respectively, as displayed below right.*

$$\begin{tikzcd} \mathbf{disc}(B_0) \arrow[r, bend left, "k_0"] \arrow[r, bend right, "k_0"'] \arrow[r, phantom, "\overline{\alpha}\Downarrow\ \Downarrow\overline{\beta}"] & \mathbb{F}(\mathcal{G}) \end{tikzcd} \qquad \begin{tikzcd} & B_1 \arrow[dr, "(\eta_{\mathcal{G}})_1"] & \\ B_0 \arrow[ur, "i^{\mathbb{B}}"] \arrow[dr, "k_0"'] & & \mathbb{F}(\mathcal{G})_1 \\ & C_0 \arrow[ur, "i^{\mathbb{F}(\mathcal{G})}"'] & \end{tikzcd}$$

Proof. As $\mathbf{disc}(B_0)$ is discrete, it suffices to show that $\overline{\alpha}$ and $\overline{\beta}$ respect sources and targets. For α this follows from sources and targets for identities for the category $\mathbb{B}$, while for β this follows from the same axioms for the category $\mathbb{F}(\mathcal{G})$. $\square$

Lemma 3.4.4. *Let $p : \mathbb{F}(\mathcal{G}) \to \mathbb{I}$ be the coequifier of $\overline{\alpha}$ and $\overline{\beta}$. There is a parallel pair of natural transformations $\overline{\gamma}, \overline{\delta} : p \cdot k_2 \cdot m^{\mathbb{B}} \Rightarrow p \cdot m^{\mathbb{F}(\mathcal{G})} \cdot \eta_{\mathcal{G}_1} : \mathbf{disc}(B_2) \to \mathbb{I}$ as displayed below left, whose component assigning morphisms $\gamma, \delta : B_2 \to \mathbb{I}_1$ are given by $p_1 \cdot (\eta_{\mathcal{G}})_1 \cdot m^{\mathbb{B}}$ and $m^{\mathbb{I}} \cdot p_2 \cdot (\eta_{\mathcal{G}})_2$ respectively, as displayed below right.*

$$\begin{tikzcd} \mathbf{disc}(B_2) \arrow[r, bend left, "p \cdot k_2 \cdot m^{\mathbb{B}}"] \arrow[r, bend right, "p \cdot m^{\mathbb{F}(\mathcal{G})} \cdot \eta_{\mathcal{G}_1}"'] \arrow[r, phantom, "\overline{\delta}\Downarrow\ \Downarrow\overline{\gamma}"] & \mathbb{I} \end{tikzcd} \qquad \begin{tikzcd} & B_1 \arrow[dr, "p_1 \cdot (\eta_{\mathcal{G}})_1"] & \\ B_2 \arrow[ur, "m^{\mathbb{B}}"] \arrow[dr, "p_2 \cdot (\eta_{\mathcal{G}})_2"'] & & I_1 \\ & I_2 \arrow[ur, "m^{\mathbb{I}}"'] & \end{tikzcd}$$

Proof. The proof is similar to that for Lemma 3.4.3, now using sources and targets for composition for the category $\mathbb{B}$ to prove that γ respects sources and targets, and sources and targets for the category $\mathbb{I}$ to prove that δ respects sources and targets. $\square$

Lemma 3.4.5. *Let $t : \mathbb{I} \to \mathbb{C}$ be the coequifier of the natural transformations $\overline{\gamma}$ and $\overline{\delta}$ of Lemma 3.4.4. The morphism of graphs displayed below is well-defined as an internal functor.*

$$Q := (\mathbb{B} \xrightarrow{\ k\ } \mathbb{F}(\mathcal{G}) \xrightarrow{\ p\ } \mathbb{I} \xrightarrow{\ t\ } \mathbb{C})$$

Proof. Respect for identities is witnessed by the commutativity of the following diagram, in which the left region commutes by the definition of the coequifier $p : \mathbb{F}(\mathcal{G}) \to \mathbb{I}$, and the other regions commute by functoriality of p and t.

$$\begin{tikzcd}
B_0 \arrow[r, "k_0=q_0"] \arrow[dd, "i^{\mathbb{B}}"'] & \mathbb{F}(\mathcal{G})_0 \arrow[r, "p_0"] \arrow[d, "i^{\mathbb{F}}"] & I_0 \arrow[r, "t_0"] \arrow[dd, "i^{\mathbb{I}}"] & C_0 \arrow[dd, "i^{\mathbb{C}}"] \\
 & \mathbb{F}(\mathcal{G})_1 \arrow[dr, "p_1"] & & \\
B_1 \arrow[r, "k_1=\eta_{\mathcal{G}_1}"'] & \mathbb{F}(\mathcal{G})_1 \arrow[r, "p_1"'] & I_1 \arrow[r, "t_1"'] & C_1
\end{tikzcd}$$

Respect for composition is witnessed by the commutativity of the following diagram, in which the region on the left commutes by definition of the coequifier $t : \mathbb{I} \to \mathbb{C}$ and the region on the right commutes by functoriality of t.

$$\begin{tikzcd}
B_2 \arrow[r, "k_2:=\eta_{\mathcal{G}_2}"] \arrow[dd, "m^{\mathbb{B}}"'] & \mathbb{F}(\mathcal{G})_2 \arrow[r, "p_2"] & I_2 \arrow[r, "t_2"] \arrow[d, "m^{\mathbb{I}}"] & C_2 \arrow[dd, "m^{\mathbb{C}}"] \\
 & & I_1 \arrow[dr, "t_1"] & \\
B_1 \arrow[r, "k_1:=\eta_{\mathcal{G}_1}"'] & \mathbb{F}(\mathcal{G})_1 \arrow[r, "p_1"'] & I_1 \arrow[r, "t_1"'] & C_1
\end{tikzcd}$$

$\square$

Proposition 3.4.6. *The internal functors $F, G : A_0 \to \mathbb{B}$ in $\mathbf{Cat}(\mathcal{E})$ have a coequaliser given by $Q : \mathbb{B} \to \mathbb{C}$, where this internal functor is defined as in Lemma 3.4.5.*

Proof. Given an internal functor $R : \mathbb{B} \to \mathbb{D}$ such that $RF = RG$, we show that there exists a unique internal functor $S : \mathbb{C} \to \mathbb{D}$ satisfying $SQ = R$.

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, "Q"] \arrow[dr, "R"'] & \mathbb{C} \arrow[d, dashed, "S"] \\
 & & \mathbb{D}
\end{tikzcd}$$

Define $S_0 : C_0 \to D_0$ by the universal property of k_0 as the coequaliser on objects. Note that there is a morphism of graphs $W := (S_0, R_1) : \mathcal{G} \to \mathcal{U}\mathbb{D}$ as exhibited by the commutativity of the following diagrams:

$$\begin{tikzcd}
B_1 \arrow[rr, "R_1"] \arrow[dd, "k_0 \cdot d_0"'] \arrow[dr, "d_0^{\mathbb{B}}"] & & D_1 \arrow[dd, "d_0^{\mathbb{D}}"] \\
 & B_0 \arrow[dl, "Q_0"] \arrow[dr, "R_0"] & \\
C_0 \arrow[rr, "S_0"'] & & D_0.
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[rr, "R_1"] \arrow[dd, "k_0 \cdot d_1"'] \arrow[dr, "d_1^{\mathbb{B}}"] & & D_1 \arrow[dd, "d_1^{\mathbb{D}}"] \\
 & B_0 \arrow[dl, "Q_0"] \arrow[dr, "R_0"] & \\
C_0 \arrow[rr, "S_0"'] & & D_0.
\end{tikzcd}$$

Hence, by the adjunction $\mathbb{F} \dashv \mathcal{U}$, there exists a unique internal functor $W^\# : \mathbb{F}(\mathcal{G}) \to \mathbb{D}$ such that $\mathcal{U}(W^\#)\eta_{\mathcal{G}} = W$. The commutativity of the following diagram shows that $W^\#$ coequifies the natural transformations in Equation 3.4.3, which induces a unique functor $Y : \mathbb{I} \to \mathbb{D}$.

$$
\begin{tikzcd}
B_0 \arrow[r, "i^{\mathbb{B}}"] \arrow[dd, "Q_0"'] \arrow[dr, "R_0"] & B_1 \arrow[r, "\eta_{\mathcal{G}_1}"] \arrow[ddr, "R_1"] & \mathbb{F}(\mathcal{G})_1 \arrow[dd, "W_1^{\#}"] \\
 & D_0 \arrow[dr, "i^{\mathbb{D}}"'] & \\
C_0 \arrow[ur, "S_0"] \arrow[r, hook, "i^{\mathbb{F}}"'] & \mathbb{F}(\mathcal{G})_1 \arrow[r, "W_1^{\#}"'] & D_1
\end{tikzcd}
$$

The commutativity of the following diagram shows that Y coequifies the natural transformations in Equation 3.4.4, which induces a unique functor $Z : \mathbb{C} \to \mathbb{D}$.

$$
\begin{tikzcd}
B_2 \arrow[r, "m^{\mathbb{B}}"] \arrow[dd, "\eta_{\mathcal{G}_2}"'] \arrow[dr, "R_2"] & B_1 \arrow[r, "\eta_{\mathcal{G}_1}"] \arrow[ddrr, "R_1"] & \mathbb{F}(\mathcal{G})_1 \arrow[r, "p_1"] \arrow[ddr, "W_1^{\#}"] & I_1 \arrow[dd, "Y_1"] \\
 & D_2 \arrow[ddrr, "m^{\mathbb{D}}"'] & & \\
\mathbb{F}(\mathcal{G})_2 \arrow[ur, "W_2^{\#}"] \arrow[r, "p_2"'] & I_2 \arrow[u, "Y_2"] \arrow[r, "m^{\mathbb{I}}"'] & I_1 \arrow[r, "Y_1"'] & D_1
\end{tikzcd}
$$

By construction, $ZQ = R\mathbb{B} \to \mathbb{C}$ and $R : \mathbb{C} \to \mathbb{D}$ is the unique such functor that does this, as required.

□

3.5 Coequalisers of arbitrary pairs

In this section, we put together all the work from previous sections in order to show that for $\mathcal{E}$ an extensive category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint, the 2-category $\mathbf{Cat}(\mathcal{E})$ has coequalisers of arbitrary pairs of arrows. Moreover this gives a recipe for how to calculate coequalisers in $\mathbf{Cat}(\mathcal{E})$. We give a proof of this through Lemma 3.5.1, which is a more general statement about coequalisers in 2-categories $\mathcal{K}$ for which the inclusion of discrete objects $\mathbf{Disc}(\mathcal{K}) \to \mathcal{K}$ is sufficiently well-behaved. Our previous results allow us to apply this lemma to the 2-category $\mathbf{Cat}(\mathcal{E})$.

Lemma 3.5.1. *Let $\mathcal{K}$ be a 2-category for which the inclusion of the full-subcategory of discrete objects* $\mathbf{disc} : \mathbf{Disc}(\mathcal{K}) \to \mathcal{K}$ *has a right adjoint $(-)_0$ with counit $\varepsilon : \mathbf{disc}((-)_0) \to 1_{\mathcal{K}}$ and unit which is given component-wise by identities. Suppose $\mathcal{K}$ has coequalisers of any parallel pair $f, g : A \to B$ for which either of the following conditions hold.*

(i) $f_0 = g_0$, *or*

(ii) *A is in the image of* **disc**.

Then $\mathcal{K}$ has all coequalisers.

Proof. Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a parallel pair. By condition (2), $\mathcal{K}$ has the coequaliser of $f \cdot \varepsilon_A$ with $g \cdot \varepsilon_A$. Let $q : B \to C$ denote this coequaliser; it has the property that $qf \cdot \epsilon_A = qg \cdot \epsilon_A$. Applying $(-)_0$ to this, and by noting that $A_0 = \textbf{disc}(A_0)_0$ since the unit has identities as its components and by the triangle identities for the adjunction, it follows that $(\epsilon_A)_0 = 1_{A_0}$, so $(qf)_0 = (qf \cdot \epsilon_A)_0 = (qg \cdot \epsilon_A)_0 = (qg)_0$, so by condition (1), qf and qg have a coequaliser, $p : C \to D$. We claim that $qp : B \to D$ is the required coequaliser of f and g. Certainly, $qpf = qpg$ as they agree on objects and arrows by construction, so it remains to show the universal property of the coequaliser holds. Given $r : B \to E$ such that $rf = rg$, then $rf \cdot \epsilon_A = rg \cdot \epsilon_A$ and so by the universal property of C as a coequaliser of $f \cdot \epsilon_A$ and $g \cdot \epsilon_A$ we get an induced unique arrow $t : C \to E$. But then $t(qf) = rf = rg = t(qg)$ so by the universal property of D as the coequaliser of qf and qg, we get an induced unique arrow $w : D \to E$ such that $wpqf = wpqg$, as required.

□

We are now able to verify our main result.

Theorem 3.5.2. *Let $\mathcal{E}$ be an extensive category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \textbf{Cat}(\mathcal{E})_1 \to \textbf{Gph}(\mathcal{E})$ has a left adjoint. Then the 2-category $\textbf{Cat}(\mathcal{E})$ has finite 2-colimits.*

Proof. From the discussion in Section 3.2, it suffices to show that $\textbf{Cat}(\mathcal{E})$ has coequalisers. To do this, we verify that Lemma 3.5.1 applies to $\mathcal{K} := \textbf{Cat}(\mathcal{E})$. Recall that the functor $\textbf{disc} : \textbf{Disc}(\textbf{Cat}(\mathcal{E})) \simeq \mathcal{E} \to \textbf{Cat}(\mathcal{E})$ has a right adjoint given by $(-)_0 : \textbf{Cat}(\mathcal{E}) \to \mathcal{E}$, with $\textbf{disc}(E)_0 = E$ for any $E \in \mathcal{E}$ (Remark 2.3.1). By Proposition 3.3.1, condition (1) of Lemma 3.5.1 holds while by Proposition 3.4.6, condition (2) of Lemma 3.5.1 holds. □

Remark 3.5.3. In Section 3.7, we explore categories satisfying the required conditions to apply this theorem. Examples include when $\mathcal{E}$ is a list-arithmetic pretopos or elementary toposes with natural numbers objects. In either case, parameterised list objects in $\mathcal{E}$ are needed to form free categories on graphs, which are used in the construction of general coequalisers in $\textbf{Cat}(\mathcal{E})$. However, it is of interest to describe the coequalisers that exist in $\textbf{Cat}(\mathcal{E})$ when milder assumptions are made on $\mathcal{E}$, such as just exactness properties between limits and colimits. Let $\mathcal{E}$ have finite limits and colimits and suppose moreover that it is lextensive and has pullback-stable coequalisers. Consider a parallel pair of internal functors $F, G : \mathbb{A} \to \mathbb{B}$ and let $Q_0 : B_0 \to C_0$ denote the coequaliser of F_0 and G_0. We briefly describe, without proof, what we believe should be a sufficient condition that is weaker than the existence of the free category on the graph $\mathbb{G} := B_1 \underset{Q_0 \cdot d_0}{\overset{Q_0 \cdot d_1}{\rightrightarrows}} C_0$ but under which the coequaliser of F and G still exists in $\textbf{Cat}(\mathcal{E})$. We describe this explicitly when $\mathcal{E} := \textbf{FinSet}$ and leave the generalisation to the internal setting to the interested reader. Let $C_n \in \textbf{Gph}(\mathcal{E})$ denote the cycle of length n; this can be built by first constructing the path of length n using the terminal object and coproducts, and then using a coequaliser to identify the source and target of the path. Then the coequaliser of $F, G : \mathbb{A} \to \mathbb{B}$ exists in $\textbf{Cat}(\mathcal{E})$ if for all $n \in \mathbb{N}$ and any map $C_n \to \mathbb{G}$, the following lifting problem has a solution in $\textbf{Gph}(\mathcal{E})$.

$$\begin{tikzcd} & \mathcal{U}(\mathbb{B}) \arrow[d] \\ C_n \arrow[r] \arrow[ur, dashed] & \mathbb{G} \end{tikzcd} \tag{3.5}$$

This is to say that any cycles which appear in the graph produced by taking equivalence classes of objects in $\mathbb{B}$ already exist in the underlying graph of $\mathbb{B}$ itself. This means that the coequaliser of $F \cdot \varepsilon_{\mathbb{A}}$ and $G \cdot \varepsilon_{\mathbb{A}}$ can be formed in $\textbf{Cat}(\mathcal{E})$, without using parameterised list objects in $\mathcal{E}$. We leave detailed verification of this construction under these milder assumptions to future work.

We finish this section by showing that this method restricts to internal groupoids, which will be needed later on in this thesis.

Proposition 3.5.4. *Colimits in* $\mathbf{Gpd}(\mathcal{E})$ *are calculated as in* $\mathbf{Cat}(\mathcal{E})$*.*

Proof. Let $\mathbb{X}, \mathbb{Y} \in \mathbf{Gpd}(\mathcal{E})$. It is easy to verify that $\mathbb{X} + \mathbb{Y} \in \mathbf{Cat}(\mathcal{E})$ is a groupoid; that is we provide a map $(-)^{-1} : (\mathbb{X} + \mathbb{Y})_1 \to (\mathbb{X} + \mathbb{Y})_1$ satisfying the properties specified in Definition 2.2.11. Since $(\mathbb{X} + \mathbb{Y})_1 := X_1 + Y_1$, this is given by the universal property of the coproduct in $\mathcal{E}$, given the maps $(-)^1 : X_1 \to X_1$ and $(-)^{-1} : Y_1 \to Y_1$; it is easy to verify that this satisfies the required relations given that it does for $\mathbb{X}$ and for $\mathbb{Y}$.

Copowers by $\mathbf{2}$ are given similarly, since $(\mathbf{2} \otimes \mathbb{X})_1 \cong (\mathbf{2}_{\mathcal{E}} \times \mathbb{X})_1 \cong (\mathbf{1}+\mathbf{1}+\mathbf{1}) \times X_1$, then the required map $(-)^{-1} : (\mathbf{2} \otimes \mathbb{X})_1 \to (\mathbf{2} \otimes \mathbb{X})_1$ is given by the universal property of the product in $\mathcal{E}$, induced by the maps:

$$\begin{tikzcd}
 & & \mathbf{1}+\mathbf{1}+\mathbf{1} \\
(\mathbf{1}+\mathbf{1}+\mathbf{1}) \times X_1 \arrow[urr, bend left, "\pi_{\mathbf{1}+\mathbf{1}+\mathbf{1}}"] \arrow[rr, dashed, "(-)^{-1}"] \arrow[dr, bend right, "\pi_{X_1}"'] & & (\mathbf{1}+\mathbf{1}+\mathbf{1}) \times X_1 \arrow[u, "\pi_{\mathbf{1}+\mathbf{1}+\mathbf{1}}"'] \arrow[d, "\pi_{X_1}"] \\
 & X_1 \arrow[r, "(-)^{-1}"'] & X_1
\end{tikzcd}$$

It is easy to check that this satisfies the correct properties

Finally, for coequalisers, we use our explicit description given in this chapter. Consider the coequaliser diagram in $\mathbf{Cat}(\mathcal{E})$ in which $\mathbb{A}, \mathbb{B} \in \mathbf{Gpd}(\mathcal{E})$.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, "Q"] & \mathbb{C}.
\end{tikzcd}$$

such that $F_0 = G_0$. We induce a map $(-)^{-1}_{\mathbb{C}} : C_1 \to C_1$ by the universal property of the coequaliser as given below, recalling from Lemma B.0.1 that for a parallel pair that agrees on objects, the object C_1 is computed as the coequaliser for the parallel pair in $\mathcal{E}$ of the top and bottom rows of the commutative diagram displayed below.

$$\begin{tikzcd}
B_1 \times_{B_0} A_1 \times_{B_0} B_1 \arrow[r, shift left, "\widetilde{F}"] \arrow[r, shift right, "\widetilde{G}"'] \arrow[d, "(-)^{-1}_{\mathbb{B}} \times_{B_0} (-)^{-1}_{\mathbb{A}} \times_{B_0} (-)^{-1}_{\mathbb{B}}"'] & B_3 \arrow[r, "m^2"] \arrow[d, "((-)^{-1}_{\mathbb{B}})^3"] & B_1 \arrow[r, "Q_1"] \arrow[d, "(-)^{-1}_{\mathbb{B}}"] & C_1 \arrow[d, dashed, "(-)^{-1}_{\mathbb{C}}"] \\
B_1 \times_{B_0} A_1 \times_{B_0} \arrow[r, shift left, "\widetilde{F}"] \arrow[r, shift right, "\widetilde{G}"'] & B_3 \arrow[r, "m^2"] & B_1 \arrow[r, "Q_1"] & C_1
\end{tikzcd}$$

This map satisfies all of the required equations, since those for $(-)^{-1}_{\mathbb{B}} : B_1 \to B_1$ and $(-)^{-1}_{\mathbb{A}} : A_1 \to A_1$ do, by the universal property. Hence $\mathbb{C} \in \mathbf{Gpd}(\mathcal{E})$, as required.

Next, we argue that the free internal category of an internal groupoid is actually an internal groupoid. Let $\mathbb{B} \in \mathbf{Gpd}(\mathcal{E})$. We show that $\mathbb{F}\mathcal{U}\mathbb{B} \in \mathbf{Gpd}(\mathcal{E})$, by constructing an internal isomorphism $(-)^{-1} : \mathbb{F}\mathcal{U}\mathbb{B} \to \mathbb{F}\mathcal{U}\mathbb{B}^{op}$; this is given by the adjunct of $\mathcal{U}(\eta_{\mathbb{B}} \circ (-)^{-1}) : \mathcal{U}\mathbb{B} \to \mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B}^{\mathrm{op}}$; it can be checked that the adjunct forms an isomorphism with inverse given by the adjunct of $\mathcal{U}(\eta_{\mathbb{B}^{op}} \circ (-)^{-1}_{\mathbb{B}^{\mathrm{op}}}) : \mathcal{U}\mathbb{B}^{\mathrm{op}} \to \mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B}$.

Now, consider the coequaliser diagram in $\mathbf{Cat}(\mathcal{E})$ in which $\mathbb{B} \in \mathbf{Gpd}(\mathcal{E})$ and A_0 is a discrete object.

$$A_0 \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B} \xrightarrow{Q} \mathbb{C}.$$

Note that trivially $A_0 \in \mathbf{Gpd}(\mathcal{E})$. The process described in Section 3.4, uses free internal categories and coequalisers out of pairs which agree on objects; we have shown that these each result in groupoids and so it follows that $\mathbb{C} \in \mathbf{Gpd}(\mathcal{E})$.

Therefore, the process for calculating an arbitrary coequaliser of internal categories described in this section has been shown to restrict to internal groupoids.

$\square$

3.6 A characterisation of when $\mathbf{Cat}(\mathcal{E})$ has 2-colimits

In this section, we prove a converse to Theorem 3.5.2. It is clear that the existence of finite 2-colimits in $\mathbf{Cat}(\mathcal{E})$ implies the existence of finite colimits in $\mathcal{E}$; this is spelled out in Lemma 3.6.1. By Lemma 3.6.2, the existence of finite 2-colimits in $\mathbf{Cat}(\mathcal{E})$ implies that the functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. We isolate the property of $\mathbf{Cat}(\mathcal{E})$ having codescent coequalisers which are stable under pullback along discrete Conduché fibrations as being important as it implies pullback stability of coequalisers in $\mathcal{E}$. This is recorded in Lemma 3.6.7. We prove that the assumptions of $\mathcal{E}$ being an extensive category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint is equivalent to the 2-categorical assumptions that $\mathbf{Cat}(\mathcal{E})$ is an extensive 2-category with finite 2-colimits, pullbacks and codescent coequalisers which are stable under pullback along discrete Conduché fibrations in Theorem 3.6.8. We state this in purely 2-categorical terms without reference to the fact that the 2-category is of the form $\mathbf{Cat}(\mathcal{E})$ in Theorem 3.6.12 following [Bou10].

Lemma 3.6.1. *Let $\mathcal{E}$ be a category with pullbacks and suppose $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits. Then $\mathcal{E}$ has finite colimits.*

Proof. This is clear from the adjunctions $\mathbf{disc} \dashv (-)_0 \dashv \mathbf{indisc}$ that colimits in $\mathcal{E}$ can be calculated in $\mathbf{Cat}(\mathcal{E})$ by applying the $\mathbf{disc}$ functor, calculating the colimit and then applying $(-)_0$, since $(-)_0 \circ \mathbf{disc} = 1_{\mathcal{E}}$. $\square$

The following observation is noted in the case when $\mathcal{E} = \mathbf{Set}$ in [Bou10, Example 2.6].

Lemma 3.6.2. *Let $\mathcal{E}$ be a category with pullbacks, and suppose that $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits. Then the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint.*

Proof. Let $\mathcal{G} = (G_0, G_1, s, t)$ be an internal graph in $\mathcal{E}$. Since $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits, we can construct the coinserter of the following diagram in $\mathbf{Cat}(\mathcal{E})$:

$$\begin{tikzcd} \mathbf{disc}(G_1) \arrow[r, bend left, "\mathbf{disc}(s)"] \arrow[r, bend right, "\mathbf{disc}(t)"'] & \mathbf{disc}(G_0) \arrow[r, dashed, "Q"] & \mathbb{F}(\mathcal{G}). \end{tikzcd}$$

This universally coinserts a 2-cell $Q\mathbf{disc}(s) \Rightarrow Q\mathbf{disc}(t)$, which out of a discrete category means that in $\mathbb{F}(\mathcal{G})$, there is an actual 1-cell in $\mathbb{F}(\mathcal{G})$ for any arrow in G_1, with source and target as desired. The universal property of the coinserter in this situation is exactly the same as the universal property of the free category. $\square$

Corollary 3.6.3. *Suppose* $\mathbf{Cat}(\mathcal{E})$ *is cartesian closed and has finite* 2*-colimits. Then* $\mathcal{E}$ *is cartesian closed and has a natural numbers object.*

Proof. If $\mathbf{Cat}(\mathcal{E})$ is cartesian closed, then $\mathcal{E}$ is cartesian closed by Theorem 4.3.1. By Lemma 3.6.2, the existence of finite 2-colimits in $\mathbf{Cat}(\mathcal{E})$ implies that we have a left adjoint to $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$; by construction, this left adjoint restricts when considering one-object categories and graphs to become a left adjoint to $\mathcal{U} : \mathbf{Mon}(\mathcal{E}) \to \mathcal{E}$, so that $\mathcal{E}$ has free monoids. Note here that we are using that one-object internal graphs are simply objects of $\mathcal{E}$. In a cartesian closed category, having free monoids is equivalent to having a natural numbers object by taking the free monoid on the terminal object; a proof of this is given in [Joh77, p. 190] and is due to C.J. Mikkelsen. $\square$

Next, we prove a converse to Proposition 3.3.7, showing a correspondence of exactness conditions. The proof requires the following "two-point suspension" functor $2[-] : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$.

Definition 3.6.4. Given $X \in \mathcal{E}$, the internal category $2[X]$ has object of objects $\mathbf{1} + \mathbf{1}'$ and object of arrows $\mathbf{1} + X + \mathbf{1}'$, in which we write $\mathbf{1}'$ to distinguish the two copies of the terminal object. It has identity assigner given by a coproduct of the coprojection maps $i := (\iota_{\mathbf{1}} + \iota_{\mathbf{1}'}) : \mathbf{1} + \mathbf{1}' \to \mathbf{1} + \mathbf{1}' + X \cong \mathbf{1} + X + \mathbf{1}'$, and it has source and target maps given by $d_0 := \iota_{\mathbf{1}} + \iota_{\mathbf{1}} \cdot ! + \iota_{\mathbf{1}'} : \mathbf{1} + X + \mathbf{1}' \to \mathbf{1} + \mathbf{1}'$ and $d_1 := \iota_{\mathbf{1}} + \iota_{\mathbf{1}'} \cdot ! + \iota_{\mathbf{1}'} : \mathbf{1} + X + \mathbf{1}' \to \mathbf{1} + \mathbf{1}'$. By extensivity, $(\mathbf{1} + X + \mathbf{1}) \times_{\mathbf{1}+\mathbf{1}} (\mathbf{1} + X + \mathbf{1}) \cong \mathbf{1} + X + \mathbf{1}$ and we define $m := 1_{\mathbf{1}+X+\mathbf{1}'}$. Given a morphism $f : X \to Y$ in $\mathcal{E}$, the internal functor $2[f] : 2[X] \to 2[Y]$ is defined on objects by $2[f]_0 := 1_{\mathbf{1}+\mathbf{1}'}$ and on morphisms by $2[f]_1 := \mathbf{1} + f + \mathbf{1}$.

It is easy to verify that $\mathbf{2}[-] : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ is well-defined as a functor, and that each internal functor $\mathbf{2}[f] : \mathbf{2}[X] \to \mathbf{2}[Y]$ is a discrete Conduché fibration.

Lemma 3.6.5. *Let* $\mathcal{E}$ *be an extensive category with pullbacks. Then the functor* $2[-] : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ *preserves any coequalisers that exist in* $\mathcal{E}$*. Moreover, these coequalisers are created by the nerve* $N : \mathbf{Cat}(\mathcal{E})_1 \to [\Delta^{op}_{\leq 3}, \mathcal{E}]$.

Proof. Consider a parallel pair $f, g : X \to Y$ in $\mathcal{E}$. Denote the coequaliser in $\mathcal{E}$ by $q : Y \to C$. Denote the coequaliser of the parallel pair $2[f], 2[g] : 2[X] \to 2[Y]$ in $\mathbf{Cat}(\mathcal{E})$ by $p : 2[Y] \to \mathbb{C}$. The functor $(-)_0 : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$ preserves colimits, so $p_0 : \mathbf{1} + \mathbf{1}' \cong \mathbb{C}_0$. Certainly, $2[q]$ coequalises $2[f]$ and $2[g]$, so this induces a unique arrow $\mathbb{C} \to 2[C]$. Moreover, by extensivity, the coequaliser of $\mathbf{1}+2[f]+\mathbf{1}'$ and $\mathbf{1}+2[g]+\mathbf{1}'$ is $\mathbf{1}+2[q]+\mathbf{1}' : \mathbf{1}+2[Y]+\mathbf{1}' \to \mathbf{1}+2[C]+\mathbf{1}'$, so this induces a unique arrow $r_1 : \mathbf{1} + 2[C] + \mathbf{1}' \to \mathbb{C}_1$. It is not hard to check that (p_0, r_1) assembles into an internal functor $\mathbf{1} + 2[C] + \mathbf{1}' \to \mathbb{C}$ and that this is inverse to $p : \mathbb{C} \to \mathbf{1} + 2[C] + \mathbf{1}'$, finishing the proof. $\square$

Lemma 3.6.6. *Let* $\mathcal{E}$ *be an extensive category. Then the functor* $2[-] : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ *preserves pullbacks.*

Proof. This is straightforward to verify given that pullbacks are computed pointwise. $\square$

Proposition 3.6.7. *Let* $\mathcal{E}$ *be an extensive category with pullbacks and suppose that the* 2*-category* $\mathbf{Cat}(\mathcal{E})$ *has coequalisers of parallel pairs of internal functors that agree on objects. If these coequalisers are stable under pullback along discrete Conduché fibrations, then* $\mathcal{E}$ *has pullback-stable coequalisers.*

Proof. Let $i : A \to B$ and consider the following coequaliser diagram in $\mathcal{E}/B$:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "x"'] & Y \arrow[r, "q"] \arrow[d, "y"] & C \arrow[dl, "c"] \\
& B &
\end{tikzcd}$$

By Lemma 3.6.5, this is sent to a coequaliser diagram in $\mathbf{Cat}(\mathcal{E})/2[B]$ under $2[-]$; note that $2[f]_0 = 1_{\mathbf{1}+\mathbf{1}'} = 2[g]_0$, and so this is a coequaliser of a parallel pair of internal functors that agrees on objects. Therefore, by assumption, it is stable under pullback along $2[i] : 2[A] \to 2[B]$ which is in particular a discrete Conduché fibration, and so by Lemma 3.6.6 we get a coequaliser diagram in $\mathbf{Cat}(\mathcal{E})/2[A]$ given by the following:

$$\begin{tikzcd}
2\left[i^*X\right] \arrow[r, shift left, "2{[i^*f]}"] \arrow[r, shift right, "2{[i^*g]}"'] \arrow[dr, bend right, "2{[i^*x]}"'] & 2[i^*Y] \arrow[r, "2{[i^*q]}"] \arrow[d, "2{[i^*y]}"] & 2[i^*C] \arrow[dl, bend left, "2{[i^*c]}"] \\
& 2[A] &
\end{tikzcd} \tag{3.6}$$

Since $2[f]_0 = 2[g]_0$, by Lemma 3.6.5, this is a levelwise coequaliser, so the following diagram is a coequaliser in $\mathcal{E}/(\mathbf{1} + A + \mathbf{1})$:

$$\begin{tikzcd}
\mathbf{1} + i^*X + \mathbf{1}' \arrow[r, shift left, "\mathbf{1}+i^*f+\mathbf{1}'"] \arrow[r, shift right, "\mathbf{1}+i^*g+\mathbf{1}'"'] & \mathbf{1} + i^*Y + \mathbf{1}' \arrow[r, "\mathbf{1}+i^*q+\mathbf{1}'"] & \mathbf{1} + i^*C + \mathbf{1}'
\end{tikzcd}$$

Since coequalisers commute with coproducts, it follows that the following is a coequaliser in $\mathcal{E}/A$, as required.

$$\begin{tikzcd}
i^*X \arrow[r, shift left, "i^*f"] \arrow[r, shift right, "i^*g"'] & i^*Y \arrow[r, "i^*q"] & i^*C
\end{tikzcd}$$

$\square$

We therefore have the following characterisation of when $\mathbf{Cat}(\mathcal{E})$ has 2-colimits.

Theorem 3.6.8. *Let $\mathcal{E}$ be a category with pullbacks. Then $\mathcal{E}$ is extensive, has pullback-stable coequalisers, and the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint if and only if the 2-category $\mathbf{Cat}(\mathcal{E})$ is extensive, has 2-colimits, pullbacks and coequalisers of parallel pairs of functors that agree on objects are stable under pullback along discrete Conduché fibrations.*

Proof. Extensivity of $\mathcal{E}$ being equivalent to extensivity of $\mathbf{Cat}(\mathcal{E})$ is given by Lemma 4.4.2 and $\mathcal{E}$ has pullbacks if and only if $\mathbf{Cat}(\mathcal{E})$ has pullbacks. If $\mathcal{E}$ is a category with pullbacks and pullback-stable coequalisers in which $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint, then Theorem 3.5.2 and Proposition 3.3.7 shows that $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits and that coequalisers of parallel pairs of functors that agree on objects are stable under pullback along discrete Conduché fibrations. Conversely, suppose that $\mathbf{Cat}(\mathcal{E})$ is extensive, has 2-colimits, pullbacks and coequalisers of parallel pairs of functors that agree on objects are stable under pullback along discrete Conduché fibrations. Then Proposition 3.6.7 shows that $\mathcal{E}$ has pullback-stable coequalisers and Lemma 3.6.2 shows that $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. $\square$

Corollary 3.6.9. *There exist Conduché fibrations between double categories that are not exponentiable.*

Proof. Let $\mathcal{E} = \mathbf{Cat}$. By Theorem 3.6.8, if all Conduché fibrations of double categories (i.e. in $\mathbf{Cat}(\mathbf{Cat})$) were exponentiable then all coequaliser diagrams in $\mathbf{Cat}(\mathbf{Cat})$ would be stable under pullback along discrete conduché fibrations, which would imply that **Cat** has pullback-stable coequalisers. However, **Cat** does not have pullback-stable coequalisers— a simple counterexample is given in Example 5.3.1. □

In [Nie20], Conduché fibrations of double categories are called *pre-exponentiable* because they are shown to satisfy a lax exponentiability condition.

Remark 3.6.10. Let

$$\mathbb{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B} \xrightarrow{Q} \mathbb{C}$$

be a coequaliser diagram in $\mathbf{Cat}(\mathcal{E})$ in which $F_0 = G_0$. By Proposition 3.3.1, the coequalising map $Q : \mathbb{B} \to \mathbb{C}$ is an isomorphism on objects. In $\mathbf{Cat}(\mathcal{E})$, such functors have a special importance— they are the codescent morphisms [Bou10]. We use this to phrase Theorem 3.6.8 in a purely 2-categorical way, without reference to the fact that the 2-category is of the form $\mathbf{Cat}(\mathcal{E})$.

Definition 3.6.11. Let $\mathcal{K}$ be a 2-category. A coequaliser diagram

$$A \underset{G}{\overset{F}{\rightrightarrows}} B \xrightarrow{Q} C$$

is call a *codescent coequaliser* if $Q : B \to C$ is a codescent morphism in $\mathcal{K}$.

We collect the results of this section so far and combine them with Bourke's characterisation of 2-categories of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ in the following.

Theorem 3.6.12. *Let $\mathcal{E}$ be an extensive category with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. Then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ satisfies the conditions listed below, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$, in which $\mathcal{E}$ is extensive, has pullbacks and pullback-stable coequalisers and the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint.*

(i) $\mathcal{K}$ has pullbacks and powers by **2**.

(ii) $\mathcal{K}$ has codescent objects of cateads.

(iii) Codescent morphisms and cateads are effective in $\mathcal{K}$.

(iv) Discrete objects in $\mathcal{K}$ are BO-projective in the sense of Definition 2.4.18.

(v) For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.

(vi) $\mathcal{K}$ *is extensive.*

(vii) $\mathcal{K}$ *has finite* 2*-colimits.*

(viii) Codescent coequalisers in $\mathcal{K}$ *are stable under pullback along discrete Conduché fibrations.*

Proof. The properties (i)-(v) are the conditions listed in Theorem 4.18 of [Bou10], from which we can deduce that $\mathcal{K} \simeq$ **Cat** $(\mathcal{E})$ where $\mathcal{E} :=$ **Disc** $(\mathcal{K})$. The properties (vi)-(viii) allow us to apply Theorem 3.6.8. $\square$

3.7 Examples

In this section, we give examples of extensive categories with pullback-stable coequalisers in which the forgetful functor $\mathcal{U}$: **Cat**$(\mathcal{E}) \to$ **Gpd**$(\mathcal{E})$ has a left adjoint.

Definition 3.7.1 ([Mai10], Definition 2.4)**.** Let $\mathcal{E}$ be a category with finite limits. We say that $\mathcal{E}$ has *parametrised list objects* if for any $X \in \mathcal{E}$, there exists an object $L(X) \in \mathcal{E}$ together with morphisms $r_0^X : \mathbf{1} \to L(X)$ and $r_1^X : L(X) \times X \to L(X)$ such that for any $b : B \to Y$ and $g : Y \times X \to Y$, there exists a unique $u : B \times L(X) \to Y$ making the following diagram commute:

$$\begin{tikzcd}
B \arrow[r, "(1_B, r_0^X \cdot !_B)"] \arrow[dr, bend right, "b"'] & B \times L(X) \arrow[d, dashed, "u"] & B \times (L(X) \times X) \arrow[l, "1_B \times r_1^X"'] \arrow[d, dashed, "(u \times 1_X) \cdot \sigma"] \\
& Y & Y \times X \arrow[l, "g"]
\end{tikzcd}$$

in which $\sigma : B \times (L(X) \times X) \to (B \times L(X)) \times X$ is the associative isomorphism of the cartesian product.

Remark 3.7.2. We note that for any category $\mathcal{E}$ with parametrised list objects, the assignment $X \mapsto L(X)$ extends to a functor $L : \mathcal{E} \to \mathcal{E}$; on morphisms $f : X \to Y$, we define $L(f) : L(X) \to L(Y)$ by the universal property of the parametrised list objects, taking $B = \mathbf{1}, Y = L(Y), b = r_0^Y$ and $g = \pi_{L(Y)} : L(Y) \times X \to L(Y)$ in the above definition. Moreover, there is a multiplication action $\mu_X : L(X) \times L(X) \to L(X)$ defined by the universal property by taking $B = L(X), Y = L(X), b = 1_{L(X)}$ and $g = r_1^X$. We also have a unit $\nu_X : X \to L(X)$ given by the the composite:

$$X \xrightarrow{(r_0^X \cdot !, 1_X)} L(X) \times X \xrightarrow{r_1^X} L(X).$$

The maps μ_X, ν_X furnish $L(X)$ with the structure of a monoid in $(\mathcal{E}, \times, \mathbf{1})$.

Example 3.7.3. Useful intuition is provided by the case $\mathcal{E} =$ **Set**. For any set X, $L(X)$ is defined to be the set of words with alphabet X, otherwise known as the free monoid generated by X. The morphism $r_0^X : \mathbf{1} \to L(X)$ is given by the empty list. The morphism $r_1^X : L(X) \times X \to L(X)$ takes a word $(x_1 ... x_n)$ and an element $y \in X$ and outputs the word $(x_1 ... x_n y)$. The morphism $\mu_X : L(X) \times L(X) \to L(X)$ concatenates two words $((x_1 ... x_n), (y_1 ... y_m)) \mapsto (x_1, ... x_n y_1 ... y_m)$. The morphism $\nu_X : X \to L(X)$ takes an element $x \in X$ and forms the singleton word $(x) \in L(X)$.

Remark 3.7.4. Any category with parametrised list objects has a parametrised natural numbers object by taking $X = \mathbf{1}$. We also remark that if $\mathcal{E}$ is cartesian closed, then the existence of parametrised lists objects (resp. a parametrised natural numbers object) is equivalent to the existence of list objects (resp. a natural numbers object) [Joh02a].

Definition 3.7.5. A *locos* is a lextensive category $\mathcal{E}$ with parameterised list objects. If $\mathcal{E}$ is also regular, we call it a *regular locos*. If it is exact, we call it a *list-arithmetic pretopos*. If it is additionally locally cartesian closed, we call it an *arithmetic* Π*-pretopos.*

In particular, we have the following useful properties.

- Any locos is extensive and has finite products, so it is distributive [CLW93]. It also has finite coproducts.
- Any list-arithmetic pretopos has coequalisers [Mai10, §3.9], and therefore has finite colimits.

We show that our main result, Theorem 3.5.2 is satisfied by any locos with pullback-stable coequalisers; more specifically, we show that for a locos $\mathcal{E}$, the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. This is essentially proven in [Mai10, Proposition 7.3], which shows this result for list-arithmetic pretoposes; the properties of exactness or even regularity are not used for the construction given there; quotients are not used in the proof, which we describe below. We note that, by the comments in Section 3.1.3, a locos with pullback stable coequalisers is in particular a regular category; however, being a regular locos is not sufficient for Theorem 3.5.2.

3.7.1 The free internal category on an internal graph

Throughout this section, let $\mathcal{E}$ be a locos, with notation as given in Section 3.7. In this section, we recall the free internal category on an internal graph given in Definition 7.2 of [Mai10]. The description we give is equivalent but uses categorical language to describe the structure rather than the internal type theory of a list-arithmetic pretopos. In Proposition 7.3 of [Mai10], it is proven that this forms a left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$.

Let $\mathcal{G} = (G_0, G_1, s, t)$. Define $\mathbb{F}\mathcal{G}_0 := G_0$ and $\mathbb{F}\mathcal{G}_1$ as the equaliser of the following diagram:

$$\begin{tikzcd} & LG_0 \times G_0 \arrow[dr, "r_1^{G_0}"] & \\ G_0 \times L(G_1) \times G_0 \arrow[ur, "! \times L(t) \times 1_{G_0}"] \arrow[dr, "1_{G_0} \times L(s) \times !"'] & & L(G_0) \\ & G_0 \times L(G_0) \arrow[ur, "r_1^{G_0} \cdot \rho"'] & \end{tikzcd} \tag{3.7}$$

where ρ denotes the symmetry isomorphism of the cartesian product $\rho : G_0 \times L(G_0) \cong L(G_0) \times G_0$ and $! : G_0 \to \mathbf{1}$ is the unique map to the terminal object. The identity assigner $i : \mathbb{F}\mathcal{G}_0 \to \mathbb{F}\mathcal{G}_1$ is induced by the universal property of the equaliser, given that $1_{G_0} \times r_0^{G_1} \cdot ! \times 1_{G_0} : G_0 \to G_0 \times L(G_1) \times G_0$ equalises Diagram (3.7). We define $d_1, d_0 : \mathbb{F}\mathcal{G}_1 \to G_0$ by the following composites:

$$d_1 := \left(\; \mathbb{F}\mathcal{G}_1 \longrightarrow G_0 \times LG_1 \times G_0 \xrightarrow{\pi_0} G_0 \; \right)$$

$$d_0 := \left(\; \mathbb{F}\mathcal{G}_1 \longrightarrow G_0 \times LG_1 \times G_0 \xrightarrow{\pi_2} G_0 \; \right).$$

The following map

$$\begin{tikzcd}
\mathbb{F}\mathcal{G}_1 \times_{G_0} \mathbb{F}\mathcal{G}_1 \arrow[d] \\
(G_0 \times LG_1 \times G_0) \times_{G_0} (G_0 \times LG_1 \times G_0) \arrow[d, "\cong"] \\
G_0 \times LG_1 \times LG_1 \times G_0 \arrow[d, "1_{G_0} \times \mu_{G_1} \times 1_{G_0}"] \\
G_0 \times LG_1 \times G_0.
\end{tikzcd}$$

equalises Diagram (3.7). This therefore induces a map $m : \mathbb{F}\mathcal{G}_1 \times_{G_0} \mathbb{F}\mathcal{G}_1 \to \mathbb{F}\mathcal{G}_1$.

Definition 3.7.6. [Mai10, p. 7.2] Given an internal graph $\mathcal{G} = (G_0, G_1, s, t)$, we define an internal category $\mathbb{F}\mathcal{G} := (\mathbb{F}\mathcal{G}_0, \mathbb{F}\mathcal{G}_1, d_1, d_0, i, m)$.

Moreover, this internal category is the *free* internal category on an internal graph, forming an adjunction as recorded below. The unit of this adjunction $\eta_{\mathcal{G}}$ is defined by $\eta_{\mathcal{G}_0} := 1_{G_0} : G_0 \to \mathbb{F}\mathcal{G}_0$ and $\eta_{\mathcal{G}_1} : G_1 \to \mathbb{F}\mathcal{G}_1$ which is induced by the universal property of the equaliser, given that $(d_1, \nu_{G_1}, d_1) : G_1 \to G_0 \times LG_1 \times G_0$ equalises Diagram (3.7). The counit of the adjunction does an internal version of taking a string of composable arrows and composing them.

Theorem 3.7.7. [Mai10, Proposition 7.3] Let $\mathcal{E}$ be a locos. The assignment $\mathcal{G} \mapsto \mathbb{F}\mathcal{G}$ provides a left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$.

Remark 3.7.8. If $\mathcal{E}$ has countable coproducts, then it is not too hard to prove that for a graph $\mathcal{G} := (G_0, G_1, s, t)$, the object $\mathbb{F}\mathcal{G}_1 \cong \Sigma_{n \in \mathbb{N}} G_n$, where for $n > 1$, G_n is its object of composable n-arrows:

$$G_n := \underbrace{G_1 \times_{G_0} \ldots \times_{G_0} G_1}_{n \text{ times}}.$$

In this case, the proof of Theorem 3.7.7 using the internal type theory of $\mathcal{E}$ corresponds to a proof using the universal property of the coproduct; internal induction becomes external universal property. This proof is categorically elegant, and illuminates that the proof in [Mai10] does not need regularity or exactness conditions.

We do not ask for $\mathcal{E}$ to have countable coproducts as this is not an elementary condition, despite the fact that arithmetic Π-pretoposes with finite colimits which do not have countable coproducts are hard to construct and do not interact well with other toposes— see, for example, [Joh02b, p. D5.1.7].

Remark 3.7.9. As mentioned, the description we give for the free internal category on an internal graph is different, but equivalent, to the one given by Maietti in [Mai10]. We choose this description as it does not rely on using the internal language of a list-arithmetic pretopos, and it does not use coproducts which are indeed not needed for the construction of free internal categories on graphs. We briefly describe how to see the equivalence between the different descriptions, although a full proof is left to the interested reader. The key to this proof is in noting that the object of non-empty lists of G_1, denoted $L^*(G_1)$ and described in [Mai10] using the internal language of $\mathcal{E}$, is isomorphic to $L(G_1) \times G_1$; the isomorphism between them is given by the maps $r_1^X : L(G_1) \times G_1 \to L^*(G_1)$ and $(\mathrm{Bck}, \mathrm{Las}) : L^*(G_1) \to L(G_1) \times G_1$, where $\mathrm{Las} : L^*(G_1) \to G_1$ internally takes the last element of a non empty list and $\mathrm{Bck} : L^*(G_1) \to L(G_1)$ takes all elements except for the last one. These maps are described inductively using the internal language of $\mathcal{E}$ in [Mai10, Appendix A]. One direction of the isomorphism is shown using the universal property of the product and the list object. The other direction is shown using internal induction on list elements, using the internal language of $\mathcal{E}$. The proof then proceeds by using the fact that $L(G_1) \cong \mathbf{1} + G_1 \times L(G_1)$. This is shown in [Joh02b]. The proof is finished by noticing that the equalising diagrams constructed give the same equaliser.

3.7.2 Examples of locoses with finite pullback-stable coequalisers

Below, we record some examples of suitable categories.

Locally cartesian closed locoses with coequalisers

Let $\mathcal{E}$ be a locally cartesian closed locos with coequalisers. By Theorem 3.7.7, the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. By local cartesian closedness, for any $f : X \to Y$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ has right adjoint, and so preserves all colimits, in particular coequalisers. Therefore, any example of such categories allows us to apply Theorem 3.5.2 and conclude that $\mathbf{Cat}(\mathcal{E})$ has all finite 2-colimits. Some key examples of interest are given by fixing A a partial combinatory algebra and consider the category of assemblies $\mathbf{Asm}_A$ over this. This also holds in the category of *modest* assemblies, $\mathbf{Mod}_A$. These examples will be examined in future work with Sam Speight. Note that these are not elementary topos, nor are pretoposes as they are not exact but merely regular.

List-arithmetic pretoposes

A list-arithmetic pretopos is by definition an exact, extensive category with parametrised list objects. Moreover, it has pullback-stable coequalisers by the following argument, communicated to us by Peter LeFanu Lumsdaine.

Lemma 3.7.10. *Let $\mathcal{E}$ be a list-arithmetic pretopos. Then $\mathcal{E}$ has pullback-stable coequalisers.*

Proof. Examine the proof of the existence of coequalisers in [Mai10, Proposition 3.10]. This process uses coproducts, list objects, pullbacks, image factorisation and a quotient by an equivalence relation. All of these are preserved by any functor between list-arithmetic pretoposes. Moreover, all of these conditions are local, and so are preserved by slicing. Therefore, these steps are preserved by pullback. □

A class of examples of list-arithmetic pretoposes are given by the the syntactic category for any univalent universes of dependent type theory that satisfies axiom K and are closed under the empty type, unit type, sum types, dependent sum types, propositional truncations, quotient sets, and parameterised natural numbers type. Conversely, any list-arithmetic pretopos gives a model of Martin-Löf type theory which satisfies UIP [Str93]. Maietti proposes that list-arithmetic pretoposes are a suitable setting to formulate Joyal's arithemtic universes [Mai10], and one can formulate many logical (in)completeness theorems internally to them.

Arithmetic Π-pretoposes

An arithmetic Π-pretopos is a list-arithmetic pretopos which is locally cartesian closed. As a consequence of [Joh02b, Theorem 2.5.17], any locally cartesian closed positive coherent category with natural numbers object has (parametrised) list objects, so we can replace the need of parametrised list-objects with the existence of a natural numbers object in this case. Hence for $\mathcal{E}$ a model of Palmgren's constructive elementary theory of the category of sets [Pal12], which give categorical models of Bishop's constructive set theory, $\mathbf{Cat}(\mathcal{E})$ has finite 2-colimits; hence we can deduce that any model of the constructive elementary theory of the 2-category of small categories (see Chapter 5) has finite 2-colimits.

Elementary toposes with natural numbers object

Any elementary topos with natural numbers object is an example of an arithmetic Π-pretopos. Hence elementary toposes with natural numbers object are a suitable setting for our results. This recovers [JW78, Corollary 6.10], which shows that for an elementary topos with natural numbers object, $\mathbf{Cat}(\mathcal{E})$ has coequalisers. Their proof is different from ours, and we generalise their method in Appendix B.

Hence, any model of the elementary theory of the category of sets [LM05] is a suitable setting for this work too. This is of interest in relation to Chapter 4 as it proves that any model of the elementary theory of the 2-category of small categories has finite 2-colimits.

Grothendieck toposes

Any Grothendieck topos is an elementary topos and has a natural numbers object given by the constant sheaf on the natural numbers in **Set**. However, these examples are already covered by Proposition 3.1.2 by noting that Grothendieck toposes are locally finitely presentable.

Chapter 4

The elementary theory of the 2-category of small categories

4.1 Introduction

Lawvere's Elementary Theory of the Category of Sets (hereafter ETCS) [Law64] provides a set theory which axiomatises the properties of function composition rather than those of a global set membership relation. It provides an important fragment of a category-theoretic foundation of mathematics, but is strictly weaker than the traditional foundation of mathematics given by Zermelo Fraenkel Set Theory with the Axiom of Choice (hereafter ZFC). Precisely, ZFC is equiconsistent with ETCS augmented with the axiom schema of replacement [Osi74].

In his PhD thesis [Law63], Lawvere also gave an elementary, first order axiomatisation of the category of categories and functors. He later advocated for the first order theory of the category of categories as a foundation of mathematics (CCAF) [Law66]. Unfortunately, this contained an error noted by [Isb67], in the so-called "category description theorem" [Law66, p. 15] which attempted to deduce from his theory that categories are built out of discrete objects as a 1-dimensional quotient; [BP75] then constructed a model of Lawvere's theory which showed that the category of discrete objects did not form a model of Lawvere's ETCS, which was a central result in [Law66]. In an address at the 2015 Category Theory conference in Aveiro, he called for an "improved axiomatisation" to an explicit formulation of the principles of category theory [Law]. Our work is a step towards this goal, which re-expresses Lawvere's foundational framework as one for category theory rather than one for set theory. We create a 2-dimensional theory such that the discrete objects form a model of ETCS and conversely given $\mathcal{E}$ a model of ETCS, a full subcategory of $[\Delta^{\mathrm{op}}, \mathcal{E}]$ models this 2-dimensional theorem, proving a correct version of the main theorem in [Law66, p. 16]. We do not explain how to do "internal category theory" in this work, although we imagine that such a 2-category has the ability to synthetically reconstruct many of the results and theory in [Mac71], for example; this would be a good next step towards Lawvere's goal.

In this chapter, we propose a different categorification of ETCS which captures the natural two-dimensional structure of the 2-category of small categories. This is the elementary theory of the 2-category of small categories (ET2CSC) of the title. Our main result establishes that the theory of such 2-categories is 'Morita biequivalent' with ETCS, meaning that the two theories have biequivalent 2-categories of models.

ETCS lacks the expressive power needed to support certain important set theoretical constructions, such as transfinite recursion. Nonetheless, it does support many of the set theoretic constructions that most mathematicians use in everyday practise. Indeed, Lawvere's aim in giving the definition was to capture more closely those aspects of set theory which are more broadly used. It is a structuralist foundation, which prioritises the perspective of how sets relate to one another, rather than a materialist one such as ZFC which prioritises how sets are built, such as via well-founded trees. While philosophical considerations are not the focus of this chapter, a reader interested in these matters should consult Chapters 1 and 5 of [Lan17], and the references therein. ET2CSC clarifies the position of the ordinary theory of small categories within Street's programme towards a formal category theory [Str80; Str74]. It facilitates a structuralist framework in which many simple category theoretical constructions can be performed, just as ETCS does for many simple set theoretical constructions. In Chapter 7, we extend the present axiomatisation of the 2-category of small categories by adding a discrete opfibration classifier that satisfies a categorified version of the axiom of replacement. This provides a 2-dimensional analogue of categories of small maps [JM95], and extends the present theory to encompass ZFC and facilitate more sophisticated categorical constructions.

4.1.1 Outline of main results

Our main contribution is giving an elementary theory for the 2-category of small categories, and showing that the 2-category of models for this theory is biequivalent to that for Lawvere's elementary theory of the category of sets, as recalled in Definition 4.1.1, to follow.

Definition 4.1.1. [Law64] A category $\mathcal{E}$ is said to *model the elementary theory of the category of sets* if the following conditions are satisfied.

1. $\mathcal{E}$ has finite limits.

2. $\mathcal{E}$ is cartesian closed.

3. The terminal object $\mathbf{1}$ is a generator for $\mathcal{E}$, as recalled in Definition 4.4.7 part (1).

4. $\mathcal{E}$ has a natural numbers object, as recalled in Definition 4.5.1 part (1).

5. $\mathcal{E}$ has a subobject classifier, as recalled in Definition 4.6.1 per the discussion in Remark 4.6.2.

6. $\mathcal{E}$ satisfies the external Axiom of Choice, as recalled in Definition 4.7.1.

See [Lei14] for a gentle introduction to ETCS, and [LM05] for technical details. For Definition 4.1.1 part $n \in \{1, ..., 6\}$, Section $4.(n+1)$ exhibits a condition on the 2-category $\mathbf{Cat}(\mathcal{E})$ that is equivalent to the condition on $\mathcal{E}$ listed as axiom n above. In particular, the main results of each of these sections are Proposition 4.2.2, Theorem 4.3.1, Theorem 4.4.13, Theorem 4.5.4, Theorem 4.6.7, and Theorem 4.7.13. We collate these results in Theorem 4.8.2 to characterise up to 2-equivalence those 2-categories which are of the form $\mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a model of ETCS. This is expressed in terms of the elementary theory of the 2-category of small categories, which we introduce in Definition 4.8.1. Theorem 4.8.7 builds upon this result to give a characterisation of morphisms of models, and finally Theorem 4.8.13 establishes the biequivalence between the 2-categories of models of ETCS and ET2CSC.

4.1.2 Key ideas and techniques

Internal Category Theory and Bourke's characterisation of $\mathbf{Cat}(\mathcal{E})$

Chapter 2 establishes our notation and conventions in internal category theory, and catalogues various concepts that will be used in constructions and proofs. Specifically, Subsection 2.2 describes internal categories, functors and natural transformations via their truncated nerves, and also describes the 2-category structure that these data comprise.

Sections 4.2 (resp. 4.3) review the well known relationships between finite limits in $\mathcal{E}$ and finite 2-limits in $\mathbf{Cat}(\mathcal{E})$ (resp. cartesian closedness of $\mathcal{E}$ and cartesian closedness of $\mathbf{Cat}(\mathcal{E})$). Our work relies heavily on Bourke's characterisation up to 2-equivalence of 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ with pullbacks, recalled in Proposition 2.4.19.

Generating families

The following new results in Section 4.4 are important stepping stones.

- Lemma 4.4.2 shows that $\mathcal{E}$ has extensive coproducts if and only if $\mathbf{Cat}(\mathcal{E})$ does.
- Theorem 4.4.5 part (2) shows that if in addition to the previous point $\mathcal{E}$ is also cartesian closed, then the 2-category $\mathbf{Cat}(\mathcal{E})$ also has copowers by $\mathbf{2}$.

As well as simplifying subsequent proofs by allowing two-dimensional aspects of limit-like universal properties to be deduced from their one-dimensional counterparts, copowers by $\mathbf{2}$ are used to construct generators in $\mathbf{Cat}(\mathcal{E})$ from those in $\mathcal{E}$. This is shown in Corollary 4.4.8, a result that we think is of independent interest. Definition 4.4.11 introduces a definition of a 2-category $\mathcal{K}$ being 2-well-pointed. This is a two-dimensional analogue of well-pointedness for categories, and is a novel concept.

Adjunctions and full subobject classifiers

Sections 4.5 (resp. 4.6) relate natural numbers objects (resp. subobject classifiers) in $\mathcal{E}$ to their appropriate counterparts in $\mathbf{Cat}(\mathcal{E})$. The proofs in Sections 4.4, 4.5 and 4.6 use routine calculations involving the adjunctions $\Pi_0 \dashv \mathbf{disc} \dashv (-)_0 \dashv$ **indisc**, which are reviewed in Subsection 2.3. Section 4.6 introduces the definition of a full subobject classifier, which is a different two-dimensional analogue of a subobject classifier to the discrete opfibration classifiers of [Web07]. These are a new concept, introduced in Definition 4.6.1.

Orthogonal factorisations and the Categorified Axiom of Choice

In Section 4.7 we first give a condition on 2-categories of the form $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ which is equivalent to the external Axiom of Choice in $\mathcal{E}$, and then re-express this condition in 2-categorical terms without relying on being able to recognise $\mathcal{K}$ as $\mathbf{Cat}(\mathcal{E})$. The internal formulation involves fully-faithfulness and the condition of being an epimorphism on objects. Whilst the first of these properties can be recognised representably in any 2-category, the second cannot. Although we could appeal to Proposition 2.4.19 to content ourselves with recognising it via $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$, we show that epimorphism on objects internal functors are characterised by a left orthogonality property against a representably defined class of maps $\mathcal{R}'$. This follows from Proposition 4.7.7, also of independent interest, in which we show that orthogonal factorisation systems $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ give rise to orthogonal factorisation systems $(\mathcal{L}', \mathcal{R}')$ on $\mathbf{Cat}(\mathcal{E})$. Indeed, $\mathcal{R}'$ is precisely the full subobjects, for which classifiers are examined in Section 4.6.3.

4.2 Finite limits and Bourke's characterisation of $\mathbf{Cat}(\mathcal{E})$

If $\mathcal{E}$ has pullbacks then on top of pullbacks, $\mathbf{Cat}(\mathcal{E})$ also has powers by the category $\mathbf{2}$, the free-living arrow in $\mathbf{Cat}$. These are given by an internal version of arrow categories, and will be described briefly in Remark 4.2.1. A more detailed explicit internal description is given in [Bou10; Mir18]. Moreover, 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ have been characterised by Bourke— see Proposition 2.4.19. For our purposes, it suffices to know that 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ may be characterised in elementary and purely 2-categorical terms.

Remark 4.2.1. In this chapter we mostly work with 2-categories $\mathcal{K}$ which satisfy the conditions listed in Proposition 2.4.19. When doing so, Bourke's result allows us to use the techniques of internal category theory in our proofs, even when dealing with properties stated in purely 2-categorical terms.

Although readers should be able to follow our proofs by treating Proposition 2.4.19 as a 'black box', we give some brief comments on its content. The powers $\mathbb{A}^{\mathbf{2}}$ can be given an explicit description internally to $\mathcal{E}$. They have objects of objects given by A_1; the object of arrows of $\mathbb{A}$, while their objects of arrows are given by the pullback depicted below which may be thought of as the 'object of internal squares in $\mathbb{A}$'.

$$\begin{tikzcd}
A_{Sq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_2 \arrow[d, "m"] \\
A_2 \arrow[r, "m"'] & A_1
\end{tikzcd}$$

Codescent objects in a 2-category $\mathcal{K}$ are 2-categorical colimits of truncated simplicial objects, defined by the weight $\Delta_{\leq 2} \to \mathbf{Cat}$, where $\Delta_{\leq 2}$ is considered as a 2-category with only identity 2-cells. Recall that categories internal to $\mathcal{K}$ whose source and target maps form a two-sided discrete fibration are called *cateads* in $\mathcal{K}$.

Recall that an object $\mathbb{A}$ of a 2-category $\mathcal{K}$ is said to be *BO-projective* if the representable $\mathcal{K}(\mathbb{A}, -) : \mathcal{K} \to \mathbf{Cat}$ preserves codescent morphisms. This extends the one-dimensional notion, where instead the representable preserves regular epimorphisms.

Codescent morphisms for cateads in $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ are precisely those internal functors $f : \mathbb{A} \to \mathbb{B}$ for which $f : A_0 \to B_0$ are isomorphisms. One may think of a catead $\mathbb{C}$ in $\mathbf{Cat}(\mathcal{E})$ as a two-dimensional version of an equivalence relation. From this perspective, its codescent object is a two-dimensional quotient, which is equivalently given by the '0-th row' of the underlying double category in $\mathcal{E}$. If the internal category of objects of the double category $\mathbb{C}$ is called its underlying vertical category internal to $\mathcal{E}$, then the codescent object of $\mathbb{C}$ is its underlying horizontal category internal to $\mathcal{E}$.

As mentioned in Remark 2.3.3, (iso on objects, fully faithful) forms an orthogonal factorisation system on $\mathbf{Cat}(\mathcal{E})_1$. We briefly review its construction. Given an internal functor $f : \mathbb{A} \to \mathbb{B}$ in $\mathbf{Cat}(\mathcal{E})$, first form its higher kernel $\mathbf{K}(f)$ (Definition 2.4.11), which is an internal double category, or category internal to $\mathbf{Cat}(\mathcal{E})_1$.

$$
\begin{tikzcd}
f \downarrow f \downarrow f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f \downarrow f \arrow[r, shift left=2, "d_1"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "d_0"'] & \mathbb{A}
\end{tikzcd}
$$

Bourke shows that the double category just described is a catead, and that the factorisation $f = hk$ where k is given by an isomorphism between objects of objects and h is fully faithful, is given by taking $k : \mathbb{A} \to \mathbb{C}$ to be coprojection to the codescent object for this catead and $h : \mathbb{C} \to \mathbb{B}$ to be the internal functor induced by the universal property of $\mathbb{C}$. The adjective 'effective' in part (3) of Proposition 2.4.19 then amounts to the fact that h is an isomorphism in $\mathbf{Cat}(\mathcal{E})$ if and only if $f_0 : A_0 \to B_0$ is an isomorphism in $\mathcal{E}$. Finally, BO-projective covers are given by $\varepsilon_{\mathbb{A}} : \mathbf{disc}(\mathbb{A})_0 \to \mathbb{A}$; the components of the counit of the adjunction $\mathbf{disc} \dashv (-)_0$ described in Remark 2.3.1.

Proposition 4.2.2. *The* 2*-category* $\mathbf{Cat}\,(\mathcal{E})$ *has all finite* 2*-limits if and only if the category* $\mathcal{E}$ *has all finite limits.*

Proof. By Proposition 2.4.19, and the fact that $\mathbf{2}$ is a strong generator in $\mathbf{Cat}$, it suffices to show that $\mathcal{E}$ has terminal objects if and only if $\mathbf{Cat}(\mathcal{E})$ does. But this follows from the adjunctions $\mathbf{disc} \dashv (-)_0 \dashv \mathbf{indisc}$. □

4.3 Cartesian closedness

Recall that exponentials $[X, Y]$ in **Set** consist of sets whose elements are functions from X to Y, while exponentials $[\mathcal{C}, \mathcal{D}]$ in **Cat** consist of categories whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are natural transformations between these functors. In this Section we consider an $\mathcal{E}$-internal version of these functor categories, which can also be constructed in terms of exponentials and finite limits in $\mathcal{E}$.

Theorem 4.3.1. *Let* $\mathcal{E}$ *be a category with finite limits. The category* $\mathcal{E}$ *is cartesian closed if and only if the* 2*-category* $\mathbf{Cat}(\mathcal{E})$ *is cartesian closed. In this case,* $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ *preserves internal homs.*

Proof. Cartesian closedness of the category $\mathbf{Cat}(\mathcal{E})_1$ has been shown in [BE72], under the assumption that $\mathcal{E}$ has finite limits and exponentials, by viewing $\mathbf{Cat}(\mathcal{E})_1$ as the category of models of a finite limit sketch. Indeed, it is shown in Theorem 2.1.1 of [Mir18] that the nerve $N : \mathbf{Cat}(\mathcal{E})_1 \to [\Delta^{\mathrm{op}}, \mathcal{E}]$ is an inclusion of an exponential ideal. The two-dimensional aspect of the universal property of cartesian closedness for the 2-category $\mathbf{Cat}(\mathcal{E})$ follows from the universal property of powers by $\mathbf{2}$, which we denote as $\mathbf{2} \pitchfork (-)$. In particular, it is exhibited by the following natural bijections.

$$\mathbf{Cat}(\mathcal{E})_1(\mathbb{A} \times \mathbb{B}, \mathbf{2} \pitchfork \mathbb{C}) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbb{A}, (\mathbf{2} \pitchfork \mathbb{C})^{\mathbb{B}}) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbb{A}, \mathbf{2} \pitchfork \left(\mathbb{C}^{\mathbb{B}}\right))$$

Conversely, let $\mathcal{E}$ be a category with finite limits and suppose $\mathbf{Cat}(\mathcal{E})$ is cartesian closed. We show that $\mathcal{E}$ is cartesian closed with exponentials given as displayed below for $Y, Z \in \mathcal{E}$.

$$Z^Y := (\mathbf{disc}(Z)^{\mathbf{disc}(Y)})_0$$

The following calculations show that the proposed exponential satisfies the isomorphism depicted below, naturally in all $X, Y, Z \in \mathcal{E}$.

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$$

$$\begin{aligned}
\mathcal{E}(X \times Y, Z) &= \mathcal{E}(X \times Y, \mathbf{disc}(Z)_0) && \text{(unit of } \mathbf{disc} \dashv (-)_0 \text{ is the identity)} \\
&\cong \mathbf{Cat}(\mathcal{E})_1(\mathbf{disc}(X \times Y), \mathbf{disc}(Z)) && (\mathbf{disc} \text{ is fully-faithful}) \\
&\cong \mathbf{Cat}(\mathcal{E})_1(\mathbf{disc}(X) \times \mathbf{disc}(Y), \mathbf{disc}(Z)) && (\mathbf{disc} \text{ preserves products}) \\
&\cong \mathbf{Cat}(\mathcal{E})_1 \left(\mathbf{disc}(X), \mathbf{disc}(Z)^{\mathbf{disc}(Y)}\right) && (\mathbf{Cat}(\mathcal{E}) \text{ is cartesian closed}) \\
&\cong \mathcal{E}\left(X, \left(\mathbf{disc}(Z)^{\mathbf{disc}(Y)}\right)_0\right) && (\mathbf{disc} \dashv (-)_0) \\
&=: \mathcal{E}\left(X, Z^Y\right).
\end{aligned}$$

Cartesian closedness of $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ is an easy inspection given the construction of internal homs in $\mathbf{Cat}(\mathcal{E})$, and also follows from Day's reflection theorem [Day72]. ☐

4.4 Well-pointedness

Recall that in **Set**, we can test whether two functions $f, g : X \to Y$ are equal by checking if $f(x) = g(x)$ for every $x \in X$. Similarly, in **Cat**, to test if two functors $F, G : \mathcal{C} \to \mathcal{D}$ are equal it suffices to check that $Ff = Gf$ for every $f \in \mathcal{C}_1$. This amounts to **1** being a generator for **Set** and **2** being a generator for **Cat**. The aim of this section is to show that the analogous statements for $\mathcal{E}$ and $\mathbf{Cat}(\mathcal{E})$ are logically equivalent under the assumption that $\mathcal{E}$ is lextensive and cartesian closed. As we saw in Theorem 4.3.1, $\mathcal{E}$ is cartesian closed if and only if $\mathbf{Cat}(\mathcal{E})$ is. We first show a similar logical equivalence between extensivity of $\mathcal{E}$ and of $\mathbf{Cat}(\mathcal{E})$. It will follow that $\mathcal{E}$ is lextensive if and only if $\mathbf{Cat}(\mathcal{E})$ is.

Definition 4.4.1.

1. A category with pullbacks $\mathcal{E}$ is said to be *extensive* [CLW93] if it has finite coproducts and for all $A, B \in \mathcal{E}$, the functor $\mathcal{E}/A \times \mathcal{E}/B \to \mathcal{E}/(A + B)$, which takes the coproduct, is an equivalence of categories. Call an extensive category *lextensive* if it moreover has a terminal object.

2. Call a 2-category with pullbacks $\mathcal{K}$ *extensive* if it has finite coproducts and the similarly defined 2-functor is a 2-equivalence. Call an extensive 2-category $\mathcal{K}$ *lextensive* if it moreover has a terminal object and powers by **2**.

Lemma 4.4.2. *Let $\mathcal{E}$ be a category with pullbacks and products. The category $\mathcal{E}$ is extensive if and only if the* 2*-category* $\mathbf{Cat}(\mathcal{E})$ *is extensive, in which case the coproducts in* $\mathbf{Cat}(\mathcal{E})$ *are computed in* $[\Delta^{op}_{\leq 3}, \mathcal{E}]$.

Proof. It is clear from the adjunctions $\mathbf{disc} \dashv (-)_0 \dashv \mathbf{indisc}$ that $\mathcal{E}$ has an initial object if and only if $\mathbf{Cat}(\mathcal{E})_1$ does, and in this case so does the 2-category $\mathbf{Cat}(\mathcal{E})$. The functor category $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$ has whatever colimits $\mathcal{E}$ has, computed pointwise. Suppose $\mathcal{E}$ has extensive coproducts. Let $\mathbb{A}$ and $\mathbb{B}$ be categories internal to $\mathcal{E}$. Then the diagrams which need to be pullbacks for $\mathbb{A}+\mathbb{B}$ to be well-defined as an internal category are precisely the coproducts in $\mathcal{E}$ of the corresponding pullbacks which exhibit $\mathbb{A}$ and $\mathbb{B}$ as internal categories. But by extensivity of $\mathcal{E}$, these will be pullbacks as well. Thus the category $\mathbf{Cat}\left(\mathcal{E}\right)_1$ has coproducts as computed in $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$. But the two-dimensional aspect of the universal property for coproducts follows from the one-dimensional aspect, since $\mathbf{Cat}(\mathcal{E})$ has powers by $\mathbf{2}$.

Conversely, suppose that $\mathbf{Cat}(\mathcal{E})$ has extensive coproducts. For $X, Y \in \mathcal{E}$, we claim that their coproduct is given by $(\mathbf{disc}(X) + \mathbf{disc}(Y))_0$. By Remark 2.3.2, since $\mathcal{E}$ has products the functor $(-)_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ is a left adjoint and hence preserves coproducts. But $(\mathbf{disc}(X))_0 = X$ and $(\mathbf{disc}(Y))_0 = Y$. This completes the proof. □

For the remainder of this section we assume that $\mathcal{E}$, and hence $\mathbf{Cat}(\mathcal{E})$, is lextensive.

Next, we recall the construction of the free-living arrow $\mathbf{2}_{\mathcal{E}}$ as a category internal to $\mathcal{E}$. Copowers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$ can be constructed in terms of this internal category, as we will show in Theorem 4.4.5.

Remark 4.4.3. Recall that any finite limit preserving functor between finite limit categories $G : \mathcal{S} \to \mathcal{E}$ gives rise to a 2-functor $\mathbf{Cat}\left(G\right) : \mathbf{Cat}\left(\mathcal{S}\right) \to \mathbf{Cat}\left(\mathcal{E}\right)$, which acts componentwisely on all data [Mir18]. Recall also that the category of finite sets **FinSet** is the free completion under finite coproducts of the terminal category. Furthermore, for lextensive $\mathcal{E}$, the unique coproduct preserving functor $F_{\mathcal{E}} : \mathbf{FinSet} \to \mathcal{E}$ which preserves the terminal object also preserves all other finite limits.

Definition 4.4.4. Take $\mathcal{S} = \mathbf{FinSet}$ as in Remark 4.4.3 and apply the 2-functor $\mathbf{Cat}\left(F_{\mathcal{E}}\right) : \mathbf{Cat}\left(\mathbf{FinSet}\right) \to \mathbf{Cat}\left(\mathcal{E}\right)$ to the free living arrow $\mathbf{2} \in \mathbf{Cat}\left(\mathbf{FinSet}\right)$. Denote the resulting category internal to $\mathcal{E}$ as $\mathbf{2}_{\mathcal{E}}$.

The internal category $\mathbf{2}_{\mathcal{E}}$ of Definition 4.4.4 can be described explicitly as a truncated simplicial object, with n-simplices given by the $(n+2)$-fold coproduct of the terminal object $\mathbf{1} \in \mathcal{E}$; see Example 2.3.2 of [Mir18] for details. Recall that the copower by $\mathbf{2}$ of an object $A \in \mathcal{K}$, if it exists, is an object $\mathbf{2} \odot A$ equipped with isomorphisms of categories $\mathcal{K}\left(\mathbf{2} \odot A, B\right) \cong \mathbf{Cat}\left(\mathbf{2}, \mathcal{K}\left(A, B\right)\right)$ which vary 2-naturally in B. The next theorem then shows that the 2-functor $\mathbf{Cat}\left(F_{\mathcal{E}}\right) :$ $\mathbf{Cat}\left(\mathbf{FinSet}\right) \to \mathbf{Cat}\left(\mathcal{E}\right)$ preserves copowers by $\mathbf{2}$.

Theorem 4.4.5. *Let $\mathcal{E}$ be lextensive and cartesian closed, and let $\mathbf{2}_{\mathcal{E}}$ be constructed as in Definition 4.4.4.*

1. *The internal hom $[\mathbf{2}_{\mathcal{E}}, \mathbb{B}]$ has the universal property of the power of $\mathbb{B}$ by $\mathbf{2}$.*

2. *For $\mathbb{A} \in \mathbf{Cat}\left(\mathcal{E}\right)$, the internal category $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$ has the universal property of the copower of $\mathbb{A}$ by $\mathbf{2}$ in $\mathbf{Cat}\left(\mathcal{E}\right)$.*

Proof. Consider the unique non-identity naturnal transformation ρ from the category $\mathbf{1}$ to the category $\mathbf{2}$. The internal functor $[\mathbf{2}_{\mathcal{E}}, \mathbb{B}] \to \mathbb{B}^{\mathbf{2}}$ is induced by the universal property of the power by $\mathbf{2}$ given the image of ρ under the 2-functor displayed below.

$$\mathbf{Cat}(\mathbf{FinSet})^{\mathrm{op}} \xrightarrow{\mathbf{Cat}(F_{\mathcal{E}})^{\mathrm{op}}} \mathbf{Cat}(\mathcal{E})^{\mathrm{op}} \xrightarrow{[-,\mathbb{B}]} \mathbf{Cat}(\mathcal{E})$$

We describe the transpose $\mathbf{2}_{\mathcal{E}} \times \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ of the required inverse internal functor $\mathbb{B}^{\mathbf{2}} \to [\mathbf{2}_{\mathcal{E}}, \mathbb{B}]$. Recall first that $\mathbf{2}$ is the category that has, as objects, the set $\{*\} + \{*\}$ and, as arrows, the set $\{*\} + \{*\} + \{*\}$. By lextensivity of $\mathcal{E}$ and as

$\mathbf{Cat}(F_{\mathcal{E}})$ preserves coproducts, $\mathbf{2}_{\mathcal{E}} \times \mathbb{B}^{\mathbf{2}}$ has, as objects, $B_1 + B_1$ and as arrows $B_{\text{sq}} + B_{\text{sq}} + B_{\text{sq}}$. Now, between objects of objects, the functor $\mathbf{2}_{\mathcal{E}} \times \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ is given by $(d_0, d_1) : B_1 + B_1 \to B_0$ induced by the universal property of the coproduct, using the source and target maps. Between objects of arrows it is given by the morphism $B_{\text{sq}} + B_{\text{sq}} + B_{\text{sq}} \to B_1$ induced by the universal property of the coproduct by the source and target maps of $\mathbb{B}^{\mathbf{2}}$, as well as by the diagonal of the pullback square defining B_{sq}. To prove internal functoriality, one needs to check commutativity conditions for maps out of coproducts. These can in turn be verified by checking cases for each summand appearing in the coproduct. However, each of these individual cases just involves pullbacks and hence follows from the analogous property when $\mathcal{E} = \mathbf{Set}$, using the Yoneda Lemma. The proof that these internal functors are mutually inverse is similar. This proves part (1). Part (2) then follows by the following chain of isomorphisms, where the penultimate step uses part (1).

$$\begin{aligned}
\mathbf{Cat}\left(\mathbf{2}, \mathbf{Cat}\left(\mathcal{E}\right)\left(\mathbb{A}, \mathbb{B}\right)\right) &\cong \mathbf{Cat}\left(\mathcal{E}\right)\left(\mathbb{A}, \mathbb{B}\right)^{\mathbf{2}} \\
&\cong \mathbf{Cat}\left(\mathcal{E}\right)\left(\mathbb{A}, \mathbb{B}^{\mathbf{2}}\right) \\
&\cong \mathbf{Cat}\left(\mathcal{E}\right)\left(\mathbb{A}, [\mathbf{2}_{\mathcal{E}}, \mathbb{B}]\right) \\
&\cong \mathbf{Cat}\left(\mathcal{E}\right)\left(\mathbf{2}_{\mathcal{E}} \times \mathbb{A}, \mathbb{B}\right)
\end{aligned}$$

□

In particular we have that $\mathbf{2}_{\mathcal{E}}$ has the universal property of the copower by $\mathbf{2}$ of the terminal object in $\mathbf{Cat}(\mathcal{E})$.

Remark 4.4.6. Generating families, in the sense we will recall in Definition 4.4.7, can be constructed in $\mathbf{Cat}(\mathcal{E})$ using copowers by $\mathbf{2}$. To show this we will need to observe that internal natural transformations out of discrete categories correspond to morphisms into the object of arrows of their codomain internal category. We now explain why this is so.

Let $X \in \mathcal{E}$ and $\mathbb{A} \in \mathbf{Cat}\left(\mathcal{E}\right)$, and recall the adjunction $\mathbf{disc} \dashv (-)_0$ from Remark 2.3.1. Then there are the following natural bijections:

$$\mathcal{E}\left(X, A_1\right) = \mathcal{E}\left(X, \left(\mathbb{A}^{\mathbf{2}}\right)_0\right) \cong \mathbf{Cat}(\mathcal{E})_1\left(\mathbf{disc}\left(X\right), \mathbb{A}^{\mathbf{2}}\right) \cong \mathbf{Cat}\left(\mathbf{2}, \mathbf{Cat}(\mathcal{E})\left(\mathbf{disc}\left(X\right), \mathbb{A}\right)\right)_0$$

Thus morphisms from X to the object of arrows of an internal category $\mathbb{A}$ are in natural bijection with internal natural transformations between internal functors from the discrete category on X to $\mathbb{A}$.

Definition 4.4.7.

1. A family of objects $\mathcal{G}$ in a category $\mathcal{C}$ is said to be *generating* if the family of hom-functors $\mathcal{C}\left(X, -\right) : \mathcal{C} \to \mathbf{Set}$ for $X \in \mathcal{G}$ are jointly faithful.

2. A family of objects $\widehat{\mathcal{G}}$ in a 2-category $\mathcal{K}$ is said to be *generating* if the family of hom-functors $\mathcal{K}\left(X, -\right) : \mathcal{K} \to \mathbf{Cat}$ for $X \in \mathcal{G}$ are jointly faithful on 1-cells and 2-cells.

Corollary 4.4.8. *Suppose that $\mathcal{E}$ has finite limits, extensive coproducts, and a generating family of objects $\mathcal{G}$. Form the family of internal categories $\widehat{\mathcal{G}} := \{\mathbf{2}_{\mathcal{E}} \times \mathbf{disc}\left(X\right) | X \in \mathcal{G}\}$. Then $\widehat{\mathcal{G}}$ is a generating family for $\mathbf{Cat}\left(\mathcal{E}\right)$.*

Proof. Let $f, g : \mathbb{A} \to \mathbb{B}$ be internal functors and assume that $fh = gh$ for all internal functors $h : \mathbf{2}_{\mathcal{E}} \times \mathbf{disc}\,(X) \to \mathbb{A}$ where $X \in \mathcal{G}$. By Proposition 2.2.6 part (1), to show that $\widehat{\mathcal{G}}$ is a generating family, it suffices to show that $f_1 = g_1$ under this assumption. Denote by $\alpha : X \to A_1$ the component assigner of the internal natural transformation which corresponds to h via the universal property of the copower by $\mathbf{2}$. Then the whiskerings $f\overline{\alpha} = g\overline{\alpha}$ are also equal in $\mathbf{Cat}(\mathcal{E})$. But by Remark 4.4.6, any morphism $X \to A_1$ is $\mathcal{E}$ corresponds to an internal natural transformation between internal functors from $\mathbf{disc}\,(X)$ to $\mathbb{A}$ This amounts to saying that $f_1\alpha = g_1\alpha$ for all $\alpha : X \to A_1$, and hence $f_1 = g_1$ as $X \in \mathcal{G}$.

This shows that the family of 2-functors $\mathbf{Cat}(\mathcal{E})(G, -) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$ for $G \in \widehat{\mathcal{G}}$ are jointly faithful on 1-cells. But joint faithfulness on 2-cells follows from joint faithfulness on 1-cells as $\mathbf{Cat}(\mathcal{E})$ has powers by $\mathbf{2}$. A parallel pair of internal natural transformations as depicted below left corresponds to a parallel pair of internal functors as depicted below right.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\overline{\gamma}\Downarrow\ \Downarrow\overline{\beta}"] & \mathbb{B}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathbb{B} \\
\mathbb{A} \arrow[ur, "f"] \arrow[r, shift left, "\tilde{\beta}"] \arrow[r, shift right, "\tilde{\gamma}"'] \arrow[dr, "g"'] & \mathbb{B}^{\mathbf{2}} \arrow[u, "d_1"'] \arrow[d, "d_0"] \\
 & \mathbb{B}
\end{tikzcd}
$$

By the one-dimensional aspect of $\widehat{\mathcal{G}}$ being a generator, the equality of such a pair of internal functors can be detected via $\mathbf{Cat}(\mathcal{E})(G, -) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$. As such, the equality of the original parallel pair internal natural transformations can also be detected via these representables.

□

Example 4.4.9. Let $\mathcal{C}$ be a small category and $\mathcal{E} := [\mathcal{C}^{\text{op}}, \mathbf{Set}]$. Then $\mathcal{E}$ has a generating family given by the representables $\mathcal{G} := \{\mathcal{C}(-, X) : \mathcal{C}^{\text{op}} \to \mathbf{Set} | X \in \mathcal{C}\}$. Now, the 2-functor $\mathbf{Cat}(-) : \mathbf{LEX} \to 2\text{-}\mathbf{CAT}$ of Proposition 3.1.5 in [Mir18] preserves powers by small categories, and as such there is an isomorphism of 2-categories $\mathbf{Cat}([\mathcal{C}^{\text{op}}, \mathbf{Set}]) \cong [\mathcal{C}^{\text{op}}, \mathbf{Cat}]$, where the second of these is the $\mathcal{V}$-enriched functor category with $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{C}$ considered as a 2-category with only identity 2-cells. Since $[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$ is an enriched fuctor category, it has copowers computed pointwisely in $\mathbf{Cat}$. Corollary 4.4.8 then says that the following is a generating family for the 2-category $[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$. This coincides with the generating family for $[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$ in terms of representables and copowers by the strong generator $\{\mathbf{2}\} \subseteq \mathbf{Cat}$.

$$
\left\{ Y \mapsto \coprod_{f \in \mathcal{C}(Y,X)} \mathbf{2} \;\middle|\; X \in \mathcal{C} \right\}
$$

Remark 4.4.10. If certain colimits exist in $\mathcal{E}$, then a generating family $\mathcal{G}' \subseteq \mathbf{Cat}(\mathcal{E})$ also gives rise to a generating family on $\mathcal{E}$. Specifically, we need $\mathcal{E}$ to have coequalisers for all reflexive pairs of source and target morphisms where $\mathbb{G} \in \mathcal{G}'$.

$$
\begin{tikzcd}
G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[r, "q_{\mathbb{G}}"] & \Pi_0(\mathbb{G})
\end{tikzcd}
$$

In this case, the partial adjunction $\mathcal{E}(\Pi_0(\mathbb{G}), X) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbb{G}, \mathbf{disc}(X))$ exists for all $\mathbb{G} \in \mathcal{G}$. The generating family in $\mathcal{E}$ is then given by $\mathcal{G} := \{\Pi_0(\mathbb{G}) | \mathbb{G} \in \mathcal{G}\}$. We give a detailed proof only of a special case in Theorem 4.4.13, since this will be enough for our main results and since generating families are in practice typically easier to construct in $\mathcal{E}$ than in $\mathbf{Cat}(\mathcal{E})$. The proof of this special case requires no extra colimit assumptions on $\mathcal{E}$. We leave the straightforward generalisation to the setting described here to the interested reader.

Recall that a category $\mathcal{E}$ is called *well-pointed* if it has a terminal object $\mathbf{1}$ and initial object $\mathbf{0}$ such that $\mathbf{0} \not\cong \mathbf{1}$ and the family containing just $\mathbf{1} \in \mathcal{E}$ is a generator. We introduce the following categorified version of this definition.

Definition 4.4.11. A 2-category $\mathcal{K}$ is called 2-*well-pointed* if the following conditions hold.

1. $\mathcal{K}$ has a terminal object $\underline{\mathbf{1}}$.
2. $\mathcal{K}$ has an initial object $\mathbf{0}$ and $\mathbf{0} \not\cong \mathbf{1}$.
3. The copower $\mathbf{2} \odot \underline{\mathbf{1}}$ exists in $\mathcal{K}$.
4. The family containing just $\mathbf{2} \odot \underline{\mathbf{1}}$ is a generator for $\mathcal{K}$, in the sense of Definition 4.4.7 part (2).

There is one final lemma that we will need before we are ready to prove the main result of this section.

Lemma 4.4.12. *Let $\mathcal{C}$ be a category with finite products. For $A \in \mathcal{C}$, consider the diagram displayed below in which the morphisms $\Delta_A : A \to A \times A$ denotes the diagonal $(1_A, 1_A)$ and $A \xleftarrow{\pi_1} A \times A \xrightarrow{\pi_2} A$ denote the product projections. This diagram is an equaliser.*

$$A \xrightarrow{\Delta_A} A \times A \underset{\Delta_A \times \pi_2}{\overset{\pi_1 \times \Delta_A}{\rightrightarrows}} A \times A \times A$$

Proof. This is straightforward to check when $\mathcal{C} = \mathbf{Set}$: the functions being equalised send (x, y) to (x, x, y) and (x, y, y) respectively. These outputs are indeed equal precisely when $x = y$. The claim then follows representably for a general $\mathcal{C}$ with finite limits. $\square$

Theorem 4.4.13. *Let $\mathcal{E}$ be a lextensive, cartesian closed category. Then $\mathcal{E}$ is well-pointed if and only if $\mathbf{Cat}(\mathcal{E})$ is 2-well-pointed in the sense of Definition 4.4.11.*

Proof. Recall that by Theorem 4.4.5, the copower $\mathbf{2} \odot \underline{\mathbf{1}} \in \mathbf{Cat}(\mathcal{E})$ may be taken as $\mathbf{2}_{\mathcal{E}}$. Corollary 4.4.8 therefore specialises to show that $\mathcal{E}$ being well-pointed implies that $\mathbf{Cat}(\mathcal{E})$ is 2-well-pointed by taking $\mathcal{G} := \{\mathbf{1}\}$. For the converse, recall from Remark 2.3.4 that a natural bijection $\mathcal{E}(\Pi_0(\mathbb{A}), B) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbb{A}, \mathbf{disc}(B))$ exists if the source and target morphisms of $\mathbb{A}$ have a coequaliser in $\mathcal{E}$. Recall from the discussion after Definition 4.4.4, with further details found in Example 2.3.2 of [Mir18], that the internal category $2_{\mathcal{E}}$ has object of n-simplices given by the $(n+2)$-fold coproduct of the terminal object $\mathbf{1}$. Now, Lemma 4.4.12 applies to $\mathcal{C} := \mathcal{E}^{\mathrm{op}}$ with $A = \mathbf{1}$, and shows that this coequaliser does exist in $\mathcal{E}$, so that $\Pi_0(\mathbf{2}_{\mathcal{E}}) \cong \mathbf{1}$. Therefore $\mathcal{E}(\Pi_0(\mathbf{2}_{\mathcal{E}}), B) \cong \mathbf{Cat}(\mathcal{E})_1(\mathbf{2}_{\mathcal{E}}, \mathbf{disc}(B))$ and so there is a bijection between diagrams of the following forms for $f, g \in \mathcal{E}(X, Y)$.

$$\mathbf{1} \longrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad \mathbf{2}_{\mathcal{E}} \longrightarrow \mathbf{disc}(X) \underset{\mathbf{disc}(g)}{\overset{\mathbf{disc}(f)}{\rightrightarrows}} \mathbf{disc}(Y)$$

Hence if $\{\mathbf{2}_{\mathcal{E}}\}$ is a generator in $\mathbf{Cat}(\mathcal{E})$ then $\mathbf{1}$ is a generator in $\mathcal{E}$. This completes the proof. $\square$

Observe that the assumptions of Theorem 4.4.13 hold if $\mathcal{E}$ is an elementary topos. In Theorem 4.6.7 we will characterise this stronger property for $\mathcal{E}$ in terms of $\mathbf{Cat}(\mathcal{E})$. Observe also that copowers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$ exist under assumptions which have already been shown to be equivalent for $\mathcal{E}$ and $\mathbf{Cat}(\mathcal{E})$, namely lextensivity and cartesian closedness. Since $\mathbf{Cat}(\mathcal{E})$ has copowers by $\mathbf{2}$, two-dimensional aspects of universal properties for 2-limits can be inferred from the one-dimensional aspects of these universal properties. This is dual to the argument for faithfulness on 2-cells of the family of 2-functors $\mathbf{Cat}(\mathcal{E})(G,-) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$ for $G \in \widehat{\mathcal{G}}$, given in the proof of Corollary 4.4.8. As such we will herein omit verification of two-dimensional aspects of universal properties for limits.

4.5 Natural numbers objects

We show that $\mathcal{E}$ has a natural numbers object if and only if $\mathbf{Cat}(\mathcal{E})$ has a natural numbers object, in the sense of Definition 4.5.1, to follow. In particular, the work of this section shows that a natural numbers object in $\mathbf{Cat}(\mathcal{E})$ is discrete on the natural numbers object of $\mathcal{E}$. Throughout this section we assume only that $\mathcal{E}$ has finite limits.

Definition 4.5.1. 1. Let $\mathcal{C}$ be a category with a terminal object $\mathbf{1}$. The data

$$\mathbf{1} \xrightarrow{z} N \xrightarrow{s} N$$

is called a *natural numbers object* in $\mathcal{C}$ if for any $\mathbf{1} \xrightarrow{f} X \xrightarrow{g} X$ there is a unique $u : N \to X$ making the diagram below commute.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "z"] \arrow[dr, bend right, "f"'] & N \arrow[r, "s"] \arrow[d, dashed, "u"] & N \arrow[d, dashed, "u"] \\
& X \arrow[r, "g"'] & X
\end{tikzcd}$$

2. Let $\mathcal{K}$ be a 2-category with a terminal object $\underline{\mathbf{1}}$. The data $\underline{\mathbf{1}} \xrightarrow{z} \underline{N} \xrightarrow{s} \underline{N}$ is called a *natural numbers object* in $\mathcal{K}$ if it is a natural numbers object for the underlying 1-category of $\mathcal{K}$ and, additionally, if given $\underline{\mathbf{1}} \xrightarrow{f} X \xrightarrow{g} X$ and $\underline{\mathbf{1}} \xrightarrow{f'} X \xrightarrow{g} X$ which have corresponding maps $u, u' : \underline{N} \to X$ respectively, whenever we have a 2-cell

$$\begin{tikzcd}
\underline{\mathbf{1}} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] & X
\end{tikzcd}$$

then there is a unique 2-cell as depicted below left, making the pasting diagram depicted below right commute.

$$\begin{tikzcd}
\underline{N} \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\overline{\phi}"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbf{1}} \arrow[r, "z"] \arrow[dr, "f"] \arrow[dr, bend right=50, "f'"'] \arrow[dr, phantom, bend right=25, "\Swarrow\overline{\alpha}"] & \underline{N} \arrow[r, "s"] \arrow[d, bend left, "u"] \arrow[d, bend right, "u'"'] \arrow[d, phantom, "\Leftarrow\overline{\phi}"] & \underline{N} \arrow[d, bend left, "u"] \arrow[d, bend right, "u'"'] \arrow[d, phantom, "\Leftarrow\overline{\phi}"] \\
& X \arrow[r, "g"'] & X
\end{tikzcd}$$

We first give a proof of the following standard result.

Lemma 4.5.2. *Let $\mathcal{C}, \mathcal{D}$ be categories with a terminal object and suppose $\mathcal{D}$ has a natural numbers object $(N, z : \mathbf{1} \to N, s : N \to N)$. If $L : \mathcal{D} \to \mathcal{C}$ is a left adjoint such that the unique morphism $j : L\mathbf{1} \to \mathbf{1}$ is invertible, then $(LN, L(z) \circ j^{-1} : \mathbf{1} \to LN, Ls : LN \to LN)$ is a natural numbers object for $\mathcal{C}$.*

Proof. Let $R : \mathcal{C} \to \mathcal{D}$ be the right adjoint of L. As a right adjoint, the unique morphism $k : R\mathbf{1} \to \mathbf{1}$ is invertible. Let $\mathbf{1} \xrightarrow{f} X \xrightarrow{g} X$ be in $\mathcal{C}$. By the adjunction $L \dashv R$, there is a bijection between diagrams of the forms depicted below.

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "z"] \arrow[dr, bend right, "R(f)\circ k^{-1}"'] & N \arrow[r, "s"] \arrow[d, dashed, "v"] & N \arrow[d, dashed, "v"] \\
 & RX \arrow[r, "Rg"'] & RX
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "L(z)\circ j^{-1}"] \arrow[dr, bend right, "f"'] & LN \arrow[r, "Ls"] \arrow[d, dashed, "u"] & LN \arrow[d, dashed, "u"] \\
 & X \arrow[r, "g"'] & X
\end{tikzcd}
$$

By the universal property of the natural numbers object $(N, z : \mathbf{1} \to N, s : N \to N)$ in $\mathcal{D}$, there is a unique such $v : N \to RX$. Hence such a $u : LN \to X$ exists and is unique, as required. $\square$

We obtain the following for one-dimensional natural number objects.

Corollary 4.5.3. *Let $\mathcal{E}$ be a category with terminal object and pullbacks. Then $\mathcal{E}$ has a natural numbers object if and only if $\mathbf{Cat}(\mathcal{E})_1$ has a natural numbers object.*

Proof. Apply Lemma 4.5.2 to $\mathbf{disc}(-) \dashv (-)_0$ for one implication, and to $(-)_0 \dashv \mathbf{indisc}(-)$ for the converse. $\square$

We extend this to a correspondence between a one-dimensional natural numbers object of $\mathcal{E}$ and a two-dimensional natural numbers object for $\mathbf{Cat}(\mathcal{E})$.

Theorem 4.5.4. *Let $\mathcal{E}$ be a category with finite limits. Then $\mathcal{E}$ has a natural numbers object if and only if the 2-category $\mathbf{Cat}(\mathcal{E})$ has a natural numbers object. In this case, the functors $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$, $(-)_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ and $\Pi_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ all preserve the natural numbers object.*

Proof. By Corollary 4.5.3, $\mathcal{E}$ has a natural numbers object if and only if $\mathbf{Cat}(\mathcal{E})_1$ does, and by Lemma 4.5.2 the functors mentioned preserve the natural numbers object. It suffices to show that

$$(\mathbf{disc}(N), \mathbf{disc}(z) : \underline{\mathbf{1}} \to \mathbf{disc}(N), \mathbf{disc}(s) : \mathbf{disc}(N) \to \mathbf{disc}(N))$$

satisfies the two-dimensional aspect of the universal property in Definition 4.5.1 part (2).

Consider a diagram

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] & \mathbb{X} \arrow[r, "g"] & \mathbb{X}
\end{tikzcd}
$$

in $\mathbf{Cat}(\mathcal{E})$ and let $u, u' : \mathbf{disc}(N) \to \mathbb{X}$ be the morphisms induced by the universal property of the natural numbers object. Since $\underline{\mathbf{1}} = \mathbf{disc}(\mathbf{1})$, the internal natural transformation $\overline{\alpha} : f \Rightarrow f'$ is uniquely determined by a map $\alpha : \mathbf{1} \to$

X_1 in $\mathcal{E}$. This along with the morphism $g_1 : X_1 \to X_1$ uniquely determines a map $\phi : N \to X_1$ giving rise to an internal natural transformation satisfying the commutativity conditions in $\mathbf{Cat}(\mathcal{E})$ depicted below left. Conversely, using the universal property of $\mathbb{X}^{\mathbf{2}}$ and the fact that $(\mathbb{X}^{\mathbf{2}})_0 = X_1$, an internal natural transformation $\overline{\phi} : u \Rightarrow u'$ satisfying $g.\overline{\phi} = \overline{\phi}.s$ corresponds to a morphism in $\mathcal{E}$ satisfying the commutativity condition depicted below right, where $\alpha := \phi.z$. Hence, $\overline{\phi}$ is unique and $(\mathbf{disc}(N), \mathbf{disc}(z) : \underline{\mathbf{1}} \to \mathbf{disc}(N), \mathbf{disc}(s) : \mathbf{disc}(N) \to \mathbf{disc}(N))$ is a natural numbers objects for the 2-category $\mathbf{Cat}(\mathcal{E})$.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "z"] \arrow[dr, "f"] \arrow[dr, bend right=50, "f'"'] \arrow[dr, phantom, bend right=25, "\Swarrow \overline{\alpha}"] & \mathbf{disc}(N) \arrow[r, "s"] \arrow[d, bend right=50, "u'"'] \arrow[d, bend left=50, "u"] \arrow[d, phantom, "\overset{\Leftarrow}{\overline{\phi}}"] & \mathbf{disc}(N) \arrow[d, bend right=50, "u'"'] \arrow[d, bend left=50, "u"] \arrow[d, phantom, "\overset{\Leftarrow}{\overline{\phi}}"] \\
& \mathbb{X} \arrow[r, "g"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "z"] \arrow[dr, bend right, "\alpha"'] & N \arrow[r, "s"] \arrow[d, dashed, "\phi"] & N \arrow[d, dashed, "\phi"] \\
& X_1 \arrow[r, "g_1"'] & X_1
\end{tikzcd}$$

□

Remark 4.5.5. We thank Ross Street for pointing us to Theorem 3.1 of [JW78]. In that theorem, $\mathcal{E}$ is assumed to be an elementary topos with a natural numbers object, and the image of the natural numbers object in $\mathbf{Cat}(\mathcal{E})$ of Corollary 4.5.3 is shown to be an up-to-isomorphism version of a natural numbers object in the 2-category of toposes bounded over $\mathcal{E}$.

4.6 Subobject classifiers

We show in this section that subobject classifiers in $\mathcal{E}$ give rise to something similar to a subobject classifier in $\mathbf{Cat}(\mathcal{E})$; rather than classifying monomorphisms as a subobject classifier would, the maps that are classified are monomorphisms which are also fully faithful. In this section we assume that $\mathcal{E}$ is lextensive and cartesian closed, so that $\mathbf{Cat}(\mathcal{E})$ has copowers by $\mathbf{2}$ as per Theorem 4.4.5. This means that the two-dimensional aspect of the universal property of pullbacks follows from the one-dimensional aspect, so we omit mention of it. Note $\mathcal{E}$ satisfying ETCS is in particular an elementary topos, and so is therefore lextensive.

Definition 4.6.1. Let $\mathcal{K}$ be a 2-category.

1. A morphism $i : A \to B$ is a *full monomorphism* if for every $X \in \mathcal{K}$ the functor $\mathcal{K}(X, i) : \mathcal{K}(X, A) \to \mathcal{K}(X, B)$ is fully faithful and injective on objects.

2. Two full monomorphisms $i : A \to B$ and $i' : A' \to B$ with the same codomain are said to be equivalent if there is an isomorphism $a : A \to A'$ satisfying $i'a = i$. A *full subobject* of B is an equivalence class of full monomorphisms into B.

3. A *full subobject classifier* is a full monomorphism $\top : \underline{\mathbf{1}} \to \underline{\Omega}$ such that for any fully faithful monomorphism $i : A \to B$, there is a unique morphism $\chi_i : B \to \underline{\Omega}$ making the following square a pullback.

$$\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, "i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \underline{\mathbf{1}} \arrow[d, "\top"] \\
B \arrow[r, "\chi_i"'] & \underline{\Omega}.
\end{tikzcd}$$

Remark 4.6.2. Note that $\underline{\top}$ being a full subobject classifier is precisely to say that it is a terminal object in the category whose objects are full subobjects in $\mathcal{K}$, and whose morphisms are pullback squares. This is indeed in analogy to the universal property defining subobject classifiers, with the 2-categorical notion of full subobjects replacing subobjects. Indeed, any monomorphism in a 1-category $\mathcal{C}$ is fully faithful as a morphism in the discrete 2-category on $\mathcal{C}$. As such, the notion of a full subobject classifier specialises to the notion of a subobject classifier in the setting where $\mathcal{K}$ has only identity 2-cells. This is in contrast to other categorifications of subobject classifiers such as discrete opfibration classifiers of [Web07]. On the other hand, full subobject classifiers in arbitrary 2-categories are typically not subobject classifiers in their underlying categories.

Note that we have not included any universal property for 2-cells into full subobject classifiers in Definition 4.6.1. It is an easy exercise to check that there is a unique internal natural transformations between any parallel pair of internal functors whose codomain is an indiscrete internal category. Since the full subobject classifiers that we construct in Proposition 4.6.3 will be indiscrete internal categories, we could have included this feature as part of the definition. We have refrained from doing so since it is not needed for Theorem 4.6.7, and also since doing so would lose subobject classifiers in 1-categories as examples.

Proposition 4.6.3. *Suppose $\mathcal{E}$ has a subobject classifier $\top : \mathbf{1} \to \Omega$. Then* $\mathbf{indisc}(\top) : \underline{\mathbf{1}} \to \mathbf{indisc}\,(\top)$ *is a full subobject classifier for* $\mathbf{Cat}(\mathcal{E})$.

Proof. Let $f : \mathbb{X} \to \mathbb{Y}$ be a full monomorphism. Then $f_0 : X_0 \to Y_0$ is a monomorphism in $\mathcal{E}$. Since $\mathcal{E}$ has a subobject classifier, we have a unique $\chi_{f_0} : Y_0 \to \Omega$ such that the square depicted below left is a pullback. Now, since $(-)_0 \dashv$ **indisc**, the adjunct of χ_{f_0} is a unique map $\chi_f : \mathbb{Y} \to \mathbf{indisc}(\Omega)$ making the square below right commute. We need to show that this square is a pullback.

$$
\begin{tikzcd}
X_0 \arrow[r, "!"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\top"] \\
Y_0 \arrow[r, "\exists ! \chi_{f_0}"'] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "!"] \arrow[d, "f"'] & \underline{\mathbf{1}} \arrow[d, "\mathbf{indisc}(\top)"] \\
\mathbb{Y} \arrow[r, "\exists ! \chi_f"'] & \mathbf{indisc}(\Omega)
\end{tikzcd}
$$

But the required square is clearly a pullback on objects, and given on morphisms as displayed below. By Proposition 2.2.6 part (2), it suffices to show that this square is a pullback. But the left square is indeed a pullback since $f : \mathbb{A} \to \mathbb{B}$ is fully faithful. The proof is complete by the pullback lemma.

$$
\begin{tikzcd}
X_1 \arrow[r, "{(d_0,d_1)}"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[r, "!"] \arrow[d, "f_0 \times f_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "{(\top,\top)}"] \\
Y_1 \arrow[r, "{(d_0,d_1)}"'] & Y_0 \times Y_0 \arrow[r, "\chi_{f_0} \times \chi_{f_0}"'] & \Omega \times \Omega
\end{tikzcd}
$$

$\square$

Example 4.6.4. Taking $\mathcal{E} = \mathbf{Set}$, the full subobject classifier in **Cat** is given by the free-living isomorphism $\mathbf{I} := \{\bot \cong \top\}$.

The proof of the converse follows easily from the adjunction $(-)_0 \dashv \mathbf{indisc}$.

Proposition 4.6.5. *Let $\mathcal{E}$ be a category with terminal object and pullbacks. Suppose* $\mathbf{Cat}(\mathcal{E})$ *has a full subobject classifier $\underline{\top} : \underline{\mathbf{1}} \to \underline{\Omega}$. Then $\underline{\top}_0 : \mathbf{1} \to \underline{\Omega}_0$ is a subobject classifier for $\mathcal{E}$.*

Proof. Let $i : A \to B$ be a monomorphism in $\mathcal{E}$. Then

$$\mathbf{indisc}(i) : \mathbf{indisc}(A) \to \mathbf{indisc}(B)$$

is clearly fully faithful and mono on objects; monomorphisms are closed under products and the maps $\mathbf{indisc}(X)_1 \to \mathbf{indisc}(X)_0 \times \mathbf{indisc}(X)_0$ are identities for $X \in \{A, B\}$, so that the relevant square defining fully-faithfulness is indeed a pullback. Hence, there exists a pullback square in $\mathbf{Cat}(\mathcal{E})$ as displayed below left. Since $(-)_0$ is a right adjoint, it preserves limits and in particular pullbacks. Hence, using the fact that $(-)_0 \circ \mathbf{indisc} = 1$, we have the pullback square in $\mathcal{E}$ depicted below right. Uniqueness also follows by adjointness.

$$\begin{tikzcd}
\mathbf{indisc}(A) \arrow[r, "!"] \arrow[d, "\mathbf{indisc}(i)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{1}} \arrow[d, "\underline{\top}"] \\
\mathbf{indisc}(B) \arrow[r, "\phi"'] & \underline{\Omega}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\underline{\top}_0"] \\
B \arrow[r, "\phi_0"'] & \underline{\Omega}_0
\end{tikzcd}$$

□

Remark 4.6.6. Note that the above proof holds without the assumption that $\mathcal{E}$ is extensive.

We have just proven the following result.

Theorem 4.6.7. *Let $\mathcal{E}$ be an extensive, cartesian closed category with finite limits. Then $\mathcal{E}$ has a subobject classifier if and only if the 2-category $\mathbf{Cat}(\mathcal{E})$ has a full subobject classifier. In this case, the 2-functor $\mathbf{indisc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$, with $\mathcal{E}$ being considered as a locally discrete 2-category, preserves full subobject classifiers.*

Proof. Combine Propositions 4.6.3 and 4.6.5. □

Remark 4.6.8. We characterise booleanness and two-valuedness of $\mathcal{E}$ in terms of properties in $\mathbf{Cat}(\mathcal{E})$. These properties follow for $\mathcal{E}$ from the axioms of ETCS. Booleanness is a consequence of the Axiom of Choice [Dia75], and in fact both of these properties are a consequence of well-pointedness (Proposition 7, Part VI of [MM94]). As such, the equivalent properties that we are about to describe in $\mathbf{Cat}(\mathcal{E})$ will also follow as a consequence of the axioms in the elementary theory of the 2-category of small categories, which we will give in Subsection 4.8.1.

Consider the two internal functors $\underline{\mathbf{1}} \to \mathbf{2}_{\mathcal{E}}$ which are the source and target of the universal 2-cell exhibiting $\mathbf{2}_{\mathcal{E}}$ as the copower of $\underline{\mathbf{1}} \in \mathbf{Cat}(\mathcal{E})$ by $\mathbf{2} \in \mathbf{Cat}$. It is easy to see that these are both full monomorphisms. Hence by Proposition 4.6.3, they determine internal functors $\mathbf{2}_{\mathcal{E}} \to \mathbf{indisc}(\Omega)$. At the level of objects, one of these is given by $(\top, \bot) : \mathbf{1} + \mathbf{1} \to \Omega$ while the other is given by $(\bot, \top)$. Recall from [Joh77, Proposition 5.14] that an elementary topos $\mathcal{E}$ is *boolean* if and only if these morphisms are invertible. As such, $\mathcal{E}$ is boolean if and only if either (hence both) of these internal functors in $\mathbf{Cat}(\mathcal{E})$ are codescent morphisms, since as discussed in Remark 4.2.1 these are precisely the internal functors which are isomorphic on objects. Similarly, recall that an elementary topos is *two-valued* if and only if the hom-set $\mathcal{E}(\mathbf{1}, \Omega)$ has exactly two morphisms, namely $\top$ and $\bot$. Hence by fully faithfulness of $\mathbf{indisc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$,

$\mathcal{E}$ is two-valued if and only if in $\mathbf{Cat}(\mathcal{E})$ there are exactly two morphisms from the terminal object to the full subobject classifier. In this case the hom-category $\mathbf{Cat}(\mathcal{E})(\underline{\mathbf{1}}, \mathbf{indisc}(\Omega))$ is the free-living isomorphism.

Remark 4.6.9. When $\mathcal{E}$ is an elementary topos, the internal functor $\mathbf{disc}(\top) : \underline{\mathbf{1}} \to \mathbf{disc}(\Omega)$ is also a classifier for a certain class of monomorphisms. These are those internal functors which are monomorphisms between objects of objects, and discrete bifibrations; a notion that can either be defined representably in $\mathbf{Cat}(\mathcal{E})$, or internally to $\mathcal{E}$ by asking $f_0 d_k = d_k f_1$ to be a pullback for $k \in \{0, 1\}$. We call such functors *strict bi-sieves*. When $\mathcal{E} = \mathbf{Set}$, such functors determine a subset of the set of connected components of their codomain, and are inclusions of full subcategories on all objects in those connected components. Indeed, the proof uses the adjunction $\Pi_0 \dashv \mathbf{disc}$ of Remark 2.3.4. We give only a sketch of the proof, since this will not be needed for any of the results in this chapter.

Via $\Pi_0 \dashv \mathbf{disc}$, a classifier $\mathbb{B} \to \mathbf{disc}(\Omega)$ corresponds to a morphism $\Pi_0(\mathbb{B}) \to \Omega$, which in turn corresponds to a monomorphism $f' : X \to \Pi_0(\mathbb{B})$ in $\mathcal{E}$. The coequaliser diagram depicted below left is sent by $\Omega^{(-)} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ to the equaliser diagram below right.

$$B_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} B_0 \xrightarrow{q_{\mathbb{B}}} \Pi_0(\mathbb{B}) \qquad\qquad \Omega^{B_1} \underset{\Omega^{d_0}}{\overset{\Omega^{d_1}}{\leftleftarrows}} \Omega^{B_0} \xleftarrow{\Omega^{q_{\mathbb{B}}}} \Omega^{\Pi_0(\mathbb{B})}$$

But the morphisms $\Omega^{d_k} : \Omega^{B_0} \to \Omega^{B_1}$ for $k \in \{0, 1\}$ correspond to pullbacks of monomorphisms. As such the monomorphism f' corresponds to a monomorphism $f_0 : A_0 \to B_0$ whose pullback along both $d_0, d_1 : B_1 \to B_0$ are the same monomorphism $f_1 : A_1 \to B_1$. These data precisely correspond to a strict bi-sieve $f : \mathbb{A} \to \mathbb{B}$. One shows that this moreover satisfies $\Pi_0(f) = f'$.

Proposition 4.6.10. *Suppose* $\mathbf{Cat}(\mathcal{E})$ *has finite* 2*-limits, is* 2*-cartesian closed and has a full subobject classifier. Then:*

1. $\mathcal{E}$ *is extensive.*

2. $\mathbf{Cat}(\mathcal{E})$ *is extensive.*

Proof. By Proposition 4.2.2, the assumptions that $\mathbf{Cat}(\mathcal{E})$ has finite 2-limits means that $\mathcal{E}$ has finite limits, and so by noting Remark 4.6.6, we can apply Proposition 4.6.5 and obtain a subobject classifier in $\mathcal{E}$. By Theorem 4.3.1, it follows that $\mathcal{E}$ is cartesian closed and so $\mathcal{E}$ is an elementary topos and therefore extensive. By Lemma 4.4.2, it follows that $\mathbf{Cat}(\mathcal{E})$ is extensive. $\square$

4.7 The Axiom of Choice

It is well known that the Axiom of Choice is equivalent to the statement that any essentially surjective on objects and fully faithful functor is part of an adjoint equivalence in $\mathbf{Cat}$ ([FS90], 1.364). The Axiom of Choice is also equivalent to the proposition that any surjective-on-objects and fully faithful functor has a section. The second of these formulations is easier to treat in the context of internal category theory. Establishing this logical equivalence is the aim of Subsection 4.7.1. Subsection 4.7.2 will consider how the property of being epimorphic-on-objects can be expressed abstractly in the 2-category $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ without reference to the fact that $\mathcal{K}$ is of this form. In particular, we will show that the class of epimorphic-on-objects internal functors in **Cat**$(\mathcal{E})$ is precisely the left orthogonality class with respect to the fully faithful monomorphisms. For this, we need the assumption that $\mathcal{E}$ has an (epi, mono)-factorisation system, which is true in any elementary (or indeed pre-)topos.

We note that in Chapter 5, we continue our study of choice principles in internal category theory. In Section 5.2.2, we provide a formulation of the Presentation Axiom and an internal version of the Law of the Excluded Middle, building on the theory developed in this section. In Section 5.5, we show that internally it is true that the external Axiom of Choice is equivalent to the fact that any essentially surjective on objects and fully faithful functor is part of an adjoint equivalence.

4.7.1 In terms of internal category theory

Definition 4.7.1. A category $\mathcal{E}$ is said to satisfy the *external Axiom of Choice* if every epimorphism $e : X \to Y$ has a section. That is, there exists a map $s : Y \to X$ satisfying $es = 1_X$.

We give a proof that the external Axiom of Choice for $\mathcal{E}$ is equivalent to the proposition that any epimorphic-on-objects functor that is fully faithful has a section. For this equivalence, we require that $\mathcal{E}$ has pullbacks and products.

Lemma 4.7.2. *Let $\mathcal{E}$ be a category with pullbacks and products and let $e : \mathbb{A} \to \mathbb{B}$ be a fully faithful internal functor. Suppose e_0 has a splitting $s_0 : B_0 \to A_0$. Then s_0 extends to an internal functor $s : \mathbb{B} \to \mathbb{A}$, with assignment on arrows given as depicted below. Moreover, $es = 1_{\mathbb{B}}$.*

$$\begin{tikzcd}
B_1 \arrow[d, "(d_0,d_1)"'] \arrow[dr, dashed, "s_1"] \arrow[drr, bend left, "1_{B_1}"] & & \\
B_0 \times B_0 \arrow[dr, "s_0 \times s_0"'] & A_1 \arrow[r, "e_1"] \arrow[d, "(d_0,d_1)"] \arrow[dr, phantom, very near start, "\lrcorner"] & B_1 \arrow[d, "(d_0,d_1)"] \\
& A_0 \times A_0 \arrow[r, "e_0 \times e_0"'] & B_0 \times B_0
\end{tikzcd}$$

Proof. By construction, s_1 is a section of $e_1 : A_1 \to B_1$ and $s := (s_0, s_1)$ forms a morphism of the underlying graphs of $\mathbb{B}$ and $\mathbb{A}$. This morphism of graphs clearly gives a splitting of e. We need to prove that this is well-defined as an internal functor. We show it respects identities using the universal property of A_1. Compatibility with the pullback projection e_1 follows from the commutativity of the diagram displayed below left, while compatibility with the other pullback projection follows from the commutativity of the diagram below right, for $k \in \{0, 1\}$.

$$\begin{tikzcd}
B_0 \arrow[r, "s_0"] \arrow[dd, "i"'] \arrow[dr, "1_{B_0}"'] & A_0 \arrow[r, "i"] \arrow[d, "e_0"] & A_1 \arrow[dd, "e_1"] \\
& B_0 \arrow[dr, "i"] & \\
B_1 \arrow[r, "s_1"'] \arrow[rr, bend left, "1_{B_1}"] & A_1 \arrow[r, "e_1"'] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, "s_0"] \arrow[dd, "i"'] \arrow[dr, "1_{B_0}"'] & A_0 \arrow[r, "i"] \arrow[ddr, "1_{A_0}"] & A_1 \arrow[dd, "d_k"] \\
& B_0 \arrow[dr, "s_0"'] & \\
B_1 \arrow[r, "s_1"'] \arrow[ur, "d_k"] & A_1 \arrow[r, "d_k"'] & B_1
\end{tikzcd}$$

Similarly, respect for composition also follows from the universal property of A_1 as per the calculations displayed below. This completes the proof.

$$
\begin{tikzcd}
B_2 \arrow[r, "s_2"] \arrow[dr, "\pi_k"] \arrow[ddd, "m"'] & A_2 \arrow[rr, "m"] \arrow[dr, "\pi_k"] & & A_1 \arrow[ddd, "d_k"] \\
& B_1 \arrow[r, "s_1"] \arrow[d, "d_k"] & A_1 \arrow[ddr, "d_k"] & \\
& B_0 \arrow[drr, "s_0"] & & \\
B_1 \arrow[ur, "d_k"] \arrow[rr, "s_1"'] & & A_1 \arrow[r, "d_k"'] & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[r, "s_2"] \arrow[dr, "1_{B_2}"'] \arrow[dd, "m"'] & A_2 \arrow[r, "m"] \arrow[d, "e_2"] & A_1 \arrow[dd, "e_1"] \\
& B_2 \arrow[dr, "m"] & \\
B_1 \arrow[r, "s_1"'] \arrow[rr, bend left, "1_{B_1}"] & A_1 \arrow[r, "e_1"'] & B_1
\end{tikzcd}
$$

□

Remark 4.7.3. By fully-faithfullness, s can be shown to be a right adjoint equivalence right inverse to e. The unit $\eta : 1_{\mathbb{A}} \Rightarrow se$ is determined by 1_e given $e = 1_{\mathbb{B}}.e = ese$ and representable fully-faithfulness of $e : \mathbb{A} \to \mathbb{B}$. Adjointness and invertibility of η follow from representable faithfulness and conservativity of e, respectively.

Proposition 4.7.4. *Let $\mathcal{E}$ be a category with pullbacks. The following are equivalent:*

1. *The external Axiom of Choice holds in $\mathcal{E}$.*

2. *Any fully faithful and epimorphism on objects functor internal to $\mathcal{E}$ has a section in the* 2*-category* **Cat**$(\mathcal{E})$*.*

Proof. Let $e : \mathbb{A} \to \mathbb{B}$ be an epi-on-objects and fully faithful functor. Assuming the external Axiom of Choice for $\mathcal{E}$, the morphism $e_0 : A_0 \to B_0$ has a splitting. The splitting for the internal functor $e : \mathbb{A} \to \mathbb{B}$ is given in Lemma 4.7.2.

Conversely, assume that every epi-on-objects and fully faithful functor has a section. Let $f : X \to Y$ be an epimorphism in $\mathcal{E}$. The internal functor **indisc**(f) : **indisc**$(X) \to$ **indisc**(Y) is fully faithful and an epimorphism on objects and hence has a section s : **indisc**$(Y) \to$ **indisc**(X) giving us $s_0 : Y \to X$, a section of f. □

Example 4.7.5. When $\mathcal{E}$ = **Set**, functors which are epi-on-objects and fully faithful are the right class of a weak factorisation system on **Cat**, with the left class being the injective on objects functors. This factorisation system features in the canonical model structure on **Cat**. See [EKL05; JT06] and Chapter 6 for more on homotopical aspects of internal category theory.

Remark 4.7.6. We briefly outline how Proposition 4.7.4 sheds light on category theory internal to categories which do not satisfy the external Axiom of Choice. When $\mathcal{E}$ does not satisfy the external Axiom of Choice, one often works with internal anafunctors, rather than internal functors, between internal categories so that 'weak equivalences' are actually adjoint equivalences [Mak96; Rob12; Rob21]. Anafunctors internal to $\mathcal{E}$ are typically defined in terms of covering families, an important example of which is the one generated by regular epimorphisms. In this setting, internal anafunctors $\mathbb{A} \twoheadrightarrow \mathbb{B}$ are spans of ordinary internal functors $\mathbb{A} \xleftarrow{l} \mathbb{F} \xrightarrow{r} \mathbb{B}$ in which l is fully faithful, and a regular epimorphism on objects. If regular epimorphisms are stable under pullback then internal anafunctors form the morphisms of a bicategory **Ana**(**Cat**$(\mathcal{E})$), with their composition involving pullbacks in **Cat**$(\mathcal{E})$. There is a canonical homomorphism of bicategories I : **Cat**$(\mathcal{E}) \to$ **Ana**(**Cat**$(\mathcal{E})$), which is the identity on objects and a full monomorphism between hom-categories. It views a functor as an anafunctor by taking the left leg $l : \mathbb{F} \to \mathbb{A}$ to be the identity on $\mathbb{A}$.

If any epimorphism in $\mathcal{E}$ is regular and $\mathcal{E}$ has an (epi, mono) orthogonal factorisation system, as is the case when $\mathcal{E}$ is an elementary topos, then by Remark 4.7.3, Proposition 4.7.4 says precisely that the external Axiom of Choice holds for $\mathcal{E}$

if and only if the left leg $l : \mathbb{F} \to \mathbb{A}$ in any internal anafunctor is in fact a left adjoint left inverse equivalence in $\mathbf{Cat}(\mathcal{E})$. In this case, the homomorphism of bicategories $I : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Ana}(\mathbf{Cat}(\mathcal{E}))$ has functors between hom-categories which are essentially surjective on objects. Thus if the external Axiom of Choice holds for $\mathcal{E}$ then the 2-category $\mathbf{Cat}(\mathcal{E})$ is biequivalent to the bicategory $\mathbf{Ana}(\mathbf{Cat}(\mathcal{E}))$. These observations will be generalised to appropriate 2-categories $\mathcal{K}$ in place of $\mathbf{Cat}(\mathcal{E})$ in Remark 4.7.14.

4.7.2 In 2-categorical terms

The property of being an epimorphism on objects may appear difficult to express in terms of the 2-categorical structure of $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, without reference to the fact that it is of this form. To fix this, we first show in Proposition 4.7.7, to follow, that orthogonal factorisation systems on $\mathcal{E}$ give rise to orthogonal factorisation systems on the 2-category $\mathbf{Cat}(\mathcal{E})$, as defined in [Day06] and described explicitly in Remark 2.3.2 of [Bou10]. This result is stated without proof in the discussion between Propositions 62 and 63 of [BG14]. We believe it to be of independent interest, and give a detailed proof in Appendix C. For our purposes, it will mean that epimorphism on objects internal functors can then be characterised via this left orthogonality property.

Proposition 4.7.7. *Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorisation system on a category $\mathcal{E}$ with pullbacks and products. Then*

$$(\mathcal{L}\textit{-on-objects}, \mathcal{R}\textit{-on-objects and fully faithful})$$

is an orthogonal factorisation system on the 2-category $\mathbf{Cat}(\mathcal{E})$.

Corollary 4.7.8. *Let $\mathcal{E}$ be a category with pullbacks, products, and an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$ in which $\mathcal{L}$ are the epimorphisms and $\mathcal{R}$ are the monomorphisms. Then $(\mathcal{L}', \mathcal{R}')$ is an orthogonal factorisation system on $\mathbf{Cat}(\mathcal{E})$, where $\mathcal{L}'$ is the class of internal functors which are epi-on-objects, and $\mathcal{R}'$ is the class of full monomorphisms.*

Proof. By Proposition 4.7.7 there is an orthogonal factorisation system $(\mathcal{L}', \mathcal{R}')$ on the 2-category $\mathbf{Cat}(\mathcal{E})$ in which $\mathcal{L}'$ is as required and $\mathcal{R}'$ is the class of internal functors which are both fully faithful and given by monomorphisms on objects. But as discussed in the beginning of Remark 2.2.10, such internal functors are precisely the full monomorphisms in $\mathbf{Cat}(\mathcal{E})$. $\square$

Remark 4.7.9. The factorisation system on $\mathbf{Cat}(\mathcal{E})$ obtained in Corollary 4.7.8 is an internal version of the factorisation system constructed in **Cat** via kernels and quotients, in 5.2 of [BG14]. The class of full monomorphisms, and its left orthogonality class, are respectively called *chronic* and *acute* in 1.1 and 1.4 of [Str82].

Definition 4.7.10. [Str82] A morphism in a 2-category $\mathcal{K}$ which is left orthogonal to all fully faithful monomorphisms in $\mathcal{K}$ will be called *acute*.

Remark 4.7.11. We note that acute maps in a 2-category are called SO-strong regular epimorphisms in [BG14]. In any SO-regular 2-category, SO-strong regular epimorphisms are the same as SO-regular epimorphism [BG14, Corollary 25], and so in an SO-regular 2-category, acute morphisms are exactly the SO-regular epimorphisms.

Definition 4.7.12. Say that a 2-category $\mathcal{K}$ satisfies the *Categorified Axiom of Choice* if any acute fully faithful morphism has a section.

Putting these results together gives the following reformulation of the external Axiom of Choice in $\mathcal{E}$ in terms of the 2-categorical structure of $\mathbf{Cat}(\mathcal{E})$.

Theorem 4.7.13. *Let $\mathcal{E}$ be a category with pullbacks, products and an (epi, mono)-orthogonal factorisation system. Then the following are equivalent.*

1. *The category $\mathcal{E}$ satisfies the external Axiom of Choice.*

2. *The* 2*-category* $\mathbf{Cat}(\mathcal{E})$ *satisfies the Categorified Axiom of Choice.*

Proof. Proposition 4.7.4 established the logical equivalence between the external Axiom of Choice in $\mathcal{E}$ and an analogue of the categorified Axiom of Choice for $\mathbf{Cat}(\mathcal{E})$ with 'epi-on-objects' in place of acute. But Corollary 4.7.8 ensures that being an epimorphism on objects characterises acute morphisms in $\mathbf{Cat}(\mathcal{E})$. □

Remark 4.7.14. The discussion in Remark 4.7.6 is also possible to rephrase in 2-categorical terms, rather than in terms of internal category theory. Let $\mathcal{K}$ be a 2-category with pullbacks and suppose that acute morphisms are stable under pullback in $\mathcal{K}$. Define an *anamorphism* in $\mathcal{K}$ to be a span whose left leg is acute and fully faithful. Then there is a bicategory $\mathbf{Ana}(\mathcal{K})$ defined in the usual way. There is also a homomorphism of bicategories $I : \mathcal{K} \to \mathbf{Ana}(\mathcal{K})$ which is given by the identity on objects and full monomorphisms between hom-categories. If the categorified Axiom of Choice holds in $\mathcal{K}$, then I moreover has functors between hom-categories which are essentially surjective on objects. Hence in this case I a biequivalence, exhibiting morphism composition as a strictification of anamorphism composition.

Remark 4.7.15. We thank Richard Garner for observing that when $\mathcal{E}$ is regular, acuteness of fully faithful internal functors is equivalent to the simpler property of being a regular epimorphism in the 1-category $\mathbf{Cat}(\mathcal{E})_1$. It is clear that if $\mathcal{E}$ has products, then since $(-)_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ is a left adjoint it preserves regular epimorphisms. Conversely, if $f_0 : \mathbb{A}_0 \to \mathbb{B}_0$ is a regular epimorphism and $\mathcal{E}$ is a regular category then f_0 is the coequaliser of its kernel pair in $\mathcal{E}$. Then $f_1 : A_1 \to B_1$ is also a regular epimorphism, since $f : \mathbb{A} \to \mathbb{B}$ is fully faithful and regular epimorphisms are closed under products and stable under pullback in $\mathcal{E}$. One verifies that f is the coequaliser of its kernel pair in $\mathbf{Cat}(\mathcal{E})_1$ using the universal property of the coequalisers f_0 and f_1 in $\mathcal{E}$ and the explicit construction of coequalisers in $\mathbf{Cat}(\mathcal{E})$ given in Chapter 3; we leave these details to the interested reader. By Remark 4.7.11, the classes of acute and fully faithful maps, fully faithful SO-regular epimorphisms in the 2-category $\mathbf{Cat}(\mathcal{E})$ and fully faithful regular epimorphisms in $\mathbf{Cat}(\mathcal{E})_1$ all coincide.

4.8 Comparing ETCS to ET2CSC

We collect the main results of previous sections and characterise 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ is a model of the elementary theory of the category of sets. Our characterisation of such 2-categories is in 2-categorical terms, rather than in terms of category theory internal to the discrete objects of $\mathcal{K}$. The theory of such 2-categories is again elementary, although we refrain from providing an explicit first order presentation as is done for ETCS on [nLa23]. Following this, in Subsection 4.8.2 we describe relationships between different models of ET2CSC, and establish a 'Morita biequivalence' between ETCS and ET2CSC.

4.8.1 A characterisation of $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ is a model of ETCS

Definition 4.8.1. We say that the 2-category $\mathcal{K}$ models the *elementary theory of the* 2*-category of small categories* (ET2CSC) if the following properties hold:

1. It satisfies the conditions listed in Proposition 2.4.19.
2. It is cartesian closed.
3. It is 2-well-pointed, in the sense of Definition 4.4.11.
4. It has a natural numbers object, in the sense of Definition 4.5.1 part (2).
5. It has a full subobject classifier, in the sense of Definition 4.6.1 part (3).
6. It satisfies the Categorified Axiom of Choice, in the sense of Definition 4.7.12.

We are now ready to combine the results so far and prove our first main result.

Theorem 4.8.2.

1. *Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ models the elementary theory of the category of sets if and only if* $\mathbf{Cat}(\mathcal{E})$ *models the elementary theory of the* 2*-category of small categories, and in this case* $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$.
2. *Conversely, let $\mathcal{K}$ be a* 2*-category. Then $\mathcal{K}$ models the elementary theory of the* 2*-category of small categories if and only if* $\mathbf{Disc}\,(\mathcal{K})$ *models the elementary theory of the category of sets, and in this case* $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$.

Proof. Proposition 2.4.19 gives the correspondence between pullbacks in $\mathcal{E}$ and the first item of Definition 4.8.1, as well as the equivalences $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$ and $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$. We describe how the results in this chapter so far give correspondences between the various other properties of ETCS and ET2CSC.

The correspondence for terminal objects is given in Proposition 4.2.2, and the correspondence for cartesian closedness is Theorem 4.3.1. Herein, assume that the category $\mathcal{E}$ (resp. the 2-category $\mathcal{K}$) satisfies the properties mentioned so far. Assuming additionally that $\mathcal{E}$ has a subobject classifier makes $\mathcal{E}$ into an elementary topos; in particular it is extensive and so by Theorem 4.6.7, $\mathbf{Cat}(\mathcal{E})$ has a full subobject classifier. Conversely, assuming that $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ has a full subobject classifier means that $\mathcal{E}$ is extensive by Proposition 4.6.10 and so by Theorem 4.6.7, we get the other direction of this correspondence. Note also that by Theorem 4.4.5, under these assumptions $\mathcal{K}$ (resp. $\mathbf{Cat}(\mathcal{E})$) has copowers by $\mathbf{2}$.

The correspondence between well-pointedness and 2-well-pointedness is Theorem 4.4.13. The correspondence for natural numbers objects is Theorem 4.5.4. Finally, the correspondence between the Axiom of Choice and the Categorified Axiom of Choice is Theorem 4.7.13. This last correspondence uses the epi-mono factorisation system on $\mathcal{E}$ (resp. $\mathbf{Disc}(\mathcal{K})$), which exists since by this stage this category is an elementary topos. $\square$

Theorem 4.8.2 will be built upon further in Subsection 4.8.2, where we will define 2-categories whose objects are models of ETCS and ET2CSC respectively, and prove that these two 2-categories are biequivalent in Theorem 4.8.13.

Remark 4.8.3. Assuming that $\mathcal{K}$ satisfies the conditions listed in Proposition 2.4.19, the one-dimensional aspects of the remaining conditions in ET2CSC are enough to imply that **Disc**$(\mathcal{K})$ satisfies ETCS, and hence that $\mathcal{K}$ satisfies the two-dimensional aspects of ET2CSC. In particular, the theory can be simplified by removing the two-dimensional aspect of cartesian closedness, the faithfulness on 2-cells aspect of 2-well-pointedness, the two-dimensional aspect of the universal property of natural numbers objects, and the two-dimensional aspect of left orthogonality in the definition of acute maps. Indeed, as discussed in Remark 4.6.2, we could have also included a two-dimensional universal property in our definition of a full subobject classifier. Such a definition would demand a representing object $\underline{\Omega}$ for the 2-functor $\mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$ which sends an object X to the indiscrete category on the set of full subobjects into X, and acts on morphisms via pullback. We chose not to give such a definition so that we retained ordinary subobject classifiers as examples.

4.8.2 Morphisms of models of ET2CSC

The notion of what a morphism of models of ETCS or of ET2CSC should be is clear from the description of these theories, but we spell it out in detail in Definition 4.8.4, to follow. The aim of this Subsection is to extend Theorem 4.8.2 to a correspondence between morphisms of models of the two theories, and to show that they have biequivalent 2-categories of models.

Definition 4.8.4.

1. Let $\mathcal{E}$ and $\mathcal{E}'$ be categories modelling ETCS. An *ETCS-morphism* is a functor $F : \mathcal{E} \to \mathcal{E}'$ which preserves finite limits, internal homs, the subobject classifier, and the natural numbers object.

2. Let $\mathcal{K}$ and $\mathcal{K}'$ be 2-categories modelling ET2CSC. An *ET2CSC-morphism* is a 2-functor $\mathbf{F} : \mathcal{K} \to \mathcal{K}'$ which preserves pullbacks, powers by $\mathbf{2}$, codescent objects of cateads, the terminal object, internal homs, the full subobject classifier, and the natural numbers object.

Proposition 4.8.5. *(Theorem 4.28 of [Bou10]) Let $F : \mathcal{E} \to \mathcal{E}'$ be a pullback preserving functor. Then* $\mathbf{Cat}\,(F)$ *preserves pullbacks, powers by* $\mathbf{2}$ *and codescent objects of cateads and there is a natural isomorphism* $F \cong \mathbf{Disc} \circ \mathbf{Cat}(F)$*. Conversely, if a 2-functor* $G : \mathbf{Cat}\,(\mathcal{E}) \to \mathbf{Cat}\,(\mathcal{E}')$ *preserves pullbacks, powers by* $\mathbf{2}$ *and codescent objects of cateads, then* $\mathbf{Disc}\,(G) : \mathbf{Disc} \circ \mathbf{Cat}\,(\mathcal{E}) \to \mathbf{Disc} \circ \mathbf{Cat}\,(\mathcal{E}')$ *preserves pullbacks and there is a 2-natural isomorphism* $G \cong \mathbf{Cat} \circ \mathbf{Disc}\,(G)$.

Remark 4.8.6. By Proposition 4.8.5 an ET2CSC-morphism is isomorphic to one of the form $\mathbf{F} = \mathbf{Cat}(F)$ for some pullback preserving functor $F : \mathcal{E} \to \mathcal{E}'$. As such, we will continue this section assuming that $\mathbf{F} \cong \mathbf{Cat}(F)$ for some such $F : \mathcal{E} \to \mathcal{E}'$. Note that since ET2CSC-morphisms preserve pullbacks, terminal objects and powers by $\mathbf{2}$, they preserves all 2-limits. The reason that well-pointedness, the Axiom of Choice and their respective analogues do not feature in Definition 4.8.4 is that these are properties rather than structure to be preserved. In any case, logical functors preserve epimorphisms and the terminal object, and once we show that **Disc**(F) for a morphism of models of ET2CSC is a logical functor, it will follow in Corollary 4.8.11 that F also preserves coproducts, copowers by $\mathbf{2}$, and acute morphisms.

Theorem 4.8.7. *A 2-functor* $\mathbf{F} : \mathcal{K} \to \mathcal{K}'$ *between categories satisfying ET2CSC is an ET2CSC-morphism if and only if is is of the form* $\mathbf{F} \cong \mathbf{Cat}(F)$ *for some* $F : \mathcal{E} \to \mathcal{E}'$ *where* F *is an ETCS-morphism.*

We prove this through a series of lemmata. In these, we repeatedly use the fact that $(-)_0$ is a 2-natural transformation from the 2-functor $\mathbf{Cat}(-) : \mathbf{Lex} \to \mathbf{Lex}$ to the identity on $\mathbf{Lex}$. Here $\mathbf{Lex}$ denotes the 2-category whose objects are categories with finite limits, whose morphisms are functors that preserve finite limits, and whose 2-cells are arbitrary 2-natural transformations. Similarly, we use that $\mathbf{disc} : 1_{\mathbf{Lex}} \to \mathbf{Cat}(-)$ is a 2-natural transformation and that $\mathbf{indisc} : 1_{\mathbf{Lex}} \to \mathbf{Cat}(-)$ is a pseudonatural transformation. See [Mir18] for proofs of these properties, although we will address preservation of the terminal object in Lemma 4.8.8 for completeness. Throughout these proofs, suppose that our 2-functor $\mathbf{F}$ preserves pullbacks, powers by $\mathbf{2}$ and codescent objects of cateads, so that it is of the form $\mathbf{F} = \mathbf{Cat}(F)$ for some $F : \mathcal{E} \to \mathcal{E}'$.

Lemma 4.8.8. *A pullback preserving functor $F : \mathcal{E} \to \mathcal{E}'$ preserves the terminal object if and only if $\mathbf{Cat}(F) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E}')$ preserves the terminal object.*

Proof. Suppose that for any $A, B \in \mathcal{E}$, we have $F\mathbf{1} \cong \mathbf{1}'$. Then by 2-naturality of $\mathbf{disc}$

$$\mathbf{Cat}(F)(\underline{\mathbf{1}}) = \mathbf{Cat}(F)(\mathbf{disc}(\mathbf{1})) = \mathbf{disc}(F(\mathbf{1})) \cong \mathbf{disc}(\mathbf{1}') = \underline{\mathbf{1}}'.$$

Conversely, suppose $\mathbf{Cat}(F)(\underline{\mathbf{1}}) \cong \underline{\mathbf{1}}'$. By 2-naturality of $(-)_0$, we have

$$F\mathbf{1} = F(\underline{\mathbf{1}})_0 = (\mathbf{Cat}(F)\underline{\mathbf{1}})_0 \cong (\underline{\mathbf{1}}')_0 = \mathbf{1}'.$$

□

In Lemma 4.8.9, to follow, we denote exponentials in $\mathcal{E}$ as $[X, Y]$ rather than Y^X, for ease of readability. Similarly, we denote exponentials in $\mathbf{Cat}(\mathcal{E})$ as $\underline{[\mathbb{X}, \mathbb{Y}]}$.

Lemma 4.8.9. *Suppose $F : \mathcal{E} \to \mathcal{E}'$ preserves finite limits. Then $F[A, B] \cong [FA, FB]'$ for all $A, B \in \mathcal{E}$ if and only if $\mathbf{Cat}(F)\underline{[\mathbb{X}, \mathbb{Y}]} \cong \underline{[\mathbf{Cat}(F)\mathbb{X}, \mathbf{Cat}(F)\mathbb{Y}]}'$ for all $\mathbb{X}, \mathbb{Y} \in \mathbf{Cat}(\mathcal{E})$.*

Proof. Suppose $F[A, B] \cong [FA, FB]'$ and recall that exponentials in $\mathbf{Cat}(\mathcal{E})_1$ are constructed in $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$. The proof that $\mathbf{Cat}(F)\underline{[\mathbb{X}, \mathbb{Y}]} \cong \underline{[\mathbf{Cat}(F)\mathbb{X}, \mathbf{Cat}(F)\mathbb{Y}]}'$ follows from the chain of isomorphisms in $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$ depicted below.

$$\begin{aligned}
\mathbf{Cat}(F)\underline{[\mathbb{X}, \mathbb{Y}]}(-) &= F \int_{[n]\in\Delta_{\leq 3}} \prod_{\phi\in\Delta(-,n)} [X_n, Y_n] && \text{definition of exponentials in } \mathbf{Cat}(\mathcal{E}), \\
&\cong \int_{[n]\in\Delta_{\leq 3}} F \prod_{\phi\in\Delta(-,n)} [X_n, Y_n] && \text{the end is a finite limit,} \\
&\cong \int_{[n]\in\Delta_{\leq 3}} \prod_{\phi\in\Delta(-,n)} F[X_n, Y_n] && \text{each hom of } \Delta_{\leq 3} \text{ is finite,} \\
&\cong \int_{[n]\in\Delta_{\leq 3}} \prod_{\phi\in\Delta(-,n)} [FX_n, FY_n]' && F \text{ preserves exponentials,} \\
&= \underline{[\mathbf{Cat}(F)\mathbb{X}, \mathbf{Cat}(F)\mathbb{Y}]}'(-) && \text{by definition of } \mathbf{Cat}(F).
\end{aligned}$$

Conversely, suppose that for any $\mathbb{X}, \mathbb{Y} \in \mathbf{Cat}(\mathcal{E})$, we have

$$\mathbf{Cat}(F)\underline{[\mathbb{X}, \mathbb{Y}]} \cong \underline{[\mathbf{Cat}(F)\mathbb{X}, \mathbf{Cat}(F)\mathbb{Y}]'}.$$

In Theorem 4.3.1, we showed that $[A, B] = ([\mathbf{disc}(A), \mathbf{disc}(B)])_0$. Let $A, B \in \mathcal{E}$. Then

$$\begin{aligned} F[A, B] &= F\underline{[\mathbf{disc}(A), \mathbf{disc}(B)]}_0 \\ &= (\mathbf{Cat}(F)\underline{[\mathbf{disc}(A), \mathbf{disc}(B)]})_0 \\ &\cong \underline{[\mathbf{Cat}(F)\mathbf{disc}(A), \mathbf{Cat}(F)\mathbf{disc}(B)]}'_0 \\ &= \underline{[\mathbf{disc}(FA), \mathbf{disc}(FB)]}'_0 \\ &= [FA, FB]' \end{aligned}$$

□

Lemma 4.8.10. *$F\Omega \cong \Omega'$ if and only if* $\mathbf{Cat}(F)(\underline{\Omega}) \cong \underline{\Omega}'$.

Proof. In Section 4.6, we characterised the full subobject classifier of $\mathbf{Cat}(\mathcal{E})$ in terms of the subobject classifier in $\mathcal{E}$, with the full subobject classifier being given by $\underline{\Omega} := \mathbf{indisc}(\Omega)$.

Assume that $F : \mathcal{E} \to \mathcal{E}'$ preserves the subobject classifier. Then there is the following chain of isomorphisms in $\mathbf{Cat}(\mathcal{E}')$, with the first being given by pseudonaturality of **indisc** in F and the second being given by the isomorphism up to which F preserves the subobject classifier.

$$\mathbf{Cat}(F)(\underline{\Omega}) = \mathbf{Cat}(F)(\mathbf{indisc}(\Omega)) \cong \mathbf{indisc}(F\Omega) \cong \mathbf{indisc}(\Omega') = \underline{\Omega}'.$$

Conversely, suppose that $\mathbf{Cat}(F)\underline{\Omega} \cong \underline{\Omega}'$. Then the calculation below demonstrates that $F : \mathcal{E} \to E'$ also preserves the subobject classifier.

$$F\Omega = F(\mathbf{indisc}(\Omega))_0 = F(\underline{\Omega})_0 = (\mathbf{Cat}(F)\underline{\Omega})_0 \cong (\underline{\Omega}')_0 = (\mathbf{indisc}(\Omega'))_0 = \Omega'.$$

□

Corollary 4.8.11. *If* **Cat** (F) *preserves pullbacks, powers by* **2** *and codescent objects of cateads, then* **Cat** (F) *preserves coproducts, copowers by* **2***, and acute morphisms as in Definition 4.7.10.*

Proof. By Proposition 4.8.5, Lemma 4.8.8, Lemma 4.8.9 and Lemma 4.8.10, it follows that $F : \mathcal{E} \to \mathcal{E}'$ is a logical functor. But logical functors preserve coproducts (Corollary 2.2.10 part (i) A2.2 [Joh02b]), and coproducts in **Cat** $(\mathcal{E})$ are computed in $[\Delta^{\mathrm{op}}, \mathcal{E}]$ so **Cat** (F) also preserves coproducts. Similarly, **Cat** (F) preserves copowers by **2** since these are built in $\mathcal{E}$ out of coproducts, terminal objects and products, all of which F preserves. Finally, by Corollary 4.7.8, acute morphisms in $\mathbf{Cat}(\mathcal{E})$ are precisely the epimorphism on objects internal functors, and logical functors also preserve epimorphisms. □

Lemma 4.8.12. $F(N) \cong N'$ *if and only if* $\mathbf{Cat}(F)(\underline{N}) \cong \underline{N}'$.

Proof. Similar to the proof of Lemma 4.8.10, but with **disc** in place of **indisc**. □

We now describe how these results combine to prove Theorem 4.8.7.

Proof. (Theorem 4.8.7).

The correspondence between preservation of pullbacks in $\mathcal{E}$, and preservation of pullbacks, powers by **2**, and codescent objects of cateads in $\mathcal{K}$ is part of Bourke's result recalled in Proposition 4.8.5. The correspondence for preservation of terminal objects is shown in Lemma 4.8.8, while the correspondence for preservation of exponentials is shown in Lemma 4.8.9. The correspondence between preservation of subobject classifiers in $\mathcal{E}$ and full subobject classifiers in $\mathcal{K}$ is shown in Lemma 4.8.10. Finally, the correspondence between preservation of natural numbers objects is shown in Lemma 4.8.12. □

Theorem 4.8.13, to follow, says that ETCS and ET2CSC have biequivalent 2-categories of models. This is the sense in which we claim to have categorified ETCS, and provided a foundation of mathematics that captures the structural aspects of categories. In contrast, ETCS is a foundation which axiomatises the structural properties of sets.

Theorem 4.8.13. *Let* **ETCS** *denote the* 2*-category whose objects are categories modelling ETCS, whose morphisms are ETCS morphisms, and whose* 2*-cells are natural isomorphisms. Let* **ET2CSC** *denote the* 2*-category whose objects are* 2*-categories modelling ET2CSC, whose morphisms are ET2CSC morphisms, and whose* 2*-cells are* 2*-natural isomorphisms. Then there is a biequivalence as depicted below.*

$$\begin{tikzcd} \mathbf{ETCS} \arrow[r, shift right, "\mathbf{Cat}(-)"'] \arrow[r, phantom, "\sim"] & \mathbf{ET2CSC} \arrow[l, shift right, "\mathbf{Disc}(-)"'] \end{tikzcd}$$

Proof. The required biequivalence is a restriction of the one in Theorem 4.28 of [Bou10]. The fact that it restricts as required follows from Theorem 4.8.2 and Theorem 4.8.7. □

4.9 Conclusions and future directions

In this chapter we have extended Bourke's characterisation of 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ of internal categories, functors and natural transformations for $\mathcal{E}$ a category with pullbacks (Proposition 2.4.19), and his characterisation of 2-functors of the form $\mathbf{Cat}(F)$ for pullback preserving functors $F : \mathcal{E} \to \mathcal{E}'$ (Proposition 4.8.5). Specifically, we have characterised 2-categories of the same form $\mathbf{Cat}(\mathcal{E})$, where $\mathcal{E}$ now models Lawvere's elementary theory of the category of sets (Theorem 4.8.2), and we have also characterised 2-functors of the form $\mathbf{Cat}(F)$ where $F : \mathcal{E} \to \mathcal{E}'$ preserves the structure in ETCS (Theorem 4.8.7). In particular, we have done so in a way that such 2-categories $\mathbf{Cat}(\mathcal{E})$ can be finitely axiomatised in first order logic, without presupposing an ambient set theory. For these reasons we have called the theory of such 2-categories 'the elementary theory of the 2-category of small categories', or ET2CSC. These results build upon Bourke's work to show that ETCS and ET2CSC have biequivalent 2-categories of models (Theorem 4.8.13). To the extent that ETCS provides a structural foundation by axiomatising the category structure of sets and functions, ET2CSC provides a structural foundation by axiomatising the 2-category structure of categories, functors and natural transformations.

ET2CSC also has the feature that it can be expressed in purely 2-categorical terms, without reference to the fact that its models are of the form $\mathbf{Cat}(\mathcal{E})$, up to equivalence. An important step towards this is Corollary 4.4.8, in which we show that generating families in lextensive $\mathcal{E}$ give rise to generating families in $\mathbf{Cat}(\mathcal{E})$. This motivated the notion of 2-well-pointedness, introduced in Definition 4.4.11 part (2), which is a key ingredient in ET2CSC. Another key ingredient in this axiomatisation is the concept of a 'full subobject', which is an abstraction of functors which include full subcategories determined by some subset of the objects of their codomain. Classifiers for full subobjects were introduced in Definition 4.6.1, and such classifiers in $\mathbf{Cat}(\mathcal{E})$ were shown in Theorem 4.6.7 to be tantamount to subobject classifiers in $\mathcal{E}$. Meanwhile, maps which are left orthogonal to these full subobjects, the so called acute maps of [Str82], played a role in expressing the Categorified Axiom of Choice abstractly rather than in terms of internal category theory, in Theorem 4.7.13. The correspondence between specific properties of $\mathcal{E}$ and analogous properties of $\mathbf{Cat}(\mathcal{E})$ often requires much less to be assumed than the remaining properties in ETCS, and we have presented our proofs accordingly so that the various intermediate results may be applied in greater levels of generality. In particular, we think the intermediate results Corollary 4.4.8 and Proposition 4.7.7 may be of independent interest.

An interesting direction for future research would be to reformulate other set theoretical conditions such as the continuum hypothesis, or large cardinal axioms, in terms of the 2-categorical structure of $\mathbf{Cat}(\mathcal{E})$. On the other hand, a related but different direction for future research could be axiomatising 2-categories of the form $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ satisfies Giraud's axioms for Grothendieck toposes (Proposition 6.1.01 of [Lur09]), Frey's axioms for realisability toposes [Fre19], or Kock's axioms for smooth toposes [Koc06]. Finally, one could try to extend our work to higher categorical settings by axiomatising the three dimensional structure that small double categories and double functors underlie [Böh20]. Similarly, higher dimensional structures comprising Kan complexes or quasicategories are already an active area of research [RV22; Ste24].

Chapter 5

The constructive elementary theory of the 2-category of small categories

5.1 Introduction

Palmgren's *Constructive Elementary Theory of the Category of Sets* (CETCS) [Pal12] is to Aczel's constructive Zermelo-Fraenkel set theory (CZF) what Lawvere's elementary theory of the category of sets (ETCS) [Law64] is to Zermelo-Fraenkel set theory with choice (ZFC). More precisely, CETCS captures the categorical properties of the category of sets in the interpretation of CZF within type theory [Acz78].

In this work, we present a categorification of this, and provide a list of purely 2-categorical axioms that capture the 2-categorical properties of the 2-category of categories in the interpretation of CZF in type theory. We say that a 2-category $\mathcal{K}$ satisfying these properties models the *Constructive Elementary Theory of the* 2*-Category of Small Categories* (CET2CSC) (Definition 5.4.1). The main theorem (Theorem 5.4.2) will establish a logical equivalence between 1-categorical assumptions on a 1-category $\mathcal{E}$ so that it models CETCS and 2-categorical assumptions on the 2-category $\mathcal{K}$ so that it models CET2CSC; in particular, we will be able to show that if $\mathcal{K}$ models CET2CSC, then $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a 1-category modelling CETCS so that the 2-categories of models of these theories are biequivalent. Therefore, CET2CSC is a suitable setting for a 2-categorical constructive and predicative foundation of mathematics. This work gives a constructive version of the Elementary Theory of the 2-Category of Small Categories given in Definition 4.8.1, which does a similar process for Lawvere's ETCS.

Our categorification also allows us to give 2-categorical axioms that characterise 2-categories of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a pretopos (Proposition 5.3.6). Further assumptions allows us to capture the case when $\mathcal{E}$ is an arithmetic Π-pretopos (Theorem 5.3.8). This somewhat clarifies the landscape for potential definitions of 2-pretoposes. We discuss this in Section 5.3.2. Arithmetic Π-pretoposes play an important role in categorical foundations and relationships between category theory and type theory, as explored in [Mai10] and in Chapter 6.

A key assumption for CETCS is the Presentation Axiom (recalled in Definition 5.2.9). This says that every object is covered by an object that has the perspective that the Axiom of Choice is true. We call such an object a *choice object* (Definition 5.2.6). The Presentation Axiom is a weaker version of the Axiom of Choice, which says that every object *is* a choice object. A lot of mathematics (for example, homology theory) does not require the full Axiom of Choice in order for the main theorems to be true; often the Presentation Axiom suffices. As such, it is important to understand what the Presentation Axiom looks like in a 2-categorical foundation of mathematics. We give a categorified version of the Presentation Axiom in Definition 5.3.11. To this end, we give definitions of choice objects and projective objects (Definition 5.3.11) in a 2-category that could be of more general interest, for example, in higher homology theory.

Another key assumption made in CETCS is that we have dependent products and quotients of equivalence relations. Categorically, this corresponds to the properties of local cartesian closedness and exactness. In Lawvere's ETCS, this is handled by the fact that models of ETCS are given by well-pointed elementary toposes with a natural numbers object satisfying the Axiom of Choice; indeed elementary toposes are locally cartesian closed and exact [Joh02a]. This categorical setting is inherently classical, however; in Palmgren's constructive setting, since we want to keep the axiom of extensionality (which corresponds categorically to well-pointedness), the setting of an elementary topos is inappropriate as an elementary topos is well-pointed if and only if it is Boolean. Therefore any well-pointed elementary topos satisfies the Law of the Excluded Middle, a classical principle. Instead, a model of CETCS is given by a well-pointed arithmetic Π-pretopos that satisfies the disjunction property and the Presentation Axiom with projective terminal object (Definition 5.2.10). It is therefore our goal to categorify each of these notions in turn. We note that the concepts of well-pointedness, extensivity and natural numbers objects were categorified in Chapter 4. Thus, the interesting remaining parts of the theory to categorify are local cartesian closure, exactness, the disjunction property.

It is well known that **Cat** is not locally cartesian closed ([Con72], see also Example 5.3.1), and so naively categorifying locally cartesian closedness does not work for us. Instead, we assume exponentiability of discrete opfibrations; this presents an interesting example wherein in order to give the categorification of a concept (such as exponentiability of maps) we have to increase the homotopical complexity of the concept (the requirement for all maps to be exponentiable is replaced by the requirement that the homotopically more complicated ones are— the discrete opfibrations). It turns out that $\mathcal{E}$ is locally cartesian closed if and only if $\mathbf{Cat}(\mathcal{E})$ has exponentiable discrete opfibrations (Proposition 5.3.2).

As explained in Section 2.4, exactness for 2-categories is a slightly more complex matter. In [BG14], notions of exactness are defined with respect to any orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$ on **Cat**, providing a whole host of possibilities for categorification. We isolate their notion of SO-exactness as being important to this theory; this is the notion of exactness that corresponds to the (surjective-on-objects, injective-on-objects and fully faithful) factorisation system on **Cat**. In this case, the categorification of regular epimorphisms is given by SO-regular epimorphisms, which in an SO-regular 2-category are precisely the acute functors given Definition 4.7.10. These are the functors that are given particular importance in Street's 2-dimensional sheaf theory [Str82]; in $\mathbf{Cat}(\mathcal{E})$ they are precisely the epimorphic-on-objects functors. We recall the required definitions in Section 5.3.2. We prove that $\mathcal{E}$ is exact if and only if $\mathbf{Cat}(\mathcal{E})$ is SO-exact in Proposition 5.3.5. This result helps clarify the roles that different 2-dimensional notions of exactness play.

The disjunction property is categorified in Definition 5.3.9 and it is shown that $\mathcal{E}$ satisfies the disjunction property if and only if $\mathbf{Cat}(\mathcal{E})$ satisfies the 2-disjunction property in Proposition 5.3.10. Much like in the case of 2-well-pointedness, this utilises the fact that properties in $\mathbf{Cat}(\mathcal{E})$ must be tested on morphisms rather than just objects.

In Section 5.4, we arrive at a purely 2-categorical list of axioms. We say that a 2-category $\mathcal{K}$ that satisfies these axioms models the *constructive elementary theory of the* 2*-category of small categories* (Definition 5.4.1). We prove that a 2-category $\mathcal{K}$ satisfies these axioms if and only if $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ in which $\mathcal{E}$ models CETCS (Theorem 5.4.2).

Whilst in CET2CSC we do not have a classifier for all full monomorphisms as in ET2CSC (Definition 4.6.1), we prove that in a model of CET2CSC, decidable-on-objects and fully faithful functors are classified (Proposition 5.4.9). This motivates the abstract definition of a decidably fully faithful morphism (Definition 5.4.10) which is a generalisation of decidable-on-objects and fully faithful functors that can be stated in an arbitrary 2-category. From this, we can give the Categorified Law of the Excluded Middle (Definition 5.4.13)— we prove that $\mathcal{E}$ satisfies the Law of the Excluded Middle if and only if $\mathbf{Cat}(\mathcal{E})$ satisfies the Categorified Law of the Excluded middle in Proposition 5.4.14. This allows us to compare the Constructive Elementary Theory of the 2-Category of Small Categories with its non-constructive counterpart in Corollary 5.4.15.

From Chapter 6, we can deduce that a restriction of CET2CSC to $(2, 1)$-categories (so that we are considering $\mathbf{Gpd}(\mathcal{E})$ in which $\mathcal{E}$ is a model of CETCS) produces a model of Martin-Löf type theory. Since in this model every groupoid represents a type, one might hope that our categorical theory could help define the notion of projective type which correspond to the SO-projective groupoids in this model. Such a definition would allow us to consider a type theoretic Presentation Axiom and Axiom of Choice. However, our abstract definitions, though stated purely 2-categorically, are often instantiated through a property that is true on objects— properties that suffer from being unstable under equivalences of categories. This unfortunately means that we cannot state corresponding definitions in MLTT. In Section 5.5, we take some first steps towards weakening these definitions. This works in the case of the Axiom of Choice and we are able to provide a Weak Categorified Axiom of Choice (Definition 5.5.4) and prove that $\mathbf{Cat}(\mathcal{E})$ satisfies it if and only if $\mathcal{E}$ satisfies the Axiom of Choice; however, in the case of the Presentation Axiom, this seems to fail showing that this approach to defining weak projective objects does not work in the strict 2-dimensional case— see Remark 5.5.7 for a discussion on this. We conclude that this might work better in the bicategorical setting.

5.2 The 1-categorical case

In this section, we recall the 1-categorical definitions needed for this chapter.

5.2.1 Basic notions

The following condition says that an element of a sum is an element of exactly one of its injections.

Definition 5.2.1. Let $\mathcal{E}$ be a category with coproducts and let $A + B$ be a coproduct of $A, B \in \mathcal{E}$ with coproduct injections $\iota_A : A \to A + B, \iota_B : B \to A + B$. We say that $\mathcal{E}$ satisfies the *disjunction property* if any $z : \mathbf{1} \to A + B$ factors through either ι_A or ι_B.

One of the main contributions of this paper is to show the relation between 1-dimensional regularity and exactness to a certain kind of notion of 2-dimensional regularity and exactness.

Recall the definition of regular epimorphism, given in Definition 2.4.1. It is not hard to see that every regular epimorphism is an epimorphism. It is not always the case that the converse is true; however under the assumptions that we will be working with in this chapter, this will be true.

For logical purposes, we are interested in categories that have quotients of equivalence relations that are well-behaved under re-indexing and coproducts that are well-behaved under re-indexing. These are given by the notions of exactness (Definition 2.4.6) and extensivity (Definition 2.5.1).

Definition 5.2.2. A pretopos is an exact and extensive category.

We note that when $\mathcal{E}$ is a pretopos, every epimorphism is a regular epimorphism. As we are assuming that our category $\mathcal{E}$ is a pretopos, we will drop the distinction from this point onwards although note that much of the following can be developed without this assumption with added care when defining things in terms of regular epimorphisms.

In a category $\mathcal{E}$ with pullbacks, any $f : X \to Y$ in $\mathcal{E}$ gives rise to a functor

$$f^* : \mathcal{E}/Y \to \mathcal{E}/X$$

by pulling back along f.

Definition 5.2.3. Let $\mathcal{E}$ be a category with pullbacks. We call a morphism f *exponentiable* if f^* has a right adjoint; in this case, we denote the right adjoint by Π_f.

$$\begin{tikzcd} \mathcal{E}/Y \arrow[rr, bend left, "f^*"] \arrow[rr, phantom, "\perp"] & & \mathcal{E}/X \arrow[ll, bend left, dashed, "\exists \Pi_f"] \end{tikzcd}$$

We call $\mathcal{E}$ locally cartesian closed if every $f : X \to Y$ in $\mathcal{E}$ is exponentiable.

5.2.2 Choice principles

In this subsection, we explain the choice principle used in CETCS as well as its relation to the Axiom of Choice and other related concepts.

First, we give the definition of a projective object.

Definition 5.2.4. Let $\mathcal{E}$ be a regular 1-category. An object $P \in \mathcal{E}$ is called *projective* if either of the equivalent following conditions is met.

1. For any regular epimorphism $q : A \twoheadrightarrow B$ and $f : P \to B$ there exists an arrow $g : P \to A$ such that $qg = f$.

$$\begin{tikzcd} & A \arrow[d, two heads, "q"] \\ P \arrow[ur, dashed, "\exists g"] \arrow[r, "f"'] & B \end{tikzcd}$$

2. $\mathcal{E}(X, -)$ preserves regular epimorphisms.

Remark 5.2.5. In Definition 5.2.4, conditions (1) and (2) are equivalent if and only if we assume the Axiom of Choice in **Set** as it requires that for any epimorphism $q : A \twoheadrightarrow B$, the function $\mathrm{Hom}(P, q) : \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ has a section. Constructively, we could instead characterise projective objects as those $P \in \mathcal{E}$ such that $\mathrm{Hom}(P, -)$ sends epimorphisms to split epimorphisms.

Definition 5.2.6. Let $\mathcal{E}$ be a pretopos. We call $C \in \mathcal{E}$ a *choice object* if any epimorphism $q : A \twoheadrightarrow C$ admits a section: that is a map $s : C \to A$ that makes the following diagram commute.

$$
\begin{tikzcd}
 & A \arrow[d, two heads, "q"] \\
C \arrow[ur, dashed, "\exists s"] \arrow[r, equal] & C
\end{tikzcd}
$$

Clearly, any projective object is a choice object by taking $f = 1_B$. The converse is true in any pretopos, as recorded by the following lemma.

Lemma 5.2.7. *Let $\mathcal{E}$ be a pretopos. Then every choice object is a projective object.*

Proof. Assume $C \in \mathcal{E}$ is a choice object. Let $q : A \twoheadrightarrow B$ be an epimorphism and $f : C \to B$. Take their pullback as displayed below.

$$
\begin{tikzcd}
A \times_B C \arrow[r, "f^*(q)"] \arrow[d, "q^*(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "q"] \\
C \arrow[r, "f"'] & B
\end{tikzcd}
$$

Now, note that $q^*(f)$ is the pullback of a (regular) epimorphism so it is again a (regular) epimorphism. As C is a choice object, there exists a section $s : C \to A \times_B C$ of $q^*(f)$. Define $g : C \to A$ by $f^*(q) \circ s$. This has the property that

$$q \circ g := q \circ f^*(q) \circ s = f \circ q^*(f) \circ s = f$$

as required. $\square$

Remark 5.2.8. Lemma 5.2.7 also holds for milder assumptions on $\mathcal{E}$; in fact we only need for pullbacks to exist and for epimorphisms to be preserved by pullbacks. In this case, we need to be careful about which notion of epimorphism we use for defining choice objects and projective objects.

Recall the definition of the external Axiom of Choice given in Definition 4.7.1. In light of Definition 5.2.6, an alternative way to phrase the external Axiom of Choice for $\mathcal{E}$ is to say that every object is a choice object, or equivalently by Lemma 5.2.7, that every object is projective.

Homological algebra often requires the formation of projective resolutions of short exact sequences, for example; that these exist follows from the Axiom of Choice. However, this is often the only use of the Axiom of Choice in homological algebra; to develop the main tools of homological algebra, we can get away with working in a weaker meta theory in which we get rid of the Axiom of Choice and instead assume just that **Set** has enough projectives.

Definition 5.2.9. Let $\mathcal{E}$ be a pretopos. We say that $\mathcal{E}$ satisfies the Presentation Axiom if for every object $X \in \mathcal{E}$, there exists a choice object P and an epimorphism $q : P \twoheadrightarrow X$.

Clearly the Axiom of Choice implies the Presentation Axiom, but the converse is not true. In fact, the Presentation Axiom is often considered constructive due to it being provable in certain constructively flavoured logics such as in the internal logic of the effective topos [Hyl82] and in Aczels' interpretation of CZF in MLTT [Acz78].

The Presentation Axiom in **Set** is often called CoSHEP (Category of Sets Has Enough Projectives).

5.2.3 The constructive elementary theory of the category of sets

The constructive elementary theory of the category of sets can be given as the following conditions on a 1-category [Pal12, Theorem 6.10]:

Definition 5.2.10. Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ is said to model the *constructive elementary theory of the category of sets* (CETCS) if:

1. It has a natural numbers object.
2. It is well-pointed.
3. It is extensive.
4. It satisfies the disjunction property.
5. It is exact.
6. It is locally cartesian closed.
7. It satisfies the Presentation Axiom
8. Its terminal object is projective.

In the following section, we will provide categorifications of each item in turn, and show that they are logically equivalent to the original assumptions on $\mathcal{E}$ by showing that the 2-dimensional property holds in $\mathbf{Cat}(\mathcal{E})$ if and only if the 1-dimensional property holds in $\mathcal{E}$.

5.3 Categorification

In this section, we categorify all the definitions in Section 5.2. We note that items (1)-(3) of CETCS have already been categorified in Chapter 4.

5.3.1 Exponentiability of discrete opfibrations

In this section, we categorify the notion of locally cartesian closedness for a 2-category. Note that $\mathcal{E}$ being locally cartesian closed does not imply that $\mathbf{Cat}(\mathcal{E})$ is locally cartesian closed; this fails even in the case that $\mathcal{E} = \mathbf{Set}$. If $\mathbf{Cat}$ was locally cartesian closed, then all colimits would be stable under pullback along any morphism, since pullback along that morphism would be a left adjoint. The following example shows that this is not the case.

Example 5.3.1 ([Shua])**.** Consider the pushout in $\mathbf{Cat}$:

$$\begin{tikzcd}
(1) \arrow[r] \arrow[d] & (1 \to 2) \arrow[d] \\
(0 \to 1) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (0 \to 1 \to 2)
\end{tikzcd}$$

Pull this back along $(0 \to 2) \to (0 \to 1 \to 2)$ gives the following diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & (2) \arrow[d] \\
(0) \arrow[r] & (0 \to 2)
\end{tikzcd}$$

which is certainly not a pushout. However, it *is* a cocomma. We conjecture that in **Cat** (and $\mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ locally cartesian closed) cocomma objects are stable under pullback, along with all other PIE-colimits [BKP89]. Note that pushouts are not PIE-colimits.

So it is not true that every morphism in **Cat** is exponentiable. In fact, the exponentiable morphisms are precisely the functors which are *Conduché fibrations*— that is they have the factorisation lifting property [Gir64, Theorem 4.4]. Unfortunately, the definition of Conduché fibration does not seem to be simply expressible in an arbitrary 2-category. Since discrete opfibrations are in particular Conduché fibrations in **Cat**, exponentiability of Conduché fibrations implies that discrete opfibrations are exponentiable in **Cat**. Conversely, exponentiability of discrete opfibrations can be used to prove that Conduché fibrations are exponentiable in **Cat**. Exponentiability of discrete opfibrations is therefore an important property of the 2-category **Cat**. The notion of discrete opfibration *can* be expressed easily in any 2-category, either by a representable definition or an internal characterisation.

Hence, to formulate a 2-categorical notion of local cartesian closure that **Cat** satisfies, instead of asking that all morphisms in $\mathcal{K}$ are exponentiable, we ask only that the discrete (op)fibrations are exponentiable.

In Chapter 6, Lemma 6.6.1 gives a 2-dimensional form of locally cartesian closure in the form of asking for groupoidal isofibrations to be exponentiable. Here, we give a slightly different (but equivalent) axiom, which replaces groupoidal isofibrations with discrete opfibrations. We argue this formulation is more inherently 2-dimensional, as discrete opfibrations are precisely the kind of map that is classified by 2-dimensional classifiers (See Chapter 7).

Proposition 5.3.2. *Let $\mathcal{E}$ have pullbacks and coequalisers. Then $\mathcal{E}$ is locally cartesian closed if and only if discrete opfibrations in* $\mathbf{Cat}(\mathcal{E})$ *are exponentiable.*

The following proof uses similar arguments to that in Theorem 6.3.15.

Proof. If $\mathcal{E}$ is locally cartesian closed, then by [SV10, Theorem 2.16], discrete opfibrations are exponentiable, as they are representably exponentiable and levelwise exponentiable.

Conversely, let $f : X \to Y$ be a morphism in $\mathcal{E}$. Consider the internal functor $\mathbf{disc}(f) : \mathbf{disc}(X) \to \mathbf{disc}(Y)$ in $\mathbf{Cat}(\mathcal{E})$; this is trivially a discrete opfibration and hence is exponentiable by assumption. Hence, we have a right adjoint to the functor

$$\mathbf{disc}(f)^* : \mathbf{Cat}(\mathcal{E})/\mathbf{disc}(Y) \to \mathbf{Cat}(\mathcal{E})/\mathbf{disc}(X)$$

which we denote by $\Pi_{\mathbf{disc}(f)} : \mathbf{Cat}(\mathcal{E})/\mathbf{disc}(Y) \to \mathbf{Cat}(\mathcal{E})/\mathbf{disc}(X)$. Construct the right adjoint to $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ by composing this series of right adjoints

$$(-)_0 \cdot \Pi_{\mathbf{disc}(f)} \cdot \mathbf{disc} : \mathcal{E}/X \to \mathcal{E}/Y$$

which is left adjoint to $\Pi_0 \cdot \mathbf{disc}(f)^* \cdot \mathbf{disc} : \mathcal{E}/Y \to \mathcal{E}/X$ which we can verify is equal to $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ since the counit of $\Pi_0 \vdash \mathbf{disc}$ can be chosen to be the identity, since $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ is fully faithful. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}/Y \arrow[r, "\mathbf{disc}"] \arrow[d, "f^*"'] & \mathbf{Cat}(\mathcal{E})/_{\mathbf{disc}(f)} \arrow[d, "\mathbf{disc}(f)^*"] \\
\mathcal{E}/X \arrow[r, "\mathbf{disc}"] \arrow[dr, equal] & \mathbf{Cat}(\mathcal{E})/_{\mathbf{disc}(X)} \arrow[d, "\Pi_0"] \\
& \mathcal{E}/X.
\end{tikzcd}$$

Note that for Π_0 to exist, we needed to assume that we had coequalisers. $\square$

Remark 5.3.3. We note that [Pal12, §7] gives a functor-free formulations of local cartesian closure via a dependent products axiom, which together with assuming the category in question is cartesian and balanced states that the category is locally cartesian closed. This is done in order to avoid using the Axiom of Choice in the metatheory. To do this, the tools of a universal Π diagram are used. This can also be done for the 2-categorical case, although it is complicated and not very enlightening, and so we proceed without doing so.

5.3.2 Exactness

In this section, we give 2-categorical properties which correspond to the 1-categorical properties of being exact. We show that $\mathcal{E}$ is exact if and only if $\mathbf{Cat}(\mathcal{E})$ is SO-exact (Definition 2.4.24).

Proposition 5.3.4. *Let $\mathcal{E}$ be a category with pullbacks. $\mathcal{E}$ is regular if and only if* $\mathbf{Cat}(\mathcal{E})$ *is SO-regular.*

Proof. If $\mathcal{E}$ is regular, then $\mathbf{Cat}(\mathcal{E})$ is SO-regular by [BG14, Proposition 64].

Conversely, suppose that $\mathbf{Cat}(\mathcal{E})$ is SO-regular. Note that regular epimorphisms in a 1-category $\mathcal{E}$ are equivalently SO-regular epimorphisms when considering $\mathcal{E}$ as a locally discrete 2-category; by [BG14, Proposition 55], it is enough to show that under our assumptions, $\mathcal{E}$ is SO-regular when considered as a locally discrete 2-category. The condition (SO1) is clear. To see that (SO2) is satisfied, note that since $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ is right adjoint, it preserves pullbacks and hence SO-kernels in $\mathcal{E}$. Therefore, SO-quotients of SO-kernels in $\mathcal{E}$ can be computed in $\mathbf{Cat}(\mathcal{E})$ under $\mathbf{disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ and then transferred back to $\mathcal{E}$ by $(-)_0$ since it is a left 2-adjoint and left 2-adjoints preserve 2-colimits and moreover $(-)_0 \circ \mathbf{disc} = \mathbf{1}_{\mathcal{E}}$; hence if $\mathbf{Cat}(\mathcal{E})$ has SO-quotients of SO-kernels then so does $\mathcal{E}$; from this, (SO4) follows too, since both SO-kernels and SO-quotients of SO-congruences are calculated in $\mathbf{Cat}(\mathcal{E})$, and they are effective therein. Finally, to show (SO3), consider the following pullback square in $\mathcal{E}$, in which $f : X \to Y$ is an SO-regular epimorphism.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] & Y. \end{tikzcd}$$

We show that $g : A \to B$ is an SO-regular epimorphism in $\mathcal{E}$. Since **disc** $: \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ is a right adjoint, this pullback square, and the SO-kernels of f and g are preserved by it. Calculate the SO-quotient of the SO-kernel of f given by $q : X \to Q\mathbf{Ker}(f)$. Since $(-)_0$ is left 2-adjoint to **indisc**, it preserves SO-regular epimorphisms; since it is right adjoint to **disc**, it preserves its SO-kernel and so it follows that $q_0 : X \to Q'\mathbf{Ker}(f)_0 \cong f : X \to Y$. Hence there is an internal functor $v' : B \to Q'\mathbf{Ker}(f)$ in $\mathbf{Cat}(\mathcal{E})$ defined on objects by $v : B \to Y$ and on morphisms by $i_{Q'\mathbf{Ker}(f)}v$. Take the pullback:

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "q"] \\ B \arrow[r, "v'"'] & Q'\mathbf{Ker}(f). \end{tikzcd}$$

Since SO-regular epimorphisms are closed under pullback in $\mathbf{Cat}(\mathcal{E})$ by assumption, p is an SO-regular epimorphism, and again, $p_0 : C_0 \to B$ is an SO-regular epimorphism in $\mathcal{E}$ by left 2-adjointness of $(-)_0$. But this pullback square is preserved by $(-)_0$, too, proving that $p_0 : C_0 \to B \cong g : A \to B$, as required. $\square$

Proposition 5.3.5. *Let $\mathcal{E}$ be a category with pullbacks. $\mathcal{E}$ is exact if and only if* $\mathbf{Cat}(\mathcal{E})$ *is SO-exact in the sense of [BG14, §5.2].*

Proof. If $\mathcal{E}$ is exact, then $\mathbf{Cat}(\mathcal{E})$ is SO-exact by [BG14, Proposition 64].

Conversely, suppose $\mathbf{Cat}(\mathcal{E})$ is SO-exact. By Proposition 5.3.4, it is enough to show that every internal equivalence relation is the kernel pair of its coequaliser. Let $(X_1, X_0, d_1, d_0, i, m)$ be an internal equivalence relation on X_0 in $\mathcal{E}$. We construct an SO-congruence in $\mathbf{Cat}(\mathcal{E})$ as follows:

$$\begin{tikzcd} & X_1 \arrow[d, equal] & \\ X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "\pi_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_1"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "i" description] \end{tikzcd}$$

in which every object is considered a discrete internal category.

Since $\mathbf{Cat}(\mathcal{E})$ is SO-exact, SO-congruences are effective and so this is the SO-kernel of its SO-quotient; since all objects in this diagram are discrete, both this SO-kernel and its SO-quotient is preserved by the left 2-adjoint $(-)_0$. By [BG14, Proposition 55], the SO-regular epimorphism in a locally discrete 2-category is equivalently the quotient of the underlying equivalence relation. Hence every internal equivalence relation is the kernel pair of its coequaliser, as required. $\square$

This section culminates to give us a complete characterisation of arithmetic Π-pretoposes in purely 2-dimensional terms. We first prove a characterisation for pretoposes.

Proposition 5.3.6. *Let $\mathcal{E}$ be a pretopos. Then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ satisfies the conditions listed below, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$ is a pretopos.*

1. *$\mathcal{K}$ has pullbacks and powers by $\mathbf{2}$.*

2. *$\mathcal{K}$ has codescent objects of cateads.*

3. *Cateads and Codescent morphisms are effective in $\mathcal{K}$.*

4. *Discrete objects in $\mathcal{K}$ are BO-projective, in the sense of Definition 2.4.18.*

5. *For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*

6. *$\mathcal{K}$ is extensive.*

7. *$\mathcal{K}$ is SO-exact.*

Proof. Suppose $\mathcal{E}$ is an arithmetic Π-pretopos. Then $\mathbf{Cat}(\mathcal{E})$ satisfies (1)-(5) by [Bou10], (6) by Lemma 4.4.2 and (7) by [BG14].

Conversely, suppose $\mathcal{K}$ is a 2-category satisfying (1)-(7). Again, (1)-(5) shows that $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$, (6) shows that $\mathcal{E}$ is extensive by Lemma 4.4.2. Exactness of $\mathcal{E}$ given (7) is Proposition 5.3.5. Hence $\mathcal{E}$ is a pretopos. □

Definition 5.3.7. Let $\mathcal{K}$ be a 2-category. We say that $\mathcal{K}$ satisfies the *elementary theory of the 2-category of small abstract categories* if it satisfies the conditions listed below.

1. $\mathcal{K}$ has pullbacks and powers by $\mathbf{2}$.

2. $\mathcal{K}$ has codescent objects of cateads.

3. Cateads and Codescent morphisms are effective in $\mathcal{K}$.

4. Discrete objects in $\mathcal{K}$ are BO-projective, in the sense of Definition 2.4.18.

5. For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.

6. $\mathcal{K}$ is extensive.

7. $\mathcal{K}$ is SO-exact.

8. Discrete opfibrations in $\mathbf{Cat}(\mathcal{E})$ are exponentiable.

9. $\mathcal{K}$ has finite 2-colimits.

If $\mathcal{K}$ is a $(2,1)$-category, then we say that it satisfies the *elementary theory of the* $(2,1)$*-category of small abstract groupoids* (ETCSAG).

Theorem 5.3.8. *Let $\mathcal{E}$ be an arithmetic Π-pretopos. Then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the elementary theory of the 2-category of small abstract categories. Conversely, if $\mathcal{K}$ satisfies the elementary theory of the 2-category of small abstract categories, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}\,(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}\,(\mathcal{K})$ is an arithmetic Π-pretopos.*

Proof. Recall that an arithmetic Π-pretopos is simply a pretopos which is locally cartesian closed and has a natural numbers object, since in any local cartesian closed pretopos, having a natural numbers object is equivalent to having an parametrised list objects by [Joh02b, Theorem 2.5.17]. Therefore, by Proposition 5.3.6 it remains to treat items (8) and (9).

Suppose $\mathcal{E}$ is an arithmetic Π-pretopos. Then $\mathbf{Cat}(\mathcal{E})$ satisfies (8) by Proposition 5.3.2 and (9) by Theorem 3.5.2.

Conversely, suppose $\mathcal{K}$ satisfies (1)-(9). Then since it satisfies (1)-(7), $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ in which $\mathcal{E}$ is a pretopos. Property (8) implies that $\mathcal{E}$ is locally cartesian closed by Proposition 5.3.2. Property (9) implies that $\mathcal{K}$ has a natural numbers object by Corollary 3.6.3. $\square$

5.3.3 The 2-disjunction property

In **Set**, coproducts are disjoint; that is: an element $x \in A + B$ is either in A or in B but not both. In **Cat**, we can either say this levelwise, or we can test it on morphisms: $f : x \to x'$ is a morphisms in $\mathbb{A}+\mathbb{B}$ means that f is either a morphism in $\mathbb{A}$ or a morphism in $\mathbb{B}$, but not both.

Definition 5.3.9. Let $\mathcal{K}$ be a lextensive 2-category that is powered by $\mathbf{2}$ and such that the copower $\mathbf{2} \odot \mathbf{1}$ exists in $\mathcal{K}$. We say $\mathcal{E}$ satisfies *the* 2*-disjunction property* if for any $A, B \in \mathcal{K}$ we have $\mathcal{K}(\mathbf{2} \odot \mathbf{1}, A + B) \cong \mathcal{K}(\mathbf{2} \odot \mathbf{1}, A) + \mathcal{K}(\mathbf{2} \odot \mathbf{1}, B)$.

Proposition 5.3.10. *Let $\mathcal{E}$ be a lextensive cartesian closed category. Then $\mathcal{E}$ satisfies the disjunction property if and only if* $\mathbf{Cat}(\mathcal{E})$ *satisfies the* 2*-disjunction property.*

Proof. Recall from Section 4.4 that in $\mathbf{Cat}(\mathcal{E})$, the copower $\mathbf{2} \odot \mathbf{1}$ is given by $\mathbf{2}_{\mathcal{E}}$.

Firstly, suppose $\mathbf{Cat}(\mathcal{E})$ satisfies the 2-disjunction property. For any $A, B \in \mathcal{E}$, we have the following calculation, which shows that $\mathcal{E}$ satisfies the disjunction property.

$$\begin{aligned}
\mathcal{E}(\mathbf{1}, A+B) &\cong \mathcal{E}(\Pi_0(\mathbf{2}_{\mathcal{E}}), A+B) \\
&\cong \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}}, \mathbf{disc}(A+B)) \\
&\cong \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}}, \mathbf{disc}(A) + \mathbf{disc}(B)) \\
&\cong \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}}, \mathbf{disc}(A)) + \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}}, \mathbf{disc}(B)) \\
&\cong \mathcal{E}(\Pi_0(\mathbf{2}_{\mathcal{E}}), A) + \mathcal{E}(\Pi_0(\mathbf{2}_{\mathcal{E}}), B) \\
&\cong \mathcal{E}(\mathbf{1}, A) + \mathcal{E}(\mathbf{1}, B).
\end{aligned}$$

Here, the first line uses that $\Pi_0(\mathbf{2}_\mathcal{E}) = \mathbf{1}$, the second line uses $\Pi_0 \dashv$ **disc**, the third line uses **disc** $\dashv (-)_0$ and that left adjoints preserve colimits and the fourth line uses the 2-disjunction property.

Conversely, suppose $\mathcal{E}$ satisfies the disjunction property. Given a functor $f : \mathbf{2}_\mathcal{E} \to \mathbb{A} + \mathbb{B}$. By applying $(-)_0 : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$ we obtain $f_0 : \mathbf{1} + \mathbf{1} \to A_0 + B_0$; write $\iota_l f_0$ and $\iota_r f_0$ for its restriction to the left and right inclusion so that $\iota_l f_0 + \iota_r f_0 = f_0$. Since $\mathcal{E}$ satisfies the disjunction property, each of $\iota_l f_0$ and $\iota_r f_0$ are to either A_0 or B_0; similarly for the maps $f_1 : \mathbf{1} + \mathbf{1} + \mathbf{1} \to A_1 + B_1$ and $f_2 : \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1} \to A_2 + B_2$. The compatibility of f with source, target and composition maps force that this choice is uniform: they are either all in A_i or all in B_i; in either case, this constitutes a functor $f : \mathbf{2}_\mathcal{E} \to \mathbb{A}$ or $f : \mathbf{2}_\mathcal{E} \to \mathbb{B}$. Therefore $\mathbf{Cat}(\mathcal{E})$ satisfies the 2-disjunction property. □

5.3.4 Projectivity and choice objects

In this section, we first introduce a categorified version of projective objects and show that this corresponds to the 1-dimensional notion.

Definition 5.3.11. Let $\mathcal{K}$ be a 2-category. We say that an object $\mathbb{P} \in \mathcal{K}$ is *SO-projective* if for any fully faithful SO-regular epimorphism $Q : \mathbb{A} \to \mathbb{B}$ and any $F : \mathbb{P} \to \mathbb{B}$, there exists a morphism $G : \mathbb{P} \to \mathbb{A}$ such that $GF = Q$.

We say that an object $\mathbb{C} \in \mathcal{K}$ is an *SO-choice object* if for any fully faithful SO-regular epimorphism $Q : \mathbb{A} \to \mathbb{C}$ there exists a $S : \mathbb{C} \to \mathbb{A}$ such that $QS = 1_\mathbb{C}$ and $SQ \cong 1_\mathbb{A}$.

We say that $\mathcal{K}$ has *enough SO-projectives* if for any $X \in \mathcal{K}$ there is an SO-projective object $P \in \mathcal{K}$ and a fully faithful SO-regular epimorphism $P \to X$.

We say that a 2-category $\mathcal{K}$ with enough SO-projectives satisfies the *Categorified Presentation Axiom*.

Every SO-projective is an SO-choice object by taking $F = 1_\mathbb{B}$. Just like in the 1-dimensional case (Lemma 5.2.7), under certain assumptions on a 2-category $\mathcal{K}$, every SO-choice object is an SO-projective object too.

Proposition 5.3.12. *Let $\mathcal{K}$ be an SO-regular 2-category. Then every SO-choice object is SO-projective.*

Proof. The proof is exactly the same as in Lemma 5.2.7 given that in an SO-regular category, fully faithful SO-regular epimorphisms are stable under pullback. □

Remark 5.3.13. In the cases we will be considering, the fully faithful SO-regular epimorphisms will be precisely the acute and fully faithful morphisms— see Remark 4.7.11. These are exactly the fully faithful morphisms which have the left lifting property with respect to full monomorphisms in $\mathcal{K}$.

Example 5.3.14. Let $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a regular category. The SO-regular epimorphisms are precisely the acute morphisms in $\mathbf{Cat}(\mathcal{E})$; that is they are the (regular) epimorphism-on-object functors. Therefore an object $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$ is SO-projective if and only if X_0 is projective in $\mathcal{E}$.

Remark 5.3.15. In [Bou10], a natural notion of projectivity is considered with respect to the concept of BO-exactness; for a 2-category $\mathcal{K}$, an object $X \in \mathcal{K}$ is called *BO-projective* if $\mathcal{K}(X, -)$ preserves codescent morphisms (BO-regular epimorphisms). Our definition instead generalises the notion of projectivity with respect to SO-exactness. For $\mathcal{K} =$

Cat($\mathcal{E}$) for suitable $\mathcal{E}$, the SO-projective objects are precisely the cofibrant objects in the regular epimorphism topology model structure on **Cat**($\mathcal{E}$) considered in [EKL05], and so they are also homotopically interesting.

Note that the notions of BO-projective and SO-projective do not coincide.

Lemma 5.3.16. *Let $\mathcal{K}$ be a 2-category. $X \in \mathcal{K}$ is SO-projective if and only if $\mathcal{K}(X, -)$ sends acute and fully faithful maps to split epimorphism on objects and fully faithful functors in* **Cat***.*

We show that the notion of SO-projectively strictly categorifies the notion of projectivity in $\mathcal{E}$.

Theorem 5.3.17. *Let $\mathcal{E}$ be a regular 1-category. Then $\mathcal{E}$ has enough projectives if and only if* **Cat**($\mathcal{E}$) *has enough SO-projectives.*

Proof. Suppose $\mathcal{E}$ has enough projectives. Consider $\mathbb{X} \in$ **Cat**($\mathcal{E}$). Since $\mathcal{E}$ has enough projectives, there exists a projective object P_0 and regular epimorphism $F_0 : P_0 \to X_0$. Construct P_1 as the pullback of $F_0 \times F_0 : P_0 \times P_0 \to X_0 \times X_0$ and $(d_1, d_0) : X_1 \to X_0 \times X_0$. In a standard way (see Section 6.4.1 for an explicit proof) this assembles into an internal category $\mathbb{P}$ and an internal functor $F : \mathbb{P} \to \mathbb{X}$. By construction, this morphism is regular epi-on-objects and fully faithful, and therefore a fully faithful SO-regular epimorphism, since $\mathcal{E}$ is a regular category (see Example 5.3.14).

Conversely, suppose **Cat**($\mathcal{E}$) has enough SO-projectives. For $X \in \mathcal{E}$, consider the internal category **disc**(X). As **Cat**($\mathcal{E}$) has enough SO-projectives, there exists an SO-projective object $\mathbb{P}$ and a fully faithful SO-regular morphism $F : \mathbb{P} \to$ **disc**(X). In particular, since SO-regular epimorphisms are regular epimorphisms on objects, we have a regular epimorphism $F_0 : P_0 \to X$. We show that P_0 is a choice object, completing the proof. Consider some regular epimorphism $r : A \to P_0$ in $\mathcal{E}$. Form the pullback of $(d_1, d_0) : P_1 \to P_0 \times P_0$ with $r \times r : A \times A \to P_0 \times P_0$; in a standard way, this assembles into a fully faithful and regular epimorphic on object functor $\mathbb{A} \to$ ¶, which is an SO-regular epimorphism because $\mathcal{E}$ is regular. Now, since ¶ is SO-projective, it has a splitting. On objects, this gives the required splitting of $r : A \to P_0$. $\square$

5.4 The constructive elementary theory of the 2-category of small categories

In this section, we collect the work of the previous sections, which is enough to prove our main Theorem.

Definition 5.4.1. We say that the 2-category $\mathcal{K}$ models the *constructive elementary theory of the 2-category of small categories* (CET2CSC) if the following properties hold:

1. $\mathcal{K}$ has pullbacks and powers by **2**.

2. $\mathcal{K}$ has codescent objects of cateads (Definition 2.4.9).

3. Cateads and Codescent morphisms are effective in $\mathcal{K}$ (see Definition 2.4.9).

4. Discrete objects in $\mathcal{K}$ are BO-projective in the sense of Definition 2.4.18.

5. For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.

6. It is 2-well-pointed in the sense of Definition 4.4.11.

7. It has a 2-natural numbers object in the sense of Definition 4.5.1 part (2).

8. It is SO-exact in the sense of Definition 2.4.24.

9. Discrete opfibrations are exponentiable.

10. It satisfies the 2-disjunction property in the sense of Definition 5.3.9.

11. It satisfies the Categorified Presentation Axiom in the sense of Definition 5.3.11.

12. The terminal object is SO-projective in the sense of Definition 5.3.11.

We are now ready to combine the results so far and prove our main result.

Theorem 5.4.2.

1. *Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ models the constructive elementary theory of the category of sets if and only if* $\mathbf{Cat}(\mathcal{E})$ *models the constructive elementary theory of the* 2*-category of small categories, and in this case* $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$.

2. *Conversely, let $\mathcal{K}$ be a* 2*-category. Then $\mathcal{K}$ models the constructive elementary theory of the* 2*-category of small categories if and only if* $\mathbf{Disc}\,(\mathcal{K})$ *models the constructive elementary theory of the category of sets, and in this case* $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$.

Proof. For this proof, we list the results which allow us to deduce this theorem to be true.

- Axioms (1)-(5) allow us to apply Proposition 2.4.19 and deduce that $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$.

- The equivalence of $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ being 2-well pointed and $\mathbf{Disc}(\mathcal{K})$ being well-pointed is given in Theorem 4.4.13.

- The equivalence of 2-natural numbers objects in $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ and natural numbers objects in $\mathbf{Disc}(\mathcal{K})$ is given by Theorem 4.5.4.

- Proposition 5.3.5 tells us that $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ is SO-exact if and only if $\mathbf{Disc}(\mathcal{K})$ is exact.

- Proposition 5.3.2 shows that $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K})$ has exponentiable discrete opfibrations if and only if $\mathbf{Disc}(\mathcal{K})$ is locally cartesian closed.

- Proposition 5.3.10 shows the equivalence in this instance between $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ satisfying the 2-disjunction property and $\mathbf{Disc}(\mathcal{K})$ satisfying the disjunction property.

- Proposition 5.3.17 shows that $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ satisfies the Categorified Presentation Axiom if and only if $\mathbf{Disc}(\mathcal{K})$ satisfies the Presentation Axiom. From this, we can also deduce that the terminal in $\mathbf{Cat}(\mathbf{Disc}(\mathcal{K}))$ is SO-projective if and only if the terminal in $\mathbf{Disc}(\mathcal{K})$ is projective, since the terminal is discrete.

□

These assumptions allow for us to do a lot of the category theory that we would like to do. For example, we have all finite limits and colimits.

Proposition 5.4.3. *Let $\mathcal{K}$ be a* 2 *category that models CET2CSC. Then $\mathcal{K}$ has* 2*-colimits.*

Proof. This follows from Theorem 5.4.2, since if $\mathcal{K}$ models CET2CSC, then **Disc**$(\mathcal{K})$ models CETCS and so is in particular an arithmetic Π-pretopos; therefore, by Theorem 3.5.2, $\mathcal{K}$ has 2-colimits. □

An elementary topos is well-pointed if and only if it is Boolean (i.e. $\Omega \cong \mathbf{1}+\mathbf{1}$) [MM94]. A CETCS category is not assumed to be Boolean; instead, we have that $\mathbf{1} \to \mathbf{1}+\mathbf{1}$ is a *decidable subobject classifier*. In the case that the CETCS category is Boolean, every subobject is decidable— this is the Law of the Excluded Middle. In the following, we develop a 2-dimensional form of this statement.

Let $\mathcal{I}$ denote the walking isomorphism in **Cat** as described in Remark 2.0.1. Recall that for $\mathcal{E}$ an extensive category with finite coproducts, we have a functor $\underline{(-)}$: **Cat**(**FinSet**) $\to$ **Cat**$(\mathcal{E})$ which is defined levelwise as the partial left adjoint to the functor $\mathrm{Hom}(\mathbf{1},-) : \mathcal{E} \to$ **Set**; this partial left adjoint is defined on finite sets. In **Cat**$(\mathcal{E})$, the internal category $\underline{\mathcal{I}}$ has the universal property of the copower $\mathcal{I} \odot \mathbf{1}$.

We define the notion of a decidable full subobject classifier which in **Cat**$(\mathcal{E})$ is given by $\mathbf{1} \to \underline{\mathcal{I}}$. It classifies those full subobjects that are decidable, i.e. decidable full subcategories.

Definition 5.4.4. [GSS22] In an extensive 1-category $\mathcal{E}$, we call a morphism $i : A \to B$ *decidable* if there exists some $j : C \to B$ which exhibits B as the coproduct of A and C. i.e. $i \cong \iota_A : A \to A + C$.

Note that this is sometimes called a *complemented inclusion*. Recall that in an extensive category, coproduct inclusions are monomorphisms (see Remark 2.5.6). The internal Law of the Excluded Middle for a category $\mathcal{E}$ says that every monomorphism is decidable.

Definition 5.4.5. Let $\mathcal{E}$ be an extensive category and let $\mathbb{X} \in$ **Cat**$(\mathcal{E})$. Let $i : \mathbb{A} \to \mathbb{X}$ be a full monomorphism. If $i_0 : A_0 \rightarrowtail X_0$ is a decidable monomorphism, we call i a *decidably fully faithful.*

The above definition makes sense since full monomorphisms are fully faithful and monomorphic on objects.

Remark 5.4.6. We note that decidable-on-object functors are the cofibrations in the model structure on **Cat**$(\mathcal{E})$ given in Chapter 6. The decidably fully faithful morphisms are therefore fully faithful cofibrations.

Remark 5.4.7. For a decidable full monomorphism $i : \mathbb{A} \to \mathbb{X}$, fully faithfullness of it and $i_0 : A_0 \to X_0$ implies that $i_1 : A_1 \to X_1$ is also a decidable monomorphism, since coproduct injections are stable under pullback in an extensive category, and so the definition of a decidably fully faithful morphism could equivalently be described as being levelwise decidable and full.

Lemma 5.4.8. *Suppose* **Cat**$(\mathcal{E})$ *is extensive. Then decidably fully faithful morphisms are closed under pullback.*

Proof. Since $\mathbf{Cat}(\mathcal{E})$ is extensive, so is $\mathcal{E}$; complemented inclusions are stable under pullback in an extensive category by definition. Hence in $\mathbf{Cat}(\mathcal{E})$, since limits are computed pointwise, complemented inclusion-on-objects functors are stable under pullback. Fully faithful maps are stable under pullback since they are the right class of the (bijective on objects, fully faithful) factorisation system. $\square$

Proposition 5.4.9. *Let $\mathcal{E}$ be an extensive category and let $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$. For any decidable full monomorphism $i : \mathbb{A} \hookrightarrow \mathbb{X}$, there exists a unique map $\chi_i : \mathbb{X} \to \underline{\mathcal{I}}$ such that the diagram below is a pullback square.*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "!"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\top"] \\ \mathbb{X} \arrow[r, "\exists! \chi_i"'] & \underline{\mathcal{I}}. \end{tikzcd}$$

Proof. Let $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$. By the adjunction $\mathbf{indisc} \vdash (-)_0$, we have the following string of equations:

$$\begin{aligned} \mathbf{Cat}(\mathcal{E})(\mathbb{X}, \underline{\mathcal{I}}) &\cong \mathbf{Cat}(\mathcal{E})(\mathbb{X}, \mathbf{indisc}(\mathbf{1}+\mathbf{1})) \\ &\cong \mathcal{E}(X_0, \mathbf{1}+\mathbf{1}) \end{aligned}$$

which shows that given any full monomorphism $i : \mathbb{A} \to \mathbb{X}$, we can construct a unique map $\chi_i : \mathbb{X} \to \underline{\mathcal{I}}$ and vice versa, as required. $\square$

We now define the notion of decidable full monomorphism in an arbitrary extensive, complete and cocomplete 2-category, without reference to the morphisms being "decidable-on-objects". Recall that any complete and cocomplete 2-category $\mathcal{K}$ can be equipped with a model structure in which the weak equivalences and fibrations are defined representably to be equivalences of categories and isofibrations in **Cat** [Lac07, §3.5]; this is called the trivial model structure on a 2-category.

Definition 5.4.10. Let $\mathcal{K}$ be an extensive, complete and cocomplete 2-category. A 1-cell is $F : \mathbb{X} \to \mathbb{Y}$ is called *decidably fully faithful* if it is a fully faithful cofibration in the trivial model structure on $\mathcal{K}$.

Note that cofibrations are defined to be those morphisms which have the left lifting property with respect to the representable trivial fibrations. In $\mathbf{Cat}(\mathcal{E})$, these are precisely the complemented-inclusion on objects functors (see Chapter 6). Therefore, if $\mathcal{E}$ is a category in which coproduct injections are not monomorphisms, such as in the category of commutative rings (see the map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ which cannot be a monomorphism as $\mathbb{Z}/2X \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \mathbf{0}$), then the fully faithful cofibrations are not monomorphisms.

Example 5.4.11. For $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, the fully faithful cofibrations are precisely the fully faithful and decidable-on-objects functors, and so this definition is consistent with Definition 5.4.5. The details of this are worked out in Chapter 6.

This allows us to phrase the definition of decidable full subobject classifier in an arbitrary extensive, complete and cocomplete 2-category. Note that if $\mathcal{K}$ is complete then it is copowered by the walking isomorphism, and so there exists an object $\mathbf{I} \odot \mathbf{1} \in \mathcal{K}$. The map $\mathbf{1} \to \underline{\mathcal{I}}$ in **Cat** induces a map $\mathbf{1} \to \underline{\mathcal{I}} \odot \mathbf{1}$ in $\mathcal{K}$; we note this is always a (trivial) cofibration in the trivial model structure on $\mathcal{K}$ since its universal property ensures that it precisely lifts against isofibrations. Note that this does not imply that every pullback is a trivial cofibration, since trivial cofibrations are not automatically stable under pullback; however, the pullback will always be a fully faithful cofibration.

Definition 5.4.12. Let $\mathcal{K}$ be an extensive, complete and cocomplete 2-category. We say that $\mathcal{K}$ has a *decidable full subobject classifier* if for any $\mathbb{X} \in \mathcal{K}$ and any decidably fully faithful morphism $i : \mathbb{A} \hookrightarrow \mathbb{X}$, there exists a unique map $\chi_i : \mathbb{X} \to \mathcal{I} \odot \mathbf{1}$ such that the diagram below is a pullback square.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "!"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\top"] \\ \mathbb{X} \arrow[r, "\exists! \chi_i"'] & \mathcal{I} \odot \mathbf{1}. \end{tikzcd}$$

In the 1-dimensional case, mild assumptions ensure that $\mathcal{E}$ has a decidable subobject classifier. In future work, we will investigate the conditions which ensure the existence of a decidable full subobject classifier for a 2-category, and investigate its internal language.

Our concept of decidable full monomorphism allows us to phrase the Law of the Excluded Middle in purely 2-categorical terms.

Definition 5.4.13. Let $\mathcal{K}$ be an extensive, complete and cocomplete 2-category. We say $\mathcal{K}$ satisfies the *Categorified Law of the Excluded Middle* (2LEM) if every full monomorphism is a decidable full monomorphism.

Proposition 5.4.14. *Let $\mathcal{E}$ be an arithmetic Π-pretopos. Then $\mathcal{E}$ satisfies the Law of the Excluded Middle if and only if* $\mathbf{Cat}(\mathcal{E})$ *satisfies the Categorified Law of the Excluded Middle.*

Proof. If $\mathcal{E}$ satisfies LEM, then it is Boolean and $\mathbf{1} + \mathbf{1}$ is its subobject classifier, and therefore $\mathbf{1} \to \underline{\mathcal{I}}$ is the full subobject classifier in $\mathbf{Cat}(\mathcal{E})$, showing that $\mathbf{Cat}(\mathcal{E})$ satisfies 2LEM. Conversely, if $\mathbf{Cat}(\mathcal{E})$ satisfies 2LEM and we have some monomorphism $A \hookrightarrow X$ in $\mathcal{E}$, we can make this into an injective-on-objects and fully faithful functor in the same way as in Proposition 5.3.17; $K(i) \to \mathbf{disc}(X)$. By 2LEM, this is decidable on objects, as required. □

This result allows us to compare our work with the non-constructive version explored in Chapter 4.

Corollary 5.4.15. *CET2CSC + 2LEM + 2AC and ET2CSC have the same theorems.*

Proof. By Proposition 5.4.14, CET2CSC+2LEM implies that we $\mathcal{K}$ has a full subobject classifier; the remaining gap between CET2CSC and ET2CSC is the Categorified Axiom of Choice. □

5.5 Towards a bicategorical theory

So far in this chapter, we have developed a 2-categorical theory that characterises 2-categories of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a model of CETCS using elementary strict 2-categorical axioms. By restricting to $(2,1)$-categories, so that we are looking at internal groupoids rather than internal categories, we will see in Chapter 6 that any $(2,1)$-category modelling CET2CSC is a model of Martin-Löf type theory. However, there are more axioms in CET2CSC than are necessary to build a model of type theory, and further investigations will research the effect of these additional axioms.

It turns out that for type theoretic considerations, it is appropriate to look at weak categorification of Palmgren's CETCS rather than a strict one; this is because in type theory, it is more natural to consider definitions up to propositional equality instead of definitional equality. For example, there is a problem when we try to interpret having enough SO-projectives in type theory; this should correspond to a type theoretic Presentation Axiom, however, when spelling this out in the syntax, it becomes clear that one would like a definition of projective object with respect to ESO-morphisms (that is—morphisms with the bicategorical left lifting property with respect to fully faithful arrows). Below, we give some definitions that would be interesting to study in this context. We show that under certain assumptions on our 2-category $\mathcal{K}$, these are equivalent; consequently, any ESO-projective object can be strictified into an SO-projective object.

Definition 5.5.1. Let $\mathcal{K}$ be a 2-category. An ESO-morphism is a morphism $f : A \to B$ with the property that for any fully faithful $m : C \to D$, the following diagram is a bipullback:

$$\begin{tikzcd} \mathcal{K}(B,C) \arrow[r, "m\circ -"] \arrow[d, "-\circ f"'] & \mathcal{K}(B,D) \arrow[d, "-\circ f"] \\ \mathcal{K}(A,C) \arrow[r, "m\circ -"'] & \mathcal{K}(A,D). \end{tikzcd}$$

Example 5.5.2. For $\mathcal{K} = \mathbf{Cat}$, this recovers the definition of essentially surjective on objects functor. Let $F : \mathbb{A} \to \mathbb{B}$ be an ESO-morphism in **Cat**. Use the bijective-on-objects and fully faithful factorisation [SW73] to obtain a commutative square in which the arrow on the right is a fully faithful inclusion.

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "F"'] & \mathrm{img}(F) \arrow[d, hook] \\ \mathbb{B} \arrow[r, equal] \arrow[ur, dashed, "\ell"] & \mathbb{B} \end{tikzcd}$$

Since F is an ESO-morphism, the dashed arrow $\ell : \mathbb{B} \to \mathrm{img}(F)$ exists making the two triangles commute up to isomorphism. This means that for any $b \in \mathbb{B}$, there exists an isomorphism $\ell b \cong b$ by the lower triangle. Since the top arrow is bijective-on-objects, there exists an $a \in \mathbb{A}$ with $Fa = \ell b$. Putting these together says precisely that F is essentially surjective on objects. The converse can be argued in a standard way by constructing a lift on objects and then defining the lift on morphisms using fully-faithfulness; we leave the details as an exercise for the interested reader.

Remark 5.5.3. By the above, in **Cat** the ESO-morphisms are precisely those morphisms with the weak left lifting property with respect to fully faithful morphisms. The morphisms which have the strong (also called orthogonal) lifting property with respect to fully faithful morphisms are the bijective-on-objects functors. By [BG14, Proposition 4], in any BO-regular 2-category $\mathcal{K}$, the maps with the orthogonal left lifting property with respect to fully-faithful morphisms are the so-called *codescent morphisms*, which play an important role in the theory of 2-dimensional exactness for strict 2-categories, as explained in Section 2.4. As such, we might expect that ESO-morphisms play just as much of an important role in the theory of exactness for bicategories; indeed, we expect that ESO-morphisms are precisely the *bicolimit* of cateads, as this give the essential image of a functor in **Cat**, and so will give a generalisation of the essential image in a 2-category.

Being a bit more careful, the argument in Example 5.5.2 also works for $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, recovering there the definition of essentially surjective on objects functor given below.

Definition 5.5.4. In $\mathbf{Cat}(\mathcal{E})$, we define an internal functor $F : \mathbb{A} \to \mathbb{B}$ to be *essentially surjective on objects* if in the following diagram, the top arrow $d_0\hat{F_1}$ is an epimorphism.

$$\begin{tikzcd} P_F \arrow[r, "\hat{F_1}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Iso}(\mathbb{Y}) \arrow[r, "d_0"] \arrow[d, "d_1"] & Y_0 \\ X_0 \arrow[r, "F_0"'] & Y_0 & \end{tikzcd}$$

Definition 5.5.5. Let $\mathcal{K}$ be a 2-category. We call $\mathbb{X} \in \mathcal{K}$ *ESO-projective* if any fully faithful ESO-morphism $\mathbb{A} \to \mathbb{X}$ is an equivalence.

We say that $\mathcal{K}$ satisfies the *Weak Categorified Axiom of Choice* if every object is ESO-projective.

We say that $\mathcal{K}$ has *enough ESO-projectives* if for every $\mathbb{X} \in \mathcal{K}$ there exists an ESO-projective object $\mathbb{P}$ and a fully faithful ESO-morphism $\mathbb{P} \to \mathbb{X}$.

The Weak Categorified Axiom of Choice agrees with the Categorified Axiom of Choice in $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$. This is because in this setting, the Weak Categorified Axiom of Choice is the same as the statement that every fully-faithful ESO-morphism is an equivalence of internal categories.

Theorem 5.5.6. *Let $\mathcal{E}$ be a category with pullbacks, products and an (epi, mono)-orthogonal factorisation system. The following are equivalent:*

1. *$\mathcal{E}$ satisfies the Axiom of Choice.*
2. $\mathbf{Cat}(\mathcal{E})$ *satisfies the categorified Axiom of Choice.*
3. $\mathbf{Cat}(\mathcal{E})$ *satisfies the weak categorified Axiom of Choice.*
4. *Assuming additionally that $\mathcal{E}$ is extensive, any fully faithful ESO-morphism is equivalent to a split epi-on-objects and fully faithful morphism. That is: for any equivalence of internal categories $F : \mathbb{A} \to \mathbb{B}$ there is a commutative diagram*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "w"] \arrow[dr, "F"'] & & \hat{\mathbb{A}} \arrow[dl, "\hat{F}"] \\ & \mathbb{B} & \end{tikzcd}$$

such that w is an equivalence of categories and $\hat{F}$ is a split epi-on-objects fully faithful functor.

Proof. $(1) \iff (2)$ was proven in Theorem 4.7.13.

We prove $(1) \implies (2)$. Suppose $\mathcal{E}$ satisfies the Axiom of Choice and let $F : \mathbb{X} \to \mathbb{Y}$ be a fully faithful ESO-morphism. By definition, this means that the top arrow in the following is an epimorphism.

$$
\begin{tikzcd}
P_F \arrow[r, "\hat{F_1}"] \arrow[d, "\pi_{X_0}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Iso}(\mathbb{Y}) \arrow[r, "d_0"] \arrow[d, "d_1"] & Y_0 \\
X_0 \arrow[r, "F_0"'] & Y_0
\end{tikzcd}
$$

Since $\mathcal{E}$ satisfies the Axiom of Choice, this arrow has a section $s : Y_0 \to P_F$. We define $G_0 := \pi_{X_0} s : Y_0 \to X_0$.

Using s, we define a morphism $k : Y_1 \to Y_3$ by its universal property given the commutativity of the outside of the following diagram.

$$
\begin{tikzcd}
 & & Y_1 \arrow[dl, "d_1"] \arrow[dr, "d_0"] \arrow[ddddd, "1_{Y_1}"] & & \\
 & Y_0 \arrow[dl, "s"] \arrow[ddddd, "1_{Y_0}"] & & Y_0 \arrow[dr, "s"] \arrow[ddddd, "1_{Y_0}"] & \\
P_F \arrow[d, "\hat{F_1}"'] & & & & P_F \arrow[d, "\hat{F_1}"] \\
\mathbf{Iso}(\mathbb{Y}) \arrow[dd, hook] \arrow[dddr, "d_0"] & & & & \mathbf{Iso}(\mathbb{Y}) \arrow[d, "(-)^{-1}"] \arrow[dddl, "d_0"'] \\
 & & & & \mathbf{Iso}(\mathbb{Y}) \arrow[d, hook] \\
Y_1 \arrow[dr, "d_0"'] & & Y_1 \arrow[dl, "d_1"] \arrow[dr, "d_0"'] & & Y_1 \arrow[dl, "d_1"] \\
 & Y_0 & & Y_0 &
\end{tikzcd}
$$

Which makes the following diagram commute.

$$
\begin{tikzcd}
Y_1 \arrow[rr, "k"] \arrow[d, "{(d_1,d_0)}"'] & & Y_3 \arrow[r, "m^2"] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\
Y_0 \times Y_0 \arrow[r, "s\times s"'] & P_F \times P_F \arrow[r, "\pi_{X_0}\times\pi_{X_0}"'] & X_0 \times X_0 \arrow[r, "F_0\times F_0"'] & Y_0 \times Y_0
\end{tikzcd}
$$

Due to fully faithfulness of F, the commutativity of the diagram above induces an arrow $G_1 : Y_1 \to X_1$. By construction, $G := (G_0, G_1)$ is a morphism of internal graphs, and it is not hard to check that it is also an internal functor $G : \mathbb{Y} \to \mathbb{X}$. We construct internal natural isomorphisms $\alpha : GF \Rightarrow 1_{\mathbb{X}}$ and $\beta : GF \Rightarrow 1$. Let $\hat{\alpha} : X_0 \to X_1$ be the composite

$$
X_0 \xrightarrow{F_0} Y_0 \xrightarrow{s} P_F \xrightarrow{\hat{F}} \mathbf{Iso}(\mathbb{Y}) \hookrightarrow Y_1 \xrightarrow{G_1} X_1
$$

And define $\hat{\beta} : Y_0 \to Y_1$ by the composite

$$
Y_0 \xrightarrow{s} P_F \xrightarrow{\hat{F}} \mathbf{Iso}(\mathbb{Y}) \hookrightarrow Y_1
$$

It is left to the reader to verify that these satisfy the required equations to form internal natural isomorphisms $\alpha : GF \Rightarrow 1_{\mathbb{X}}$ and $\beta : FG \Rightarrow 1_{\mathbb{Y}}$, as required. Hence G exhibits F as an equivalence of internal categories.

Conversely, we prove $(3) \implies (2)$. Assume that $\mathbf{Cat}(\mathcal{E})$ satisfies the weak categorified Axiom of Choice. Let $f : A \to B$ be an epimorphism in $\mathcal{E}$. Consider the internal category $K(f)$ as constructed in Definition 2.4.4.

$$A \times_B A \times_B A \overset{p_1}{\underset{p_2}{\underset{-m\to}{\rightrightarrows}}} A \times_B A \overset{d_1}{\underset{d_0}{\underset{\leftarrow i -}{\rightrightarrows}}} A$$

There is a functor $K : K(f) \to \mathbf{disc}(B)$ for which $K_0 := f$ and $K_1 := fd_1$. This is epi-on-object and therefore an ESO-morphism since f was an epimorphism and it is moreover fully faithful since the pullback of f along itself is equally the pullback of $\Delta : B \to B \times B$ along $f \times f : A \times A \to B \times B$. Hence, by the weak categorified Axiom of Choice, K is an equivalence, and so has an equivalence inverse $L : \mathbf{disc}(B) \to K(f)$ with $\beta : KL \Rightarrow 1_{\mathbf{disc}(B)}$. Moreover, this must be the identity, since $\mathbf{disc}(B)$ is discrete; hence L_0 is a splitting of f as required.

The deduction $(4) \implies (3)$ is obvious as it says that every fully faithful ESO-morphisms is equivalent to an equivalence of categories and so is therefore itself an equivalence (by 3-for-2, for example.)

Finally, we prove $(3) \implies (4)$. Factorise F as a cofibration followed by trivial fibration, which exists on $\mathbf{Cat}(\mathcal{E})$ and for which the trivial fibrations are exactly the split-epi-on-objects and fully faithful functors (see Proposition 6.3.6). Since F is an equivalence by assumption, it follows from 3-for-2 that the cofibration is also an equivalence, as required. $\square$

In particular, item (4) says that we can strictify fully faithful ESO-morphisms into acute morphisms in the presence of the Axiom of Choice.

Remark 5.5.7. One could ask for a result that says that $\mathcal{E}$ has enough projective if and only if $\mathbf{Cat}(\mathcal{E})$ has enough ESO-projectives (a weak version of Proposition 5.3.17). However, it seems that this does not work— this appears to boil down to the fact that we cannot strictify fully-faithful ESO-morphisms. Indeed in order to prove that $\mathbf{Cat}(\mathcal{E})$ has enough ESO-projectives implies that $\mathcal{E}$ has enough projectives, we might wish to proceed as in the proof of $(3) \implies (1)$ in Theorem 5.5.6. Given an $X \in \mathcal{E}$, we can form an internal category $\mathbf{disc}(X)$ in $\mathbf{Cat}(\mathcal{E})$. Since $\mathbf{Cat}(\mathcal{E})$ is assumed to have enough projectives, there is a fully-faithful ESO-morphism $q : \mathbb{P} \to \mathbf{disc}(X)$ such that $\mathbb{P}$ is ESO-projective. Given that $\mathbf{disc}(X)$ is discrete, we can indeed conclude that $q_0 : P_0 \to X$ is an epimorphism. However, it does not seem to be true that P_0 is projective in $\mathcal{E}$; given any epimorphism $f : A \to P_0$ we can construct the category $K(f)$ and a fully faithful ESO-morphism $K(f) \to \mathbb{P}$, which by assumption is an equivalence; however, without the Axiom of Choice, it does not seem to be possible to deduce that this provides a splitting for f.

We conjecture that the concept of ESO-projectivity would be more well-behaved in the weaker setting, for example in considering the bicategory of pseudo-internal categories (in which the equations are only required to hold up to invertible 2-cells). In future work, we will investigate a bicategorical formulation of CET2CSC. Our modular formulation of CET2CSC suggests ways in which each of the axioms can be suitably weakened. The hope is that this will be a more suitable setting for investigating axioms in type theory: statements about essentially surjective on objects functors are expressible in Martin-Löf type theory whilst statements about surjective-on-objects functors are not. Hence we can state what it means to be an ESO-projective type within Martin-Löf type theory, but not what it means to be an SO-projective object.

Chapter 6

The internal groupoid model of Martin-Löf type theory

6.1 Introduction

6.1.1 Context and motivation

The groupoid model of Martin-Löf type theory (MLTT) was introduced by Hofmann and Streicher in 1998 [HS98], proving that the Uniqueness of Identity Proofs principle (UIP) cannot be derived in this type theory and thus revealing a deep connection between logic, (higher) category theory and homotopy theory (see also the work of Lamarche [Lam]). This key observation led to the development of homotopy type theory and univalent foundations [Uni13]. Since then, many different categorical models of (homotopy) type theory have been studied with a particular focus on those which have an abstract homotopy theory [Voe06; AW09; BCH14; RS17; Ber18; Ber20; Awo26].

One of the earliest and most effective ways to abstractly provide a category with an abstract homotopy theory is given by Quillen's notion of a model structure [Qui67]. Loosely, this is a category with three distinguished classes of maps called weak equivalences, fibrations and cofibrations. From a categorical point of view these suffer from certain defects, for example cofibrations (resp. fibrations) are not closed under colimits (resp. limits) in the arrow category. Riehl's notion of an algebraic model category [Rie11] builds on the notion of an algebraic weak factorisation system [GT06; Gar09] and provides a solution to this by considering cofibrations and fibrations as retracts of coalgebras for a comonad and algebras for a monad respectively, since (co)algebras for a (co)monad are closed under (co)limits. The philosophy is that by keeping track of algebraic data, certain calculations become canonical and therefore more categorically well-behaved.

An example of key interest to us is given by the classical model structure on **Cat**, the category of small categories and functors and the restriction of this to the category of small groupoids, **Gpd** [And78; Rez96]. Here, the weak equivalences are equivalences of categories; fibrations are isofibrations; cofibrations are functors which are injective on objects. Because this model structure is cofibrantly generated, it underlies an algebraic model structure by a process described in [Rie11], although the explicit details of this for the classical model structure on **Cat** has not been worked out until now.

Hofmann and Streicher's groupoid model of MLTT lives in the context of the homotopy theory given by the classical model structure on **Gpd**; dependent types are modelled by isofibrations between groupoids. In fact, there are a couple of coherence issues with this model; substitution and identity types are defined only up to isomorphism rather than equality. One suggested approach to fixing this problem is using precisely the algebraic perspective described above. By keeping track of algebraic data there are canonical choices for substitutions and identity types in the model. To this end, Gambino and Larrea introduced the notion of a type theoretic algebraic weak factorisation system [GL23], which is an algebraic weak factorisation system with extra structure.

In this chapter, we shall extend these ideas to the setting of internal categories and internal groupoids. These notions have recently witnessed an increase in interest partially due to their connection with double category theory, 2-dimensional universal algebra and alternative foundations of mathematics [Joh02b; EKL05; JT06; FPP08; Bou10; NP19]. The homotopy theory of $\mathbf{Cat}(\mathcal{E})$ has been studied in [EKL05]. In [EKL05, §7], a model structure is considered in which the fibrations are the representable isofibrations and the weak equivalences are the representable equivalences of categories. We call this the *natural model structure* on $\mathbf{Cat}(\mathcal{E})$. Note that this is not in general the same as the model structure on internal categories considered in [JT06]. The goal of this chapter is to extend this to an algebraic model structure and use it to obtain models of MLTT.

6.1.2 Main results

This chapter makes two main contributions. The first contribution is to prove some nice properties of the natural model structure on $\mathbf{Cat}(\mathcal{E})$. We show that it is a cofibrantly generated, monoidal model structure which underlies an algebraic model structure (Theorem 6.3.15) and give explicit descriptions of the algebraic structure. As an application, we conclude that for an internal monoidal category $\mathbb{M}$, there is a cofibrantly generated monoidal model structure on the category of $\mathbb{M}$-modules. This is Theorem 6.3.16.

The second contribution of this chapter is that by restricting to $\mathbf{Gpd}(\mathcal{E})$, we obtain an algebraic model of MLTT. We show that the algebraic weak factorisation system consisting of maps with a trivial cofibration structure and maps with a fibration structure forms a type theoretic algebraic weak factorisation system in the sense of [GL23]. This is Theorem 6.5.9.

The main benefit of this approach is that we can apply these results to a variety of different examples. In some cases, this gives an improved perspective on results that have been previously considered in the literature; in other cases, it gives novel results. For example, we are able to apply our result to groupoids internal to **Set**, the category of (modest) assemblies and the effective topos, and presheaves on a locally small category $\mathbb{C}$, giving us constructive versions of classical results, realisability models of type theory, and an indexed version of the groupoid model of type theory respectively.

6.1.3 Related work

Algebraic groupoid models of MLTT are also considered in [GL23, Theorem 5.5]. This chapter works in the more general setting of groupoids internal to some category $\mathcal{E}$, and places this in the larger structure of an algebraic model structure. Taking $\mathcal{E} = \mathbf{Set}$, the precise relationship between these models is explained in Section 6.7.3; in particular the fibrations considered there are *normal* isofibrations in contrast to ours, which deals with *cloven* isofibrations.

Relatedly, [Agw25] constructs a model of type theories with groupoids internal to the category of assemblies over some partial combinatory algebra and also an algebraic model structure using different methodology. Again, the fibrations of this model structure considered are normal isofibrations in contrast to ours which looks at cloven isofibrations. As a result, both the model structure and the model of type theory will be different. In Section Section 6.5.5, we comment about generalising this approach from groupoids internal to the category of assemblies to more general internal groupoids.

The method of looking at groupoids internal to the effective topos as a model of type theory is also related to the work of both Awodey and Emmenegger [AE25] and Speight, who are both considering 2-dimensional forms of the effective topos.

[EPR21] studied how $\mathcal{E}$-enriched groupoids can be equipped with a type theoretic algebraic weak factorisation system, similar in spirit to my result, which shows that $\mathcal{E}$-internal groupoids can be equipped with a type theoretic algebraic weak factorisation system. However, the definitions of $\mathcal{E}$-enriched groupoids and $\mathcal{E}$-internal groupoids do not overlap unless $\mathcal{E} = \mathbf{Set}$, in which case this is already been studied by Gambino and Larrea.

[Woe21] studies how cloven isofibations in **Gpd** provide an algebraic weak factorisation system modelling type theory; as such, mine presents an internal generalisation of this work as opposed to of Gambino and Larrea's.

6.1.4 Outline

Section 6.2 establishes the tools which we will exploit in our proofs including an enrichment of $\mathbf{Cat}(\mathcal{E})$ over $(\mathcal{E}, \times, \mathbf{1})$ which allows us to run the enriched small object argument on a pair of classes of maps in $\mathcal{E}$. In Section 6.3, we show that these form the generated (trivial) cofibrations of the model structure of [EKL05, §7], recalled in Section 6.2.4; whence the model structure is cofibrantly generated. Using this, we can easily show that the model structure is symmetric monoidal and admits an algebraic model structure. This is recorded in Theorem 6.3.15. We show that this restricts to $\mathbf{Gpd}(\mathcal{E})$ in Theorem 6.4.17.

In Section 6.4, we work through the details of the algebraic structure explicitly which allows us to show in Section 6.5 that for $\mathcal{E}$ a locally cartesian closed locos with coequalisers, the data of the algebraic model structure on $\mathbf{Gpd}(\mathcal{E})$ has the extra structure of a type theoretic algebraic weak factorisation system, in the sense of [GL23, Definition 4.10]. This is Theorem 6.5.9, and proves that we have an internal groupoid model of MLTT for any locally cartesian closed locos with coequalisers.

We conclude with Section 6.7, which applies the results of the chapter to various examples giving both novel results and extensions of previously known results.

6.2 Preliminaries

Throughout this chapter, we will assume that $\mathcal{E}$ is a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint. These assumptions ensure that $\mathbf{Cat}(\mathcal{E})$ is finitely cocomplete by Theorem 3.5.2.

Examples of such a setting are given in Section 3.7 and include any locally cartesian closed locos with coequalisers such as an arithmetic Π-pretopos or an elementary topos with a natural numbers object. These are key motivational examples for us. Note that we do not require exactness nor a subobject classifier, although any locally cartesian closed locos with coequalisers is regular (see Section 3.1.3).

Remark 6.2.1. We could instead have chosen $\mathcal{E}$ to be a lextensive category that is locally finitely presentable and cartesian closed, such as **Cat**, **Gph** and **Ab**— this would also ensure that $\mathbf{Cat}(\mathcal{E})$ is finitely cocomplete by Proposition 3.1.2. Hence, the existence of an algebraic model structure can be applied instead when $\mathcal{E}$ is a lextensive category that is locally finitely presentable and cartesian closed. However, being locally finitely presentable is rather restrictive and furthermore not an elementary condition. Moreover, when modelling type theory, we want isofibrations of internal groupoids to be exponentiable; in fact, we show in Proposition 6.6.1 that isofibrations in $\mathbf{Gpd}(\mathcal{E})$ are exponentiable if and only if $\mathcal{E}$ is locally cartesian closed. As such, for type theoretic aspects of this work, we cannot take $\mathcal{E}$ to be either **Cat** or **Ab** due to their lack of local cartesian closure.

6.2.1 Notation

The notation in this chapter is as in the rest of the thesis with one notable exception: since everything we deal with in this chapter is 1-categorical, we denote the 1-category of internal categories and internal functors by $\mathbf{Cat}(\mathcal{E})$ as opposed to $\mathbf{Cat}(\mathcal{E})_1$.

Recall that $\Delta_{\leq 3}$ is the category of non-empty ordinals up to and including the ordinal with four elements, whose more formal definition is given in Section 2.2.

It will be useful to have notation for the following categories.

Notation 6.2.2.

$$\mathbf{1} := \bullet$$

$$\mathbf{2} := \bullet \longrightarrow \bullet$$

$$\mathcal{P} := \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

$$\mathcal{I} := \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

For a $\mathcal{V}$-enriched category $\mathcal{C}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$, we denote its hom-object by $\mathrm{Hom}_{\mathcal{V}}(X, Y)$. We note that $\mathbf{Cat}(\mathcal{E})$ is **Cat**-enriched and make use of this in defining certain concepts representably with respect to hom-categories.

For $E \in \mathcal{E}$ and $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, we write $E \times \mathbb{X} := \mathbf{disc}(E) \times \mathbb{X}$ and $\mathbb{X}^E := \mathbb{X}^{\mathbf{disc}(E)}$.

For (T, η) a pointed endofunctor (resp. $\mathbb{T}$ a monad), we denote its category of algebras by (T, η)-**Alg** (resp. $\mathbb{T}$-**Alg**).

For $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$, we distinguish between $(u, v) : f \to g$ and $\alpha : f \Rightarrow g$; the first is a commutative square from f to g i.e. $u : \mathrm{dom}(f) \to \mathrm{dom}(g)$ and $v : \mathrm{cod}(f) \to \mathrm{cod}(g)$ such that $gu = vf$. This is a morphism in the category $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$. The latter is an internal natural transformation, which only makes sense if $\mathbb{A} = \mathbb{X}$ and $\mathbb{B} = \mathbb{Y}$.

6.2.2 Enrichment over $\mathcal{E}$

We define an enrichment of $\mathbf{Cat}(\mathcal{E})$ over $(\mathcal{E}, \times, \underline{\mathbf{1}})$. The functor $\mathrm{Hom}_{\mathbf{Set}}(\underline{\mathbf{1}}, -) : \mathcal{E} \to \mathbf{Set}$ has a partial left adjoint, defined by mapping any finite set S to

$$\underline{S} := S \cdot \underline{\mathbf{1}} = \coprod_{s \in S} \underline{\mathbf{1}}.$$

This can also be described as a (genuine) functor $\underline{(-)} : \mathbf{FinSet} \to \mathcal{E}$.

We extend this levelwise to become a partial functor $\underline{(-)} : \mathbf{Cat} \to \mathbf{Cat}(\mathcal{E})$, defined on *finite categories*, equivalently a genuine functor $\underline{(-)} : \mathbf{Cat}(\mathbf{FinSet}) \to \mathbf{Cat}(\mathcal{E})$. This is well-defined by extensivity of $\mathcal{E}$, so that in $\mathcal{E}$ the coproduct commutes with finite limits [CLW93]; in particular, for $\mathbb{C} \in \mathbf{Cat}$, we have a well-defined composition operation

$$\underline{m} : \underline{\mathbb{C}_1} \times_{\underline{\mathbb{C}_0}} \underline{\mathbb{C}_1} \to \underline{\mathbb{C}_1}.$$

Hence we have a partial adjunction defined on the class of small categories with finite set of objects and finite set of morphisms.

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{\underline{1}}, -) \vdash \underline{(-)} : \mathbf{Cat}(\mathcal{E}) \leftrightarrows \mathbf{Cat}$$

Remark 6.2.3. For a set of finite categories $\mathbf{X} := \{\mathbb{C}_0, \mathbb{C}_1, ..., \mathbb{C}_n\}$ we define the set of internal categories $\underline{\mathbf{X}} := \{\underline{\mathbb{C}_0}, \underline{\mathbb{C}_1}, ..., \underline{\mathbb{C}_n}\}$.

It is often useful to consider internal categories as simplicial objects. In our setting, since we do not want to assume countable limits, we consider internal categories as simplicial objects truncated at the fourth level. We can do this without loss of any information about the internal category because the highest data that internal categories have is associativity, which occurs at the fourth level of the simplicial nerve.

Lemma 6.2.4. *The truncated internal nerve functor* $N : \mathbf{Cat}(\mathcal{E}) \hookrightarrow \left[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}\right]$ *is fully faithful.*

Proof. This is immediate from the definition of internal functor and the definition of morphism in $\left[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}\right]$. □

Consider $\Delta^{\mathrm{op}}_{\leq 3}$ with the free $\mathcal{E}$-enrichment as described in Example 3.3.4 of [Rie14] in which

$$\mathrm{Hom}_{\mathcal{E}}\left([n], [m]\right) := \underline{\mathrm{Hom}_{\mathbf{Set}}\left([n], [m]\right)}$$

which is well-defined since $\mathrm{Hom}_{\mathbf{Set}}\left([n], [m]\right)$ is finite.

Let $\mathbb{X}, \mathbb{Y}$ be internal categories. Then $N\mathbb{X}$ and $N\mathbb{Y}$ are $\mathcal{E}$-valued presheaves. We define the following:

$$\mathrm{Hom}_{\mathcal{E}}(\mathbb{X}, \mathbb{Y}) := \int_{[n] \in \Delta^{\mathrm{op}}_{\leq 3}} NY_n^{NX_n}.$$

This is the hom-object of $\mathcal{E}$-natural transformations in $[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}]$ as described in [Rie14, Digression 7.3.1]. By fully-faithfulness of the truncated internal nerve, this provides an enrichment on $\mathbf{Cat}(\mathcal{E})$.

Remark 6.2.5. We give an equivalent description of the above enrichment. Recall from Theorem 4.3.1 that if $\mathcal{E}$ has finite limits and is cartesian closed, then $\mathbf{Cat}(\mathcal{E})$ is cartesian closed with internal hom calculated in $\mathbf{s}\mathcal{E}$, i.e. $\mathbf{Cat}(\mathcal{E})$ is an exponential ideal of $\mathbf{s}\mathcal{E}$. This factors through the truncated simplicial objects, $\left[\Delta^{\mathrm{op}}_{\leq 3}, \mathcal{E}\right]$. Explicitly, for $\mathbb{X}, \mathbb{Y} \in \mathbf{Cat}(\mathcal{E})$, we have

$$\underline{[\mathbb{X}, \mathbb{Y}]} := [N\mathbb{X}, N\mathbb{Y}] \in \mathbf{Cat}(\mathcal{E})$$

which is defined to be the functor $\Delta_{\leq 3}^{op} \to \mathcal{E}$ given by

$$k \mapsto \int\limits_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} {Y_n}^{X_n \times \Delta^{op}_{\leq 3}([n],[k])}.$$

In particular, we have

$$\begin{aligned}
\underline{[\mathbb{X},\mathbb{Y}]}_0 &:= [N\mathbb{X}, N\mathbb{Y}]([0]) \\
&= \int\limits_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} {NY_n}^{NX_n \times \Delta^{op}_{\leq 3}([n],[0])} \\
&= \int\limits_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} {NY_n}^{NX_n \times \Delta_{\leq 3}([0],[n])} \\
&\cong \int\limits_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} {NY_n}^{NX_n \times \underline{\mathbf{1}}} \\
&\cong \int\limits_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} {NY_n}^{NX_n}
\end{aligned}$$

So we equivalently have $\mathrm{Hom}_{\mathcal{E}}(\mathbb{X},\mathbb{Y}) = \underline{[\mathbb{X},\mathbb{Y}]}_0$. This different perspective will also be useful.

The following follows from a strong form of the enriched Yoneda lemma [Kel82, Theorem 2.4].

Proposition 6.2.6. *Let $\mathcal{E}$ be a cartesian closed category with finite limits. Let $X : \Delta_{\leq 3} \to \mathcal{E}$. For all $[k] \in \Delta_{\leq 3}$ we have that:*

$$X[k] \cong \int_{[n]\in\Delta_{\leq 3}} X_n^{\Delta([k],[n])}.$$

Lemma 6.2.7. *Let $\mathbb{X}$ be an internal category. We can calculate the following:*

1. $\mathrm{Hom}_{\mathcal{E}}(\underline{\emptyset}, \mathbb{X}) \cong \underline{\mathbf{1}}$.
2. $\mathrm{Hom}_{\mathcal{E}}(\underline{\mathbf{1}}, \mathbb{X}) \cong X_0$.
3. $\mathrm{Hom}_{\mathcal{E}}(\underline{\mathbf{2}}, \mathbb{X}) \cong X_1$.
4. $\mathrm{Hom}_{\mathcal{E}}(\underline{\mathbf{1}} + \underline{\mathbf{1}}, \mathbb{X}) \cong X_0 \times X_0$.
5. $\mathrm{Hom}_{\mathcal{E}}(\mathbb{X}, \underline{\mathbf{1}}) \cong \underline{\mathbf{1}}$.

Proof. (1) follows from the observation that $\underline{\emptyset}$ is the initial object in $\mathbf{Cat}(\mathcal{E})$, and so for any $E \in \mathcal{E}$ we have $E^{\underline{\emptyset}} = \underline{\mathbf{1}}$. For (2), we have that $\underline{\Delta}^{\mathrm{op}}_{\leq 3}([0], -) \cong N\underline{\mathbf{1}}$. Therefore, by Proposition 6.2.6, we have

$$\mathrm{Hom}_{\mathcal{E}}(\underline{\mathbf{1}}, \mathbb{X}) =_{\mathrm{def}} [N\underline{\mathbf{1}}, N\mathbb{X}]_0 \cong [\underline{\Delta}_{\leq 3}([0], -), N\mathbb{X}] \cong N\mathbb{X}([0]) = X_0.$$

The proof for (3) is similar. For (4), we note that for enriched hom functors, colimits in the first variable can be taken out as limits [Kel82, §3.2] and the result follows. For (5), note that $\underline{\mathbf{1}}^E \cong \underline{\mathbf{1}}$ for any $E \in \mathcal{E}$. $\square$

From the internal hom, we can define an evaluation.

Definition 6.2.8. We define a partial 2-variable functor $\mathrm{ev}_{(-)}(-) : \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat}(\mathcal{E}) \rightharpoonup \mathcal{E}$. For $\mathcal{A}$ a finite category and $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$:

$$\mathrm{ev}_{\mathcal{A}}(\mathbb{X}) := \mathrm{Hom}_{\mathcal{E}}(\underline{\mathcal{A}}, \mathbb{X}).$$

Fixing $\mathbb{X}$, we get a partial functor $\mathbb{X}(-) := \mathrm{ev}_{(-)}(\mathbb{X}) : \mathbf{Cat}^{\mathrm{op}} \rightharpoonup \mathcal{E}$.

Remark 6.2.9. Note that Lemma 6.2.7 then implies that for $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, we have:

$$\begin{aligned}\mathbb{X}(\emptyset) &\cong \underline{\mathbf{1}}\\ \mathbb{X}(\mathbf{1}) &\cong X_0\\ \mathbb{X}(\mathbf{2}) &\cong X_1\\ \mathbb{X}(\mathbf{1}+\mathbf{1}) &\cong X_0 \times X_0.\end{aligned}$$

This is an internal counterpart to statements in the ordinary setting such as maps from $\mathbf{1}$ to $\mathbb{X}$ are in bijection with its objects, X_0, which can be proven in **Cat** using the Yoneda lemma.

Recall the functor $\mathbf{Iso} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gpd}(\mathcal{E})$, the right adjoint to the inclusion $\mathbf{Gpd}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})$ defined by taking the maximal internal groupoid of an internal category $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$ (cf. [BP79]). Explicitly, this has the same object of objects and object of arrows is given and a subobject of the object of arrows $c : X_1^{\cong} \rightarrowtail X_1$ constructed as the limit of the composition arrow $m : X_1 \times_{X_0} X_1 \to X_1$ together with the maps $id_1\pi_1, id_0\pi_0 : X_1 \times_{X_0} X_1 \to X_1$. The composition map and identity assigner are induced by the universal property of the limit whilst the source and target maps are given by $d_1 c, d_0 c : X_1^{\cong} \to X_0$.

We also have the following:

Lemma 6.2.10. *Let $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$. We have $\mathbb{X}(\mathcal{I}) \cong \mathbf{Iso}(\mathbb{X})_1$.*

Proof. Firstly, Remark 6.2.5 tells us that $\mathbb{X}(\mathcal{I}) = \left(\mathbb{X}^{\underline{\mathcal{I}}}\right)_0$. The discussion of the construction of the power of $\mathbb{X}$ by the ordinary category $\mathcal{I}$ is given in section 3 of [EKL05], and has $\left(\mathbb{X}^{\mathcal{I}}\right)_0 := \mathbf{Iso}(\mathbb{X})_1$. Note that this construction is a groupoidal version of the power of an internal category by the category $\mathbf{2}$ discussed in Section 4.4 and by the same argument given there, it follows that $\mathbb{X}^{\underline{\mathcal{I}}}$ has the universal property of $\mathbb{X}^{\mathcal{I}}$. We conclude that $\mathbb{X}(\mathcal{I}) \cong \mathbf{Iso}(\mathbb{X})_1$. $\square$

We can also describe the object of pairs of parallel arrows in an internal category. Note that in **Cat**, the category $\mathcal{P}$ defined in Notation 6.2.2 can constructed by gluing two walking arrows together, as in the pushout below:

$$\begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \\ \mathbf{2} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{P} \end{tikzcd} \tag{6.1}$$

Lemma 6.2.11. *We have a pullback square*

$$\begin{tikzcd} \mathbb{X}(\mathcal{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "{(d_1,d_0)}"] \\ X_1 \arrow[r, "{(d_1,d_0)}"'] & X_0 \times X_0 \end{tikzcd}$$

Proof. First, we apply $\underline{(-)}$ to Equation (6.1) which is the defining pushout square of $\mathcal{P}$; this functor is (partial) left adjoint so preserves colimits of finite categories and results in a pushout square in $\mathcal{E}$. Now, some simple calculus using cartesian closedness and lextensivity of $\mathcal{E}$ shows that $\mathrm{Hom}_{\mathcal{E}}(-, \mathbb{X})$ turns pushouts into pullbacks. Hence, we get the above pullback square. $\square$

This enrichment of $\mathbf{Cat}(\mathcal{E})$ in $(\mathcal{E}, \times, \underline{\mathbf{1}})$ makes it copowered and powered by $\mathcal{E}$. Recall that this means for all $E \in \mathcal{E}$ and $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, there exists objects $E \odot \mathbb{X}$, (the *copower* of $\mathbb{X}$ by E) and $E \pitchfork \mathbb{X}$ (the *power* of $\mathbb{X}$ by E) such that for any $\mathbb{Y} \in \mathbf{Cat}(\mathcal{E})$ we have

$$\mathrm{Hom}_{\mathcal{E}}(E \odot \mathbb{X}, \mathbb{Y}) \cong \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Hom}_{\mathcal{E}}(\mathbb{X}, \mathbb{Y})) \cong \mathrm{Hom}_{\mathcal{E}}\left(\mathbb{X}, E \pitchfork \mathbb{Y}\right).$$

Note that this condition is also called being tensored and cotensored over $\mathcal{E}$.

Proposition 6.2.12. *For any $E \in \mathcal{E}$ and $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, the copower of $\mathbb{X}$ by E is given by $E \times \mathbb{X}$. Dually, the power of $\mathbb{X}$ by E is given by $\mathbb{X}^E$.*

Proof.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Hom}_{\mathcal{E}}(\mathbb{X}, \mathbb{Y})) &:= \mathrm{Hom}_{\mathcal{E}}\left(E, \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} NY_n^{NX_n}\right) \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} \mathrm{Hom}_{\mathcal{E}}\left(E, NY_n^{NX_n}\right) && \text{as } \mathrm{Hom}_{\mathcal{E}}(E, -) \text{ is } \mathcal{E}\text{-continuous,} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} \mathrm{Hom}_{\mathcal{E}}\left(E \times NX_n, NY_n\right) && \text{tensor-hom adjunction,} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} NY_n^{E\times NX_n} && \text{definition of enrichment of } \mathcal{E} \text{ over itself} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} NY_n^{(NE\times NX)_n} && \text{definition of tensoring and nerve,} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} NY_n^{N(E\times X)_n} \\
&=: \mathrm{Hom}_{\mathcal{E}}(E \times \mathbb{X}, \mathbb{Y}) && \text{as required.}
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Hom}_{\mathcal{E}}(\mathbb{X},\mathbb{Y})) &\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} NY_n^{NE_n\times NX_n} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} \left(NY_n^{NE_n}\right)^{NX_n} \\
&\cong \int_{[n]\in\Delta^{\mathrm{op}}_{\leq 3}} N\left(Y^E\right)_n^{NX_n} \\
&=: \mathrm{Hom}_{\mathcal{E}}\left(\mathbb{X},\mathbb{Y}^E\right) \qquad \text{as required.}
\end{aligned}$$

□

6.2.3 The classical model structure on Cat

Recall that there is a model structure on **Cat** with fibrations given by the isofibrations, cofibrations given by functors which are injective-on-objects and weak equivalences given by the equivalences of categories [JT06]. We call this the *classical model structure* on **Cat**. This model structure is monoidal with respect to the cartesian product and also cofibrantly generated by the sets

$$\mathbf{I} := \{\emptyset \to \mathbf{1}, \mathbf{1}+\mathbf{1}\to\mathbf{2}, \mathcal{P}\to\mathbf{2}\} \tag{6.2}$$

$$\mathbf{J} := \{\mathbf{1}\to\mathcal{I}\} \tag{6.3}$$

in which the relevant categories are given in Notation 6.2.2.

6.2.4 The natural model structure on Cat($\mathcal{E}$)

We recall a model structure on **Cat**($\mathcal{E}$) which we call the *natural model structure*. This was originally constructed in [EKL05], and it was later noted by Lack that it was an example of what he called a 'trivial model structure' on the 2-category of internal categories, internal functors and internal natural transformations [Lac07, §3.5].

We start by defining the fibrations and weak equivalences representably; then we give them an internal description which does not rely on the category **Set**.

Definition 6.2.13. An internal functor $f : \mathbb{X}\to\mathbb{Y}$ is called an *internal equivalence of categories* if for every $\mathbb{A}\in\mathbf{Cat}(\mathcal{E})$, the functor $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f) : \mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A},\mathbb{X})\to\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A},\mathbb{Y})$ is an equivalence of categories.

An internal functor $f : \mathbb{X}\to\mathbb{Y}$ is called an *internal isofibration* if for every $\mathbb{A}\in\mathbf{Cat}(\mathcal{E})$, the functor $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f) : \mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A},\mathbb{X})\to\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A},\mathbb{Y})$ is an isofibration in **Cat**.

The proof of the following is direct from the representable definitions.

Lemma 6.2.14.

1. *An internal functor $f : \mathbb{X} \to \mathbb{Y}$ is an internal isofibration iff the map* $\mathbf{Iso}(\mathbb{X})_1 \to \mathbf{Iso}(\mathbb{Y})_1 \times_{Y_0} X_0$ *is a split epimorphism.*

2. *An internal functor $f : \mathbb{X} \to \mathbb{Y}$ is an internal equivalence of categories if and only if there exists an internal functor $g : \mathbb{Y} \to \mathbb{X}$ and internal natural isomorphisms $gf \cong 1_{\mathbb{X}}$ and $fg \cong 1_{\mathbb{Y}}$.*

Proposition 5.9 of [EKL05] defines the cofibrations of this model structure as internal functors which on objects have the left lifting property with respect to the split epimorphisms. In Theorem 6.2.17, we include an explicit description of the cofibrations.

We note that a constructive analogue to the $(\mathbf{Inj}, \mathbf{Surj})$ weak factorisation system on **Set** is achieved by replacing the surjective maps with the split epimorphisms and replacing the injective maps by the *complemented inclusions*, as is done in [GSS22] in the setting of simplicial sets. We can formulate this concept more generally in a lextensive category $\mathcal{E}$.

Definition 6.2.15. A morphism $f : A \to B$ in $\mathcal{E}$ is called a *complemented inclusion* if there exists an object C of $\mathcal{E}$ such that f is isomorphic to $\iota_A : A \hookrightarrow A + C$.

The following is noted in [Gam+22].

Lemma 6.2.16. *Let $\mathcal{E}$ be a lextensive category. Then complemented inclusion and split epimorphisms form a weak factorisation system.*

As such, the cofibrations of the model structure can be described as functors which are a complemented inclusion on objects. We arrive at an explicit description of the model structure that does not rely on representable notions.

Theorem 6.2.17. *Let $\mathcal{E}$ be a category that is finitely complete such that* $\mathbf{Cat}(\mathcal{E})$ *is finitely cocomplete. There is a model structure on* $\mathbf{Cat}(\mathcal{E})$ *which we denote* $(\textit{\textbf{Weq}}, \textit{\textbf{Cof}}, \textit{\textbf{Fib}})$ *in which*

- ***Weq*** *is the class of internal equivalences of categories.*

- ***Cof*** *is the class of functors which are complemented inclusion on objects.*

- ***Fib*** *is the class of internal isofibrations.*

The trivial cofibrations are complemented equivalences and the trivial fibrations are the split epimorphic equivalences.

We call this the natural model structure on $\mathbf{Cat}(\mathcal{E})$.

We denote the class of trivial fibrations by **TrivFib** and the class of trivial cofibrations by **TrivCof**.

Remark 6.2.18. We note that Theorem 3.5.2 shows that any lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint satisfies the conditions required for the theorem.

For $\mathcal{E} = \mathbf{Set}$ and assuming the Axiom of Choice, we recover the classical model structure on **Cat**. Hence, this gives us an internal version of this model structure, and without the Axiom of Choice we obtain a constructive version of the classical model structure on **Cat**.

6.3 Cofibrant Generation

The well-behaved enrichment described in Section 6.2.2 allows us to apply the enriched small object argument on the sets $\underline{\mathbf{I}}$ and $\underline{\mathbf{J}}$ of maps in $\mathbf{Cat}(\mathcal{E})$, in which $\mathbf{I}$ and $\mathbf{J}$ are the sets of maps in $\mathbf{Cat}$ defined in Equation (6.2).

Proposition 6.3.1. *Let $\mathcal{E}$ be a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint. Then we have weak factorisation systems $({}^{\boxslash}(\underline{\mathbf{I}}^{\boxslash}), \underline{\mathbf{I}}^{\boxslash})$ and $({}^{\boxslash}(\underline{\mathbf{J}}^{\boxslash}), \underline{\mathbf{J}}^{\boxslash})$ on $\mathbf{Cat}(\mathcal{E})$.*

Proof. $\mathbf{Cat}(\mathcal{E})$ is finitely complete and cocomplete by Proposition 4.2.2 and Theorem 3.5.2. By Proposition 6.2.12, $\mathbf{Cat}(\mathcal{E})$ is copowered and powered by $\mathcal{E}$. Therefore, by Corollary 7.6.4 of [Rie14], $\mathbf{Cat}(\mathcal{E})$ is finitely $\mathcal{E}$-bicomplete. Therefore $\mathbf{Cat}(\mathcal{E})$ satisfies the conditions for the enriched small object argument given in Proposition 13.4.2 of [Rie14]. Finally, $\underline{\mathbf{I}}$ and $\underline{\mathbf{J}}$ are both finite sets. $\square$

We show that the $\underline{\mathbf{I}}$ and $\underline{\mathbf{J}}$ generate the model structure of Theorem 6.2.17.

Remark 6.3.2. We also have internal descriptions of the maps in $\underline{\mathbf{I}}$ and $\underline{\mathbf{J}}$ which do not rely on their description as ordinary categories. Denote the terminal object of $\mathbf{Cat}(\mathcal{E})$ by $\underline{\mathbf{1}}$. We construct $\underline{\mathbf{2}}$ as the internal category with objects $\underline{\mathbf{1}} + \underline{\mathbf{1}}$, morphisms $\underline{\mathbf{1}} + \underline{\mathbf{1}} + \underline{\mathbf{1}}$, and in general composable n-morphisms by n coproducts of $\underline{\mathbf{1}}$. The source, target, identity and composition maps are induced by the universal property of the coproduct. Note that this is precisely what is described in Definition 4.4.4 as $\mathbf{2}_{\mathcal{E}}$. Using this, we construct $\underline{\mathcal{P}}$ and $\underline{\mathcal{I}}$ as pushouts as in Equation (6.1). This recovers the internal categories required.

First, we set up some notation. Let $f : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}(\mathcal{E})$. Consider the diagrams:

$$\begin{tikzcd}
X_1 \arrow[drr, bend left, "1_{X_1}"] \arrow[dr, dashed, "\Delta"] \arrow[ddr, bend right, "1_{X_1}"'] & & \\
& \mathbb{X}(\mathcal{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "{(d_1,d_0)}"] \\
& X_1 \arrow[r, "{(d_1,d_0)}"'] & X_0 \times X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X}(\mathcal{P}) \arrow[r] \arrow[d] \arrow[drr, dashed, "f(\mathcal{P})"] & X_1 \arrow[drr, "f_1"] & & \\
X_1 \arrow[drr, "f_1"'] & & \mathbb{Y}(\mathcal{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\
& & Y_1 \arrow[r, "{(d_1,d_0)}"'] & Y_0 \times Y_0
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "d_1 \times d_0"'] & & \\
& X_0 \times X_0 \times_{Y_0 \times Y_0} Y_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "d_1 \times d_0"] \\
& X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0.
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "q"] \arrow[ddr, bend right, "\Delta"'] & & \\
& \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r, "\pi_4"] \arrow[d, "\pi_3"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "\Delta"] \\
& \mathbb{X}(\mathcal{P}) \arrow[r, "f(\mathcal{P})"'] & \mathbb{Y}(\mathcal{P}).
\end{tikzcd}$$

Definition 6.3.3. Let $f : \mathbb{X} \to \mathbb{Y}$ be an internal functor. It is called *full* if the map w is a split epi and *faithful* if w is a monomorphism. It is called *fully faithful* if it is full and faithful.

Remark 6.3.4. Note that if f is fully faithful then w is a monomorphism and a split epimorphism and is therefore an isomorphism. Therefore, an internal fully faithful functor is one such that the following commutative square in $\mathcal{E}$ is a pullback:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

So therefore it agrees with Definition 2.2.9. The definitions of faithful and fully faithful are therefore representable ones, and so an internal functor $f : \mathbb{X} \to \mathbb{Y}$ is faithful (resp. fully faithful) iff for all $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ is a faithful (resp. fully faithful) functor. Split epimorphisms are also a representable notion, so that if $f : \mathbb{X} \to \mathbb{Y}$ is full it implies that for all $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ is full as the w associated to $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ is a split epimorphism in **Set** as it was in $\mathcal{E}$. Conversely, if for all $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ is full, then it follows that for each $\mathbb{A}$ the w associated to it is a surjection. Assuming the Axiom of Choice so that surjections are exactly the split epimorphisms, this is equivalent to $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ being full.

Lemma 6.3.5. *Let $f : \mathbb{X} \to \mathbb{Y}$. The map $q : X_1 \to \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1$ is a split epi if and only if f is faithful. In this case, $X_1 \cong \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1$.*

Proof. Let $\mathrm{split}(q)$ denote the splitting of q. We must show that w is a monomorphism. By representability of monomorphisms, split epimorphisms, and the evaluation map, it is enough to show that for any $E \in \mathcal{E}$, $\mathrm{Hom}_{\mathbf{Set}}(E, w)$ is a monomorphism.

Let $E \in \mathcal{E}$ and $\alpha, \beta : E \to X_1$ such that $w\alpha = w\beta$. We show that this implies that $\alpha = \beta$. By assumption, $w\alpha = w\beta$, so in particular $(d_1, d_0)\alpha = (d_1, d_0)\beta$. By Lemma 6.2.11, we therefore have an induced arrow $(\alpha, \beta) : E \to \mathbb{X}(\mathcal{P})$. Similarly, $w\alpha = w\beta$, implies that $f_1\alpha = f_1\beta$, so we have an induced arrow $\Delta f_1\alpha : E \to \mathbb{Y}(\mathcal{P})$ that factors through Y_1. It is clear from the universal property of the pullback that $f(\mathcal{P})(\alpha, \beta) = \Delta(f_1\alpha)$, and so we have an induced map $(f_1\alpha, (\alpha, \beta)) : E \to \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1$. The following diagram commutes, exhibiting that $(\alpha, \beta) = \Delta\mathrm{split}(q)(f_1\alpha, (\alpha, \beta))$, so in particular $\alpha = \mathrm{split}(q)(f_1\alpha, (\alpha, \beta)) = \beta$, as required.

$$\begin{tikzcd}
E \arrow[r, "{(f_1\alpha,(\alpha,\beta))}"] \arrow[d, "{(\alpha,\beta)}"] & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r, "\mathrm{split}(q)"] \arrow[d, "\pi_3"] & X_1 \arrow[r, "q"] \arrow[d, "\Delta"] & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r, "\pi_4"] \arrow[d, "\pi_3"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "\Delta"] \\
\mathbb{X}(\mathcal{P}) \arrow[r, equal] & \mathbb{X}(\mathcal{P}) \arrow[r, equal] & \mathbb{X}(\mathcal{P}) \arrow[r, equal] & \mathbb{X}(\mathcal{P}) \arrow[r, "f(\mathcal{P})"'] & \mathbb{Y}(\mathcal{P})
\end{tikzcd}$$

Conversely, suppose f is faithful. We claim that the following diagram commutes providing the desired splitting for q.

$$\begin{tikzcd}
\mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r] \arrow[drr, equal] & \mathbb{X}(\mathcal{P}) \arrow[r, "\pi_1"] & X_1 \arrow[d, "q"] \\
 & & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1
\end{tikzcd}$$

We use the universal property of $\mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1$ as a pullback and show that these arrows agree on the projections to $\mathbb{X}(\mathcal{P})$ and Y_1. The commutativity of the following diagram witnesses that the arrows agree on the projections to Y_1:

$$\begin{tikzcd}
\mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r] \arrow[dd] & \mathbb{X}(\mathcal{P}) \arrow[r, "\pi_1"] \arrow[d, "f(\mathcal{P})"'] & X_1 \arrow[r, "q"] \arrow[ddr, "f_1"] & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[dd] \\
 & \mathbb{Y}(\mathcal{P}) \arrow[drr] & & \\
Y_1 \arrow[ur, "\Delta"] \arrow[rrr, equal] & & & Y_1
\end{tikzcd}$$

To check that they agree on the projection to $\mathbb{X}(\mathcal{P})$, we use the universal property of $\mathbb{X}(\mathcal{P})$ as a pullback, and show that they agree on the projections $\pi_1 : \mathbb{X}(\mathcal{P}) \to X_1$ and $\pi_2 : \mathbb{X}(\mathcal{P}) \to X_1$. The commutativity of the following diagram shows that the arrows agree on π_1 :

$$\begin{tikzcd}
\mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r] \arrow[d] & \mathbb{X}(\mathcal{P}) \arrow[r, "\pi_1"] & X_1 \arrow[r, "q"] \arrow[ddr, equal] & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[d] \\
\mathbb{X}(\mathcal{P}) \arrow[d, "\pi_1"'] & & & \mathbb{X}(\mathcal{P}) \arrow[d, "\pi_1"] \\
X_1 \arrow[rrr, equal] & & & X_1
\end{tikzcd}$$

By definition of faithfulness, $w : X_1 \to X_0 \times X_0 \times_{Y_0 \times Y_0} Y_1$ is a monomorphism; to show that the arrows agree on π_2, we show that they agree on $w_1 \pi_2 : \mathbb{X}(\mathcal{P}) \to X_0 \times X_0 \times_{Y_0 \times Y_0} Y_1$ and the result will follow. To show that they agree on this, we use again the universal property of $X_0 \times X_0 \times_{Y_0 \times Y_0} Y_1$ as a pullback and show that they agree on the projections to $X_0 \times X_0$ and Y_1. Hence, q is a split epimorphism. It is clearly also a split monomorphism since the following diagram commutes.

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[drr, equal] & \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1 \arrow[r] & \mathbb{X}(\mathcal{P}) \arrow[d, "\pi_1"] \\
 & & X_1
\end{tikzcd}$$

It follows that q is an isomorphism and $X_1 \cong \mathbb{X}(\mathcal{P}) \times_{\mathbb{Y}(\mathcal{P})} Y_1$. □

Proposition 6.3.6. *Let $f : \mathbb{X} \to \mathbb{Y}$ be an internal functor in* **Cat**$(\mathcal{E})$*. The following are equivalent:*

1. The pullback evaluation with all maps in $\underline{\mathbf{I}}$ *are split epimorphisms.*

2. $f_0 : X_0 \to Y_0$ *is a split epimorphism on objects and the canonical maps*

$$w : X_1 \to X_0 \times X_0 \times_{Y_0 \times Y_0} Y_1$$
$$q : X_1 \to X(\mathcal{P}) \times_{Y(\mathcal{P})} Y_1$$

are split epimorphisms.

3. f *is a split epimorphism on objects and internally fully faithful.*

4. f *is a split epimorphism on objects and an adjoint equivalence of categories.*

5. f *is a trivial fibration.*

Proof. By Remark 6.2.9 (1) $\iff$ (2) follows by definition of pullback evaluation and the evaluation map. The equality of (3) and (4) is given in Remark 4.7.3. The equality of (4) and (5) is definitional. By Lemma 6.3.5, the equality of (2) and (3) follows easily from the definitions of fully faithfulness.

□

Proposition 6.3.7. *The weak factorisation system* (**Cof**, **TrivFib**) *is* $\mathcal{E}$*-cofibrantly generated by* $\underline{\mathbf{I}}$.

Proof. By Proposition 6.3.1, $\underline{\mathbf{I}}$ generates an $\mathcal{E}$-enriched weak factorisation system. By Proposition 6.3.6, the right class of the underlying weak factorisation system coincides with the right class of (**Cof**, **TrivFib**), and hence these are the same weak factorisation system. □

We have a similar characterisation for the maps in $(\underline{\mathbf{J}})^{\boxslash}$.

Proposition 6.3.8. *Let* $f : \mathbb{X} \to \mathbb{Y}$ *be an internal functor in* $\mathbf{Cat}(\mathcal{E})$*. The following are equivalent:*

1. f *is an internal isofibration.*

2. $\mathbb{X}(\mathcal{I}) \to X_0 \times_{Y_0} \mathbb{Y}(\mathcal{I})$ *is a split epimorphism.*

3. *The pullback evaluation with* $\mathbf{1} \to \mathcal{I}$ *is a split epimorphism.*

4. $f \in (\underline{\mathbf{J}})^{\boxslash}$

5. *For every* $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $\mathrm{Hom}_{\mathbf{Cat}}(\mathbb{A}, f)$ *is an isofibration in* **Cat**.

Proof. (1) and (2) are the same by the definition given in [EKL05]; (2) and (3) are equivalent by Lemma 6.2.7 and definition of the pullback evaluation; (3) and (4) are equivalent by the definition of enriched lifting. (1) and (5) are equivalent as noted in 3.5 of [Lac07]. □

Remark 6.3.9. By Lemma 6.2.10, characterisation (2) of Proposition 6.3.8 means that if we restrict to $\mathbf{Gpd}(\mathcal{E})$, then this definition of internal isofibration recovers the definition studied in [NP19].

Proposition 6.3.10. *The weak factorisation system* $(\mathbf{TrivCof}, \mathbf{Fib})$ *is* $\mathcal{E}$*-cofibrantly generated by* $\underline{\mathbf{J}}$.

Proof. We apply the same proof as in Proposition 6.3.7. $\square$

Proposition 6.3.11. *The classes* **Cof** *and* **TrivCof** *are closed under cartesian products.*

Proof. An internal functor $f : \mathbb{X} \to \mathbb{Y}$ is a cofibration precisely when it is a complemented inclusion on objects, so $f_0 : X_0 \to Y_0$ is isomorphic to an inclusion map $X_0 \hookrightarrow X_0 + C$. Hence, for any $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $(\mathbb{A} \times f)_0 = 1_{A_0} \times f_0$ is isomorphic to a map $A_0 \times X_0 \to A_0 \times (X_0 + C)$. By lextensivity, $A_0 \times (X_0 + C) \cong (A_0 \times X_0) + (A_0 \times C)$, and so $\mathbb{A} \times f$ is a cofibration.

An internal functor $f : \mathbb{X} \to \mathbb{Y}$ is a trivial cofibration precisely if it is a split monomorphism which an internal adjoint equivalence and a complemented inclusion on objects. Since complemented inclusion on objects functors are closed under cartesian products, it is enough to show that split monomorphisms and internal equivalences are. Since these and the cartesian product are representable, it is enough to show that this holds in **Cat**, but this follows from $(\mathbf{Cat}, \times, \mathbf{1})$ being a monoidal model category. $\square$

Proposition 6.3.12. *The natural model structure on* $\mathbf{Cat}(\mathcal{E})$ *is cofibrantly generated by the sets* $\underline{\mathbf{I}}$ *and* $\underline{\mathbf{J}}$.

Proof. By Proposition 6.3.7 and Proposition 6.3.10, the weak factorisation systems $(\mathbf{Cof}, \mathbf{TrivFib})$ and $(\mathbf{TrivCof}, \mathbf{Fib})$ are $\mathcal{E}$-cofibrantly generated. By Lemma 6.3.11, the left classes of these are closed under cartesian product; in particular, they are closed under tensoring with objects in $\mathcal{E}$. By Proposition 13.4.2 of [Rie14], this happens if and only if the weak factorisation systems are cofibrantly generated in the ordinary sense by $\underline{\mathbf{I}}$ and $\underline{\mathbf{J}}$. $\square$

All cofibrantly generated model structures form algebraic model structures, as defined in [Rie11]. Therefore, we have the following:

Corollary 6.3.13. *There is an algebraic model structure on* $\mathbf{Cat}(\mathcal{E})$ *with underlying model structure the natural model structure on* $\mathbf{Cat}(\mathcal{E})$.

We show that this model structure interacts nicely with the cartesian product.

Proposition 6.3.14. *The natural model structure on* $(\mathbf{Cat}(\mathcal{E}), \times, \underline{\mathbf{1}})$ *is a symmetric monoidal model structure.*

Proof. Since every object is cofibrant and the monoidal product is symmetric, the condition for a monoidal model category simplifies to requiring that for every $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, $\mathbb{A} \times -$ preserves cofibrations and trivial cofibrations. This is proven in Lemma 6.3.11. $\square$

We arrive at the main result of this section.

Theorem 6.3.15. *Let* $\mathcal{E}$ *be a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor* $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ *has left adjoint. The natural model structure on* $\mathbf{Cat}(\mathcal{E})$ *has the following properties:*

1. *It is symmetric monoidal with respect to the cartesian product.*

2. *It is cofibrantly generated by the sets*

$$\underline{\mathbf{I}} := \{\underline{\emptyset} \to \underline{\mathbf{1}}, \underline{\mathbf{1}} + \underline{\mathbf{1}} \to \underline{\mathbf{2}}, \underline{\mathcal{P}} \to \underline{\mathbf{2}}\}$$
$$\underline{\mathbf{J}} := \{\underline{\mathbf{1}} \to \underline{\mathcal{I}}\}.$$

3. *There is an algebraic model structure on* **Cat**$(\mathcal{E})$ *together with the equivalences of categories whose underlying model structure is the natural model structure.*

6.3.1 A model structure on internal modules

The nice properties of being a cofibrantly generated and symmetric monoidal model structure allow the development of a lot of nice theory, as explored in [Hov98]. One such result allows us to lift the model structure on **Cat**$(\mathcal{E})$ onto the category of internal modules for an internal monoidal category in **Cat**$(\mathcal{E})$.

Recall that a monoid in **Cat**$(\mathcal{E})$ is an internal monoidal category. Let $(\mathbb{M}, \otimes, u)$ be an internal monoidal category. An $\mathbb{M}$-*module* is an $\mathbb{X} \in$ **Cat**$(\mathcal{E})$ together with an internal functor $\mu : \mathbb{M} \times \mathbb{X} \to \mathbb{X}$ such that the following diagrams commute.

$$\begin{tikzcd} \mathbb{M} \times \mathbb{M} \times \mathbb{X} \arrow[r, "1_{\mathbb{M}} \times \mu"] \arrow[d, "\otimes \times 1_{\mathbb{X}}"'] & \mathbb{M} \times \mathbb{X} \arrow[d, "\mu"] \\ \mathbb{M} \times \mathbb{X} \arrow[r, "\mu"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbf{1}} \arrow[r, "u \times x"] \arrow[dr, "\forall x"'] & \mathbb{M} \times \mathbb{X} \arrow[d, "\mu"] \\ & \mathbb{X} \end{tikzcd}$$

A morphism of left $\mathbb{M}$-modules $f : (\mathbb{X}, \mu_{\mathbb{X}}) \to (\mathbb{Y}, \mu_{\mathbb{Y}})$ is an internal functor $f : \mathbb{X} \to \mathbb{Y}$ such that $\mu_{\mathbb{Y}}(1 \times f) = f\mu_{\mathbb{X}}$. This defines a category $\mathbb{M}$-**Mod**. Suppose that $\mathbb{M}$ is furthermore a symmetric monoidal category. Then we can define $\mathbb{M}$-**Algebra** to be the category of monoids internal to $\mathbb{M}$-**Mod**.

Theorem 6.3.16. *Let $\mathcal{E}$ be a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U}$:* **Cat**$(\mathcal{E}) \to$ **Gph**$(\mathcal{E})$ *has left adjoint. Let $\mathbb{M}$ be a monoidal internal category in* **Cat**$(\mathcal{E})$*. Then there is a cofibrantly generated monoidal model structure on the category $\mathbb{M}$-***Mod** *in which a map is a weak equivalence or fibration if and only if it is a equivalence of internal categories or an internal isofibration. Furthermore, a cofibration of $\mathbb{M}$-modules is a functor which is a complemented inclusion on objects.*

*Additionally, if $\mathbb{M}$ is an internal symmetric monoidal category, the category $\mathbb{M}$-***Algebra** *is a model category.*

Proof. The first result follows by an application of Proposition 2.8 of [Hov98]. Since every object is cofibrant in $\mathbb{M}$-**Mod**, we can apply Theorem 3.3 from [Hov98] to get an actual model structure on $\mathbb{M}$-**Algebra**. $\square$

6.4 Algebraic Aspects

By Theorem 6.3.15 (3), we have an algebraic model structure on **Cat**$(\mathcal{E})$. In this section, we provide an explicit description of the two algebraic weak factorisation systems of this, leveraging Theorem 6.3.15 (3) to simplify calculations. First, we recall a well-known result from monad theory.

Definition 6.4.1. Let $(T : \mathbb{C} \to \mathbb{C}, \eta : 1_{\mathbb{C}} \Rightarrow T)$ be a pointed endofunctor and suppose there exists a left adjoint F to the forgetful functor $G : (T,\eta)$-**Alg** $\to \mathbb{C}$. We call GF the *algebraically free monad on* (T,η).

The endofunctor GF does indeed form a monad [BW85, Proposition 9.4.1]. For us, the most important property of the algebraically free monad of a pointed endofunctor will be that it has the same algebras as the pointed endofunctor. A proof of this result is given in [BW85, Proposition 9.4.5].

Proposition 6.4.2. *Let (T,η) be a pointed endofunctor on a category $\mathbb{C}$, and consider a functor $F : \mathbb{C} \to (T,\eta)$-**Alg** given on objects by $X \mapsto (TX, \mu_X : T^2X \to TX)$. The following are equivalent:*

1. *F is left adjoint to the forgetful functor (T,η)-**Alg** $\to \mathbb{C}$.*

2. *$\mathbb{T} := (T,\eta,\mu)$ is the algebraically free monad over (T,η).*

*In this case, Then $\mathbb{T}$-**Alg** $\cong (T,\eta)$-**Alg***

6.4.1 Factorisations

We now fix the setting of $\mathbb{C} = \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$. In this subsection, we describe the factorisations of [EKL05, §7] which utilise Ken Brown's factorisation lemma [Bro73]. Let $f : \mathbb{X} \to \mathbb{Y}$ be an internal functor. We construct the following factorisation

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "f"] \arrow[dr, "\mathbf{C}f"'] & & \mathbb{Y} \\ & \mathbb{E}(f) \arrow[ur, "\mathbf{TF}(f)"'] & \end{tikzcd}$$

in which $\mathbf{C}(f)$ is a cofibration and $\mathbf{TF}(f)$ is a trivial fibration. We begin on objects:

$$\begin{tikzcd} X_0 \arrow[rr, "f_0"] \arrow[dr, hook, "\mathbf{C}f_0 := \iota_{X_0}"'] & & Y_0 \\ & X_0 + Y_0 \arrow[ur, "\mathbf{TF}(f)_0 := (f_0, 1_{Y_0})"'] & \end{tikzcd}$$

Write $E(f)_0 := X_0 + Y_0$. On morphisms:

$$\begin{tikzcd} X_1 \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, dashed, "\mathbf{C}(f)_1"] \arrow[drr, bend left, "f_1"] & & \\ X_0 \times X_0 \arrow[dr, "\mathbf{C}(f)_0 \times \mathbf{C}(f)_0"'] & E(f)_1 \arrow[r, "\mathbf{TF}(f)_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\ & E(f)_0 \times E(f)_0 \arrow[r, "\mathbf{TF}(f)_0 \times \mathbf{TF}(f)_0"'] & Y_0 \times Y_0 \end{tikzcd}$$

With this, define the following:

$$\begin{tikzcd}
E(f)_0 \arrow[r, "\mathbf{TF}(f)_0"] \arrow[dr, dashed, "i"] \arrow[ddr, bend right, "(\mathbf{1}_{E(f)_0},\mathbf{1}_{E(f)_0})"'] & Y_0 \arrow[dr, "i"] & \\
& E(f)_1 \arrow[r, "\mathbf{TF}(f)_1"] \arrow[d, "(d_1,d_0)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "(d_1,d_0)"] \\
& E(f)_0 \times E(f)_0 \arrow[r, "\mathbf{TF}(f)_0\times\mathbf{TF}(f)_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

$$\begin{tikzcd}
E(f)_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(f)_1 \arrow[d, "d_0"] \\
E(f)_1 \arrow[r, "d_1"'] & E(f)_0
\end{tikzcd}$$

$$\begin{tikzcd}
E(f)_2 \arrow[rr] \arrow[d] \arrow[drr, dashed, "m"] & & E(f)_0 \arrow[dr, "\mathbf{TF}(f)_0"] & \\
E(f)_0 \arrow[drr, "(\mathbf{1}_{E(f)_0},\mathbf{1}_{E(f)_0})"'] & & E(f)_1 \arrow[r, "\mathbf{TF}(f)_1"] \arrow[d, "(d_1,d_0)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "(d_1,d_0)"] \\
& & E(f)_0 \times E(f)_0 \arrow[r, "\mathbf{TF}(f)_0\times\mathbf{TF}(f)_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

With this data, $\mathbb{E}(f) := (E(f)_0, E(f)_1, d_0, d_1, i, m)$ forms an internal category and $\mathbf{C}(f) := (\mathbf{C}(f)_0, \mathbf{C}(f)_1)$ forms an internal functor which is a complemented inclusion on objects by construction and $\mathbf{TF}(f) := (\mathbf{TF}(f)_0, \mathbf{TF}(f)_1)$ forms an internal functor which is split epi on objects and fully faithful by construction. This is the factorisation given for the $(\mathbf{Cof}, \mathbf{TrivFib})$ weak factorisation system.

The $(\mathbf{TrivCof}, \mathbf{Fib})$ factorisation requires the first factorisation system. For an internal functor $f : \mathbb{X} \to \mathbb{Y}$, we first form the following induced map:

$$\begin{tikzcd}
\mathbb{X} \arrow[r] \arrow[dr, dashed, "\mathbf{W}(f)"] \arrow[ddr, bend right, "(\mathbf{1},f)"'] & \mathbb{X}^{\mathcal{I}} \arrow[dr, "f^{\mathcal{I}}"] & \\
& \mathbf{Map}(f) \arrow[r] \arrow[d, "p_{\mathbb{X}\times\mathbb{Y}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Y}^{\mathcal{I}} \arrow[d] \\
& \mathbb{X}\times\mathbb{Y} \arrow[r, "f\times 1"'] & \mathbb{Y}\times\mathbb{Y}.
\end{tikzcd}$$

Then factorise $\mathbf{W}(f)$ using the $(\mathbf{Cof}, \mathbf{TrivFib})$ weak factorisation system and define $\mathbf{TC}(f) := \mathbf{C}(\mathbf{W}(f)) : \mathbb{X} \to \mathbb{E}(\mathbf{W}(f))$ and $\mathbf{F}(f)$ to be the composite

$$\mathbb{E}(\mathbf{W}(f)) \xrightarrow{\mathbf{TF}(\mathbf{W}(f))} \mathbf{Map}(f) \xrightarrow{p_{\mathbb{X}\times\mathbb{Y}}} \mathbb{X}\times\mathbb{Y} \xrightarrow{p_{\mathbb{Y}}} \mathbb{Y}.$$

Since projections are fibrations and trivial fibrations are fibrations, and fibrations are closed under composition, it follows that $\mathbf{F}(f)$ is a fibration. Proposition 5.10 of [EKL05] shows that $\mathbf{W}(f)$ is an equivalence of categories, in particular satisfying 2-out-of-3, and so since $\mathbf{TF}(\mathbf{W}(f))$ is a weak equivalence, it follows that $\mathbf{TC}(f)$ is too; it is also by construction a complemented inclusion on objects.

Remark 6.4.3. We note that by extensivity we have:

$$
\begin{aligned}
E(f)_1 &:= (E(f)_0 \times E(f)_0) \times_{Y_0 \times Y_0} Y_1 \\
&= ((X_0 + Y_0) \times (X_0 + Y_0)) \times_{Y_0 \times Y_0} Y_1 \\
&\cong ((X_0 \times X_0) + (X_0 \times Y_0) + (Y_0 \times X_0) + (Y_0 \times Y_0)) \times_{Y_0 \times Y_0} Y_1 \\
&\cong (X_0 \times X_0) \times_{Y_0 \times Y_0} Y_1 + X_0 \times_{Y_0} Y_1 + X_0 \times_{Y_0} Y_1 + Y_1.
\end{aligned}
$$

and also $\mathbf{Map}(f) \cong \mathbb{X} \times_{\mathbb{Y}} \mathbb{Y}^{\mathcal{I}}$.

6.4.2 Functorial factorisations

We upgrade these factorisations into functorial factorisations in the sense of [GT06]. Consider a commutative square $(u, v) : f \to g$. We supply a functor $\mathbb{E}(u, v) : \mathbb{E}(f) \to \mathbb{E}(g)$ such that we have morphisms in $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ given by the commutative squares $(u, \mathbb{E}(u, v)) : \mathbf{C}(f) \to \mathbf{C}(g)$ and $(\mathbb{E}(u, v), v) : \mathbf{TF}(f) \to \mathbf{TF}(g)$. On objects, the map $E(u, v)_0 : E(f)_0 \to E(g)_0$ is given by

$$
\begin{tikzcd}
 & A_0 + B_0 & \\
A_0 \arrow[ur, hook, bend left, "\mathbf{C}(f)_0"] & & B_0 \arrow[ul, hook', bend right] \\
X_0 \arrow[u, "u_0"] \arrow[r, hook, "\mathbf{C}(f)_0"'] & X_0 + Y_0 \arrow[uu, dashed, "E(u,v)_0"] & Y_0 \arrow[l, hook'] \arrow[u, "v_0"']
\end{tikzcd}
$$

Note that by construction this clearly provides a commutative square $(u_0, E(u, v)_0) : \mathbf{C}(f)_0 \to \mathbf{C}(g)_0$ and a map $(E(u, v)_0, v_0) : \mathbf{TF}(f)_0 \to \mathbf{TF}(g)_0$ by the universal property of the coproduct.

On morphisms, $E(u, v)_1 : E(f)_1 \to E(g)_1$ is induced by the universal property of $E(g)_1$ as a pullback:

$$
\begin{tikzcd}
E(f)_1 \arrow[rr, "\mathbf{TF}(f)_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[drr, dashed, "E(u,v)_1"] & & Y_1 \arrow[drr, "v_1"] & & \\
E(f)_0 \times E(f)_0 \arrow[drr, "E(u,v)_0 \times E(u,v)_0"'] & & E(g)_1 \arrow[rr, "\mathbf{TF}(g)_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & B_1 \arrow[d, "{(d_1,d_0)}"] \\
 & & E(g)_0 \times E(g)_0 \arrow[rr, "\mathbf{TF}(g)_0 \times \mathbf{TF}(g)_0"'] & & B_0 \times B_0.
\end{tikzcd}
$$

Again, by construction this clearly gives a commutative square $(E(u, v)_1, v_1) : \mathbf{TF}(f)_1 \to \mathbf{TF}(g)_1$, and by the universal property of the pullback, it is easy to show that it also gives a commutative square $(u_1, E(u, v)_1) : \mathbf{C}(f)_1 \to \mathbf{C}(g)_1$. This assembles into a functor as required; it preserves identities by construction and preservation of composition can be proven using the universal properties involved.

Hence we have an endofunctor $\mathbf{C} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ with copoint $\epsilon : \mathbf{C} \Rightarrow 1$ given on $f : \mathbb{X} \to \mathbb{Y}$ by

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "\mathbf{C}(f)"'] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{E}(f) \arrow[r, "\mathbf{TF}(f)"'] & \mathbb{Y}. \end{tikzcd}$$

Furthermore, we have an endofunctor $\mathbf{TF} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ with point $\eta : 1 \Rightarrow \mathbf{TF}$ given on $f : \mathbb{X} \to \mathbb{Y}$ by

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathbf{C}(f)"] \arrow[d, "f"'] & \mathbb{E}(f) \arrow[d, "\mathbf{TF}(f)"] \\ \mathbb{Y} \arrow[r, equal] & \mathbb{Y}. \end{tikzcd}$$

Proposition 6.4.4. *The (co)pointed endofunctors* $(\mathbf{C}, \epsilon)$ *and* $(\mathbf{TF}, \eta)$ *provide a functorial factorisation for the weak factorisation system* $(\mathbf{Cof}, \mathbf{TrivFib})$.

We do the same for the $(\mathbf{TrivCof}, \mathbf{Fib})$ weak factorisation system. It suffices to show that the assignment $f \mapsto \mathbf{W}(f)$ is functorial as then we can define $\mathbf{C} := \mathbf{TC} \circ \mathbf{W}$ and $\mathbf{F}$ similarly as the composition of functors. Consider the commutative square $(u, v) : f \to g$. We supply a functor $\mathbf{Map}(u, v) : \mathbf{Map}(f) \to \mathbf{Map}(g)$ induced by the universal property of the mapping path space:

$$\begin{tikzcd} \mathbf{Map}(f) \arrow[rr] \arrow[d] \arrow[drr, dashed, "\mathbf{Map}(u,v)"] & & \mathbb{Y}^{\mathcal{I}} \arrow[drr, "v^{\mathcal{I}}"] & & \\ \mathbb{X} \times \mathbb{Y} \arrow[drr, "u \times v"'] & & \mathbf{Map}(g) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbb{B}^{\mathcal{I}} \arrow[d] \\ & & \mathbb{A} \times \mathbb{B} \arrow[rr, "g \times 1"'] & & \mathbb{B} \times \mathbb{B}. \end{tikzcd}$$

This provides a commutative square $(u, \mathbf{Map}(u, v)) : \mathbf{W}(f) \to \mathbf{W}(g)$. Respect for identities follows easily from the definition and respect for composition follows from the uniqueness given by the universal property of the pullback. Hence, this assignment is functorial, and we obtain an endofunctor $\mathbf{TC} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ with copoint $\sigma : \mathbf{TC} \Rightarrow 1$ given on $f : \mathbb{X} \to \mathbb{Y}$ by:

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "\mathbf{TC}(f)"'] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{E}(\mathbf{W}(f)) \arrow[r, "\mathbf{F}(f)"'] & \mathbb{Y}. \end{tikzcd}$$

Similarly, we have an endofunctor $\mathbf{F} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ with point $\xi : 1 \Rightarrow \mathbf{F}$ given on $f : \mathbb{X} \to \mathbb{Y}$ by:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathbf{TC}(f)"] \arrow[d, "f"'] & \mathbb{E}(\mathbf{W}(f)) \arrow[d, "\mathbf{F}(f)"] \\ \mathbb{Y} \arrow[r, equal] & \mathbb{Y}. \end{tikzcd}$$

This proves that the weak factorisation system $(\mathbf{TC}, \mathbf{F})$ is functorial. Note that the factorisation of a commutative square $(u, v) : f \to g$ is given explicitly as:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "u"] \arrow[d, "\mathbf{TC}(f)"'] & \mathbb{A} \arrow[d, "\mathbf{TC}(g)"] \\
\mathbb{E}(\mathbf{W}(g)) \arrow[r, "\mathbb{E}(u,\mathbf{Map}(u,v))"] \arrow[d, "\mathbf{F}(f)"'] & \mathbb{E}(\mathbf{W}(f)) \arrow[d, "\mathbf{F}(g)"] \\
\mathbb{Y} \arrow[r, "v"] & \mathbb{B}
\end{tikzcd}$$

Proposition 6.4.5. *The (co)pointed endofunctors* $(\mathbf{TC}, \sigma)$ *and* $(\mathbf{F}, \xi)$ *provide a functorial factorisation for the weak factorisation system* $(\mathbf{TrivCof}, \mathbf{Fib})$.

6.4.3 Algebraic structure

We unpack the definition of an algebra for a pointed endofunctor to notice a correspondence between algebra structure and fibrational structure. Similarly, there is a correspondence between coalgebras for a copointed endofunctor and cofibrational structure. Let $(\mathcal{L}, \mathcal{R})$ be a functorial weak factorisation system on $\mathbf{Cat}(\mathcal{E})$ with associated (co)pointed endofunctors $L, R : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ with point $\eta : 1 \Rightarrow R$ given by:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "L(f)"] \arrow[d, "f"'] & \bullet \arrow[d, "R(f)"] \\
\mathbb{Y} \arrow[r, equal] & \mathbb{Y}
\end{tikzcd}$$

and copoint $\epsilon : L \Rightarrow 1$ given by:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "L(f)"'] & \mathbb{X} \arrow[d, "f"] \\
\bullet \arrow[r, "R(f)"'] & \mathbb{Y}
\end{tikzcd}$$

An algebra for (R, η) is therefore a pair $(f, \overrightarrow{\phi} : R(f) \to f)$ satisfying the following commutativity condition:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "L(f)"'] \arrow[rr, equal, bend left] \arrow[d, "f"'] & \bullet \arrow[r, "\phi"'] \arrow[d, "R(f)"] & \mathbb{X} \arrow[d, "f"] \\
\mathbb{Y} \arrow[r, equal] & \mathbb{Y} \arrow[r, equal] & \mathbb{Y}
\end{tikzcd}$$

This data can be rearranged into the following commutative diagram:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "L(f)"'] & \mathbb{X} \arrow[d, "f"] \\
\bullet \arrow[r, "R(f)"'] \arrow[ur, "\phi"] & \mathbb{Y}.
\end{tikzcd}$$

In other words, ϕ provides a lifting of f against its left factor. Conversely, if f lifts against its left factor, then this provides an (R, η)-algebra structure.

The dual of this is also true: f lifts against its right factor if and only if there exists an $\mathcal{L}$-coalgebra structure upon it. It is a theorem due to Garner that for a functorial weak factorisation system $(\mathcal{L}, \mathcal{R})$, (R, η)-algebras compose with (R, η)-algebras coherently if and only if (L, ϵ) extends to a comonad; dually, (L, ϵ)-coalgebras compose coherently with (L, ϵ)-coalgebras if and only if (R, η) extends to a monad [Rie11, p. 2.24].

In our setting, we have endofunctors $\mathbf{F}, \mathbf{TF}, \mathbf{C}, \mathbf{TC}$ defined in Section 6.4.1. By the above, these are all (co)pointed by the natural transformations defined above $\xi : 1 \Rightarrow \mathbf{F}$, $\eta : 1 \Rightarrow \mathbf{TF}$, $\epsilon : \mathbf{C} \Rightarrow 1$ and $\sigma : \mathbf{TC} \Rightarrow 1$ respectively. Therefore, we have the following.

Proposition 6.4.6. *Let $f : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}(\mathcal{E})$.*

1. *There exists an $(\mathbf{F}, \xi)$-algebra structure for f if and only if f is a fibration.*

2. *There exists an $(\mathbf{TF}, \eta)$-algebra structure for f if and only if f is a trivial fibration.*

3. *There exists an $(\mathbf{C}, \epsilon)$-algebra structure for f if and only if f is a cofibration.*

4. *There exists an $(\mathbf{TC}, \sigma)$-algebra structure for f if and only if f is a trivial cofibration.*

We claim that the (co)pointed endofunctors $(\mathbf{F}, \xi)$ and $(\mathbf{TF}, \eta)$ (resp. $(\mathbf{C}, \epsilon)$ and $(\mathbf{TC}, \sigma)$) extend to (co)monads; to simplify calculations, we appeal to Corollary 6.3.13, which tells us that an algebraic model structure exists, and therefore algebraic weak factorisation systems with these underlying maps exist.

For $f : \mathbb{X} \to \mathbb{Y}$, we construct an internal functor $\mu^f : \mathbb{E}(\mathbf{TF}(f)) \to \mathbb{E}(f)$. On objects, it is given by:

$$\begin{tikzcd}
 & X_0 + Y_0 & \\
X_0 + Y_0 \arrow[ur, equal, bend left] \arrow[r, hook] & X_0 + Y_0 + Y_0 \arrow[u, dashed, "\mu_0^f"] & Y_0 \arrow[l, hook'] \arrow[ul, bend right]
\end{tikzcd}$$

Note that this has the property that $\mathbf{TF}(f)_0 \mu_0^f = \mathbf{TF}^2(f)_0$, due to the definition of $\mathbf{TF}^2(f)_0$ as the unique arrow which makes the outside of the following diagram commute:

$$\begin{tikzcd}
 & Y_0 & \\
 & X_0 + Y_0 \arrow[u, "\mathbf{TF}(f)_0"] & \\
X_0 + Y_0 \arrow[uur, bend left, "\mathbf{TF}(f)_0"] \arrow[ur, equal, bend left] \arrow[r, hook] & X_0 + Y_0 + Y_0 \arrow[u, dashed, "\mu_0^f"] & Y_0. \arrow[l, hook'] \arrow[ul, bend right] \arrow[uul, equal, bend right]
\end{tikzcd} \tag{6.4}$$

On morphisms it is given by:

$$\begin{tikzcd}
\mathbb{E}(\mathbf{TF}(f))_1 \arrow[d, "{(d_1,d_0)}"] \arrow[dr, dashed, "\mu_1^f"] \arrow[drr, bend left, "\mathbf{TF}^2(f)_1"] & & \\
\mathbb{E}(\mathbf{TF}(f))_0 \times \mathbb{E}(\mathbf{TF}(f))_0 \arrow[dr, "\mu_0^f \times \mu_0^f"'] & \mathbb{E}(f)_1 \arrow[r, "\mathbf{TF}(f)_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\
 & \mathbb{E}(f)_0 \times \mathbb{E}(f)_0 \arrow[r, "\mathbf{TF}(f)_0 \times \mathbf{TF}(f)_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

for which the outer diagram commutes by functoriality of **TF** and Equation (6.4).

The map $\mu^f := (\mu_0^f, \mu_1^f)$ assembles into an internal functor and hence provides a commutative square $\overrightarrow{\mu_f} : \mathbf{TF}^2(f) \to \mathbf{TF}(f)$ given by

$$\begin{tikzcd}
\mathbb{E}(\mathbf{TF}(f)) \arrow[r, "\mu^f"] \arrow[d, "\mathbf{TF}^2(f)"'] & \mathbb{E}(f) \arrow[d, "\mathbf{TF}(f)"] \\
\mathbb{Y} \arrow[r, equal] & \mathbb{Y}.
\end{tikzcd}$$

Consequently, this forms a natural transformation $\overrightarrow{\mu} : \mathbf{TF}^2 \Rightarrow \mathbf{TF}$.

The following shows that we can give algebraic structure to $\mathbf{TF}(f)$, which exhibits it as a trivial fibration. Through the algebraic lens, this is the same as giving the free $(\mathbf{TF}, \eta)$-algebra structure to f.

Lemma 6.4.7. *For any $f : \mathbb{X} \to \mathbb{Y}$, $(\mathbf{TF}(f), \overrightarrow{\mu_f})$ is a $(\mathbf{TF}, \eta)$-algebra.*

Proof. By construction, it is clear that $\mathbf{TF}(f)\mu^f = \mathbf{TF}^2(f)$. It remains to prove that $\mu^f \mathbf{C}(f) = \mathbf{1}_{\mathbb{E}(f)}$. By construction, this is true on objects. On morphisms, we appeal to the universal property of $\mathbb{E}(f)_1$ as a pullback. That is, we show that these both agree on $\mathbf{TF}(f)_1$ and (d_1, d_0). This is witnessed by the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathbb{E}(f)_1 \arrow[r, "\mathbf{CTF}(f)_1"] \arrow[d, equal] \arrow[drr, "\mathbf{TF}(f)_1"] & \mathbb{E}(\mathbf{TF}(f))_1 \arrow[r, "\mu_1^f"] \arrow[dr, "\mathbf{TF}^2(f)_1"] & \mathbb{E}(f)_1 \arrow[d, "\mathbf{TF}(f)_1"] \\
\mathbb{E}(f)_1 \arrow[rr, "\mathbf{TF}(f)_1"'] & & Y_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}(f)_1 \arrow[rr, "\mathbf{CTF}(f)_1"] \arrow[dr, "{(d_1,d_0)}"] \arrow[dd, equal] & & \mathbb{E}(\mathbf{TF}(f))_1 \arrow[r, "\mu_1^f"] \arrow[d, "{(d_1,d_0)}"] & \mathbb{E}(f)_1 \arrow[dd, "{(d_1,d_0)}"] \\
 & \mathbb{E}(f)_0 \times \mathbb{E}(f)_0 \arrow[r, hook, "\mathbf{CTF}(f)_0"] \arrow[drr, equal] & \mathbb{E}(\mathbf{TF}(f))_0 \times \mathbb{E}(\mathbf{TF}(f))_0 \arrow[dr, "\mu_0^f \times \mu_0^f"] & \\
\mathbb{E}(f)_1 \arrow[rrr, "{(d_1,d_0)}"'] & & & \mathbb{E}(f)_0 \times \mathbb{E}(f)_0
\end{tikzcd}$$

$\square$

Consider the functor $\overline{\mathbf{TF}} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to (\mathbf{TF}, \eta)$-**Alg** defined by $f \mapsto (\mathbf{TF}(f), \overrightarrow{\mu_f})$. To show that this is well-defined on morphisms, one must show that for $(u, v) : f \to g$, $\mathbb{E}(u, v)\mu^f = \mu^g \mathbb{E}(\mathbb{E}(u, v), v)$, which follows from the universal properties of $E(\mathbf{TF}(f))_0$ as a coproduct to show equality on objects and of $E(g)_1$ as a pullback to show equality on morphisms.

Proposition 6.4.8. $\overline{\mathbf{TF}}$ *is left adjoint to the forgetful functor* $(\mathbf{TF}, \eta)$-$\mathbf{Alg} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$. *Consequently,* $\mathbb{TF} := (\mathbf{TF}, \eta, \overrightarrow{\mu})$ *is a monad.*

Proof. We describe mutually inverse functors

$$\Theta : (\mathbf{TF}, \eta)\text{-}\mathbf{Alg}((Tf, \overrightarrow{\mu_f}), (g, \alpha)) \leftrightarrows \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}(f, U(g, \alpha)) : \Lambda.$$

Given a morphism $\phi : (\mathbf{TF}(f), \overrightarrow{\mu_f}) \to (g, \alpha)$, we define $\Theta(\phi) : f \to g$ by the composition

$$f \xrightarrow{\eta_f} \mathbf{TF}(f) \xrightarrow{\phi} g$$

Conversely, given $\psi : f \to U(g, \alpha)$, we give a homomorphism of algebras $\Lambda(\psi) : (\mathbf{TF}(f), \overrightarrow{\mu_f}) \to (g, \alpha)$ by the composite

$$\mathbf{TF}(f) \xrightarrow{\mathbf{TF}(\psi)} \mathbf{TF}(g) \xrightarrow{\alpha} g$$

To show this is a well-defined homomorphism of algebras, one must show that

$$\alpha\mathbf{TF}(\alpha\mathbf{TF}(\psi)) = \alpha\mathbf{TF}(\psi)\overrightarrow{\mu_f},$$

which can be done using the universal properties of $\mathbb{E}(\mathbf{TF}(f))$ and $\mathbb{E}(g)$. To show that these maps are mutually inverse, one leverages the same universal properties and the unit property for α to be the structure of an algebra. We omit the tedious details. $\square$

This process, or the dual of it, works to extend the (co)pointed endofunctors $\mathbf{C}$, $\mathbf{TC}$ and $\mathbf{F}$ into (co)monads. Rather than repeat the explanations, we just give the construction of the (co)algebraic structure and leave the details to the interested reader.

For $f : \mathbb{X} \to \mathbb{Y}$, we construct an internal functor $\delta^f : \mathbb{E}(f) \to \mathbb{E}(\mathbf{C}(f))$. On objects it is given by:

$$\begin{tikzcd}
 & X_0 + X_0 + Y_0 & \\
X_0 \arrow[r, hook] \arrow[ur, bend left] & X_0 + Y_0 \arrow[u, dashed, "\delta_0^f"] & Y_0 \arrow[l, hook'] \arrow[ul, bend right]
\end{tikzcd}$$

On morphisms, it is given by:

$$\begin{tikzcd}
\mathbb{E}(f)_1 \arrow[d, "{(d_1,d_0)}"] \arrow[dr, dashed, "\delta_1^f"] \arrow[drr, equal, bend left] & & \\
\mathbb{E}(f)_0 \times \mathbb{E}(f)_0 \arrow[dr, "\delta_0^f \times \delta_0^f"'] & \mathbb{E}(\mathbf{C}(f))_1 \arrow[r, "\mathbf{TF}(\mathbf{C}(f))_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{E}(f)_1 \arrow[d, "{(d_1,d_0)}"] \\
& \mathbb{E}(\mathbf{C}(f))_0 \times \mathbb{E}(\mathbf{C}(f))_0 \arrow[r, "\mathbf{TF}(\mathbf{C}(f))_0 \times \mathbf{TF}(\mathbf{C}(f))_0"'] & \mathbb{E}(f)_0 \times \mathbb{E}(f)_0
\end{tikzcd}$$

We obtain a morphism $\overrightarrow{\delta_f} : \mathbf{C}(f) \to \mathbf{C}^2(f)$ given by

$$\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "\mathbf{C}(f)"'] & \mathbb{X} \arrow[d, "\mathbf{C}^2(f)"] \\
\mathbb{E}(f) \arrow[r, "\delta^f"'] & \mathbb{E}(\mathbf{C}(f)).
\end{tikzcd}$$

We therefore have a natural transformation $\overrightarrow{\delta} : \mathbf{C} \Rightarrow \mathbf{C}^2$.

Lemma 6.4.9. *For any $f : \mathbb{X} \to \mathbb{Y}$, $(f, \overrightarrow{\delta_f})$ is a $(\mathbf{C}, \epsilon)$-coalgebra.*

Consider the functor $\overline{\mathbf{C}} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to (\mathbf{C}, \epsilon)$-**Coalg** defined by $f \mapsto (\mathbf{C}(f), \overrightarrow{\delta_f})$.

Proposition 6.4.10. *$\overline{\mathbf{C}}$ is right adjoint to the forgetful functor $(\mathbf{C}, \epsilon)$-**Coalg** $\to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$. Consequently, $\mathbb{C} := (\mathbf{C}, \epsilon, \overrightarrow{\delta})$ is a comonad.*

The constructions involved in the analogous procedure for $(\mathbf{F}, \xi)$ and $(\mathbf{TC}, \sigma)$ are a little bit more involved. We start by treating $(\mathbf{F}, \xi)$. Firstly, we define a map $\beta : E(\mathbf{W}(f))_0 \to X_0$ given by

$$\begin{tikzcd}
& X_0 & \\
X_0 \arrow[ur, equal] \arrow[r, hook] & X_0 + \mathbf{Map}(f)_0 \arrow[u, dashed, "\beta"] & \mathbf{Map}(f)_0. \arrow[l, hook'] \arrow[ul, "\pi_{X_0}"']
\end{tikzcd}$$

Note that $f_0\beta = \mathbf{F}(f)_0$ by the universal property of $E(\mathbf{W}(f))_0$ as a coproduct.

We use this to define a map $\tau : \mathbf{Map}(\mathbf{F}(f))_0 \to \mathbf{Map}(f)_0$:

$$\begin{tikzcd}
\mathbf{Map}(\mathbf{F}(f))_0 \arrow[r] \arrow[d] \arrow[dr, dashed, "\tau"] & \mathbf{Iso}(\mathbb{Y})_1 \arrow[dr, equal] & \\
E(\mathbf{W}(f))_0 \times Y_0 \arrow[dr, "\beta \times \mathbf{1}"'] & \mathbf{Map}(f)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Iso}(\mathbb{Y})_1 \arrow[d] \\
& X_0 \times Y_0 \arrow[r, "f \times \mathbf{1}"'] & Y_0 \times Y_0
\end{tikzcd}$$

for which the outer diagram commutes by the commutativity of the following:

$$\begin{tikzcd}
\mathbf{Map}(\mathbf{F}(f))_0 \arrow[r] \arrow[d] & \mathbf{Iso}(\mathbb{Y})_1 \arrow[d] \arrow[dr, equal] & \\
E(\mathbf{W}(f))_0 \times Y_0 \arrow[r, "\mathbf{F}(f)_0 \times \mathbf{1}"] \arrow[dr, "\beta \times \mathbf{1}"'] & Y_0 \times Y_0 \arrow[dr, equal] & \mathbf{Iso}(\mathbb{Y})_1 \arrow[d] \\
& X_0 \times Y_0 \arrow[r, "f_0 \times \mathbf{1}"'] & Y_0 \times Y_0.
\end{tikzcd}$$

Next, we construct a map $\kappa^f : \mathbb{E}(\mathbf{WF}(f)) \to \mathbb{E}(\mathbf{W}(f))$. On objects:

$$\begin{tikzcd}
& X_0 + \mathbf{Map}(f)_0 & \\
& & \mathbf{Map}(f)_0 \arrow[ul, hook'] \\
X_0 + \mathbf{Map}(f)_0 \arrow[uur, equal] \arrow[r, hook] & X_0 + \mathbf{Map}(f)_0 + \mathbf{Map}(\mathbf{F}(f))_0 \arrow[uu, dashed, "\kappa_0^f"] & \mathbf{Map}(\mathbf{F}(f))_0 \arrow[l, hook'] \arrow[u, "\tau"']
\end{tikzcd}$$

Note that $\mathbf{F}(f)_0 \kappa_0^f = \mathbf{F}^2(f)_0$, which can be checked using the universal property of $E(\mathbf{WF}(f))_0$ as a coproduct.

We extend this to morphisms by the universal property of $E(\mathbf{W}(f))_1$ as a pullback:

$$\begin{tikzcd}
E(\mathbf{WF}(f))_1 \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, dashed, "\kappa_1^f"] \arrow[drr, bend left, "\mathbf{F}^2(f)_1"] & & \\
E(\mathbf{WF}(f))_0 \times E(\mathbf{WF}(f))_0 \arrow[dr, "\kappa_0^f \times \kappa_0^f"'] & E(\mathbf{W}(f))_1 \arrow[r, "\mathbf{F}(f)_1"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_1,d_0)}"] \\
& E(\mathbf{W}(f))_0 \times E(\mathbf{W}(f))_0 \arrow[r, "\mathbf{F}(f)_0 \times \mathbf{F}(f)_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

This provides a commutative square $\overrightarrow{\kappa_f} : \mathbf{F}^2(f) \to \mathbf{F}(f)$ given by

$$\begin{tikzcd}
\mathbb{E}(\mathbf{WF}(f)) \arrow[r, "\kappa^f"] \arrow[d, "\mathbf{F}^2(f)"'] & \mathbb{E}(\mathbf{W}(f)) \arrow[d, "\mathbf{F}(f)"] \\
\mathbb{Y} \arrow[r, equal] & \mathbb{Y}.
\end{tikzcd}$$

and therefore a natural transformation $\overrightarrow{\kappa} : \mathbf{F}^2 \Rightarrow \mathbf{F}$.

Lemma 6.4.11. *For any $f : \mathbb{X} \to \mathbb{Y}$, $(\mathbf{F}(f), \overrightarrow{\kappa_f})$ is an $(\mathbf{F}, \xi)$-algebra.*

Consider the functor $\overline{\mathbf{F}} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to (\mathbf{F}, \xi)$-**Alg** defined by $f \mapsto (\mathbf{F}(f), \overrightarrow{\kappa_f})$.

Proposition 6.4.12. *$\overline{\mathbf{F}}$ is left adjoint to the forgetful functor $(\mathbf{F}, \xi)$-**Alg** $\to$ $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$. Consequently, $\mathbb{F} := (\mathbf{F}, \xi, \overrightarrow{\kappa})$ is a monad.*

Whilst it is possible to do the same for $(\mathbf{TC}, \sigma)$ and construct algebraic structure, it requires many lengthy and unenlightening calculations due to the two step process in its factorisation. Moreover, we will not actually need the explicit description of this comonad, so we instead appeal to a universal property and prove that $(\mathbf{TC}, \sigma)$ extends to a comonad by showing that it is the underlying left factor of an algebraic weak factorisation system.

Proposition 6.4.13. *The monads $\mathbb{F}$ and $\mathbb{TF}$ are the right classes of algebraic weak factorisation systems.*

Proof. We shall treat $\mathbb{F}$; the other case is similar. We apply Theorem 6.3.15 (3); since the weak factorisation system $(\mathbf{TrivCof}, \mathbf{Fib})$ is cofibrantly generated, we can upgrade it to an algebraic weak factorisation system by Garner's algebraic version of the small object argument; this is $(\mathbb{L}, \mathbb{R})$ such that $\mathbb{R}$ is a monad whose $\mathbb{R}$-algebras are exactly the fibrations. This monad has the universal property of being algebraically free on a pointed endofunctor $\mathbf{R}$ whose algebras are fibrations. By Proposition 6.4.6 and Proposition 6.4.12, this is exactly what $\mathbb{F}$ is, whence $\mathbb{F} \cong \mathbb{R}$ and so $\mathbb{F}$ is the right class of an algebraic weak factorisation system. □

Corollary 6.4.14. *The algebraically cofree comonad on the copointed endofunctor $(\mathbf{TC}, \sigma)$ exists.*

Proof. $\mathbb{F}$ is the right class of some algebraic weak factorisation system $(\mathbb{T}, \mathbb{F})$, where $\mathbb{T} = (\mathbf{T}, a, \rho)$. This algebraic weak factorisation system has underlying ordinary weak factorisation system $(\mathbf{T}, \mathbf{F})$. Ordinary weak factorisation systems are determined by their left class, so it follows that $\mathbf{T} = \mathbf{TC}$ and $a = \sigma$, as required. □

We denote the algebraically cofree comonad on $(\mathbf{TC}, \sigma)$ by $\mathbb{TC} := (\mathbf{TC}, \sigma, \rho)$. Note that we could have used the same reasoning to construct $\mathbb{C}$, but in this case it was simple to do directly.

Corollary 6.4.15. *Both $(\mathbb{TC}, \mathbb{F})$ and $(\mathbb{C}, \mathbb{TF})$ are algebraic weak factorisation systems.*

Theorem 6.4.16. *Let $\mathcal{E}$ be a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint. The algebraic weak factorisation systems $(\mathbb{TC}, \mathbb{F})$ and $(\mathbb{C}, \mathbb{TF})$ provide an algebraic model structure on $(\mathbf{Cat}(\mathcal{E}), \mathbf{Weq})$ whose underlying model structure is the natural model structure on $\mathbf{Cat}(\mathcal{E})$.*

Proof. Proposition 6.4.15 tells us that these algebraic weak factorisation systems are isomorphic as algebraic weak factorisation systems to the ones created from Garner's small object argument. The result then follows from Corollay 6.3.13 and Theorem 6.2.17. □

An important result for us is that this algebraic model structure restricts to the category of internal groupoids. We prove this below.

Theorem 6.4.17. *Let $\mathcal{E}$ be a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint. The algebraic natural model structure on $\mathbf{Cat}(\mathcal{E})$ restricts to an algebraic natural model structure on $\mathbf{Gpd}(\mathcal{E})$.*

Proof. We need to show that given $f : \mathbb{G} \to \mathbb{H}$ in $\mathbf{Gpd}(\mathcal{E})$, the intermediate objects $\mathbb{E}(f)$ and $\mathbb{E}(\mathbf{W}(f))$ are in $\mathbf{Gpd}(\mathcal{E})$.

Firstly, we construct $E(f)_1$ by pulling back over $H_1 = \mathbf{Iso}(\mathbb{H})_1$; whence $E(f)_1 = \mathbf{Iso}(\mathbb{E})_1$ since right adjoints preserve limits.

Similarly, we construct $E(\mathbf{W}(f))_1$ by pulling back over $\mathbf{Map}(f)_1$, which contains only isomorphisms as $\mathbf{Map}(f)$ is constructed as a pullback in $\mathbf{Gpd}(\mathcal{E})$ and **Iso** is a right adjoint so preserves limits, yielding $\mathbf{Iso}(\mathbf{Map}(f))_1 = \mathbf{Map}(f)_1$, as required. $\square$

6.4.4 Equivalent algebraic descriptions

In the previous section, we described the (co)algebras for (co)monads that form an algebraic model structure. In this section, we give equivalent descriptions of these algebras that are more useful and familiar. We show that the structure of being an $\mathbb{F}$-algebra is equivalent to the structure of being a cloven isofibration, and the structure of being a $\mathbb{TC}$-coalgebra is equivalent to the structure of being a retract complemented inclusion on objects with the structure of a 2-cell witnessing the fact that is an equivalence. It is also true that the structure of being a $\mathbb{C}$-coalgebra is equivalent the structure of being a complemented inclusion on objects and the structure of being a $\mathbb{TF}$-algebra is equivalent to the structure of a splitting on objects and a 2-cell witnessing the fact that it is an equivalence.

We first treat $\mathbb{TC}$-coalgebras.

Definition 6.4.18. Let $g : \mathbb{A} \to \mathbb{Y}$ be an internal functor, $r : \mathbb{Y} \to \mathbb{A}$ be a retract of g, $j : X_0 + C \to Y_0$ be an isomorphism such that $j \cdot \iota_{X_0} = f_0$, and $\beta : gr \Rightarrow 1_{\mathbb{Y}}$ an internal natural isomorphism. We call (g, r, j, β) an *algebraic trivial cofibration*. We define a morphism of algebraic trivial cofibrations to be commutative squares that preserve this structure. These form into a category **AlgTrivCof**.

For a $\mathbb{TC}$-coalgebra (g, α), define $r : \mathbb{Y} \to \mathbb{A}$ as the following composite:

$$\mathbb{Y} \xrightarrow{\alpha} \mathbb{E}(\mathbf{W}g) \xrightarrow{\mathbf{TFW}g} \mathbf{Map}(g) \longrightarrow \mathbb{A} \times \mathbb{Y} \longrightarrow \mathbb{A}.$$

and define β as the following composite in $\mathcal{E}$:

$$Y_0 \xrightarrow{\alpha_0} E(\mathbf{W}g)_0 \xrightarrow{\mathbf{TFW}g_0} \mathbf{Map}(g)_0 \longrightarrow Y_1.$$

Moreover, α exhibits g_0 as a retract of a complement inclusion in $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$, and so g_0 is a complemented inclusion on objects with isomorphism

$$j := (f_0, \iota^*_{\mathbf{Map}(g)_0}(\alpha)) : A_0 + \mathbf{Map}(g)_0 \times_{A_0 + \mathbf{Map}(g)_0} Y_0 \to Y_0.$$

In Lemma 6.4.21, we show that (g, r, j, β) is an algebraic trivial cofibration. To prove this, it will be useful to utilise the following concept.

Definition 6.4.19. Let $g : \mathbb{A} \to \mathbb{Y}$ be a functor with $s : Y_0 \to \mathbf{Map}(g)_0$ a splitting of the map

$$\mathbf{Map}(g)_0 \longrightarrow Y_1 \xrightarrow{d_0} Y_0.$$

We call (g, s) *algebraically essentially surjective on objects*.

Lemma 6.4.20. *Let (g, α) be a $\mathbb{TC}$-coalgebra. Then $(g, \mathbf{TFW}(g)_0\alpha_0)$ is algebraically essentially surjective on objects.*

Proof. This is witnessed by the following commutative diagram.

$$
\begin{tikzcd}
Y_0 \arrow[r, "\alpha"] \arrow[ddd, equal] & E(\mathbf{W}g)_0 \arrow[r, "\mathbf{TFW}g_0"] \arrow[dddl, "Fg_0"'] & \mathbf{Map}(g)_0 \arrow[dl] \arrow[d] \\
 & A_0 \times Y_0 \arrow[d, "g_0 \times 1"] \arrow[ddl] & Y_1 \arrow[dl] \arrow[dd, "d_0"] \\
 & Y_0 \times Y_0 \arrow[dr] & \\
Y_0 \arrow[rr, equal] & & Y_0
\end{tikzcd}
$$

$\square$

Lemma 6.4.21. *Let (g, α) be a $\mathbb{TC}$-coalgebra. Then (g, r, j, β) as defined above is an algebraic trivial cofibration.*

Proof. We first show that r is a retract of g. This is witnessed by the following diagram.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "g"] \arrow[rr, bend right, "\mathbf{TC}g"'] \arrow[drr, bend right, "\mathbf{W}g"] \arrow[ddrr, bend right, "{(1,g)}"] \arrow[ddd, equal] & \mathbb{Y} \arrow[r, "\alpha"] & \mathbb{E}(\mathbf{W}g) \arrow[d, "\mathbf{TFW}g"] \\
 & & \mathbf{Map}(g) \arrow[d] \\
 & & \mathbb{A} \times \mathbb{Y} \arrow[d] \\
\mathbb{Y} \arrow[rr, equal] & & \mathbb{Y}
\end{tikzcd}
$$

Next, we show that $\beta : Y_0 \to Y_1$ is an internal natural isomorphism $gr \Rightarrow 1_{\mathbb{Y}}$. Firstly, the diagram

$$
\begin{tikzcd}
Y_0 \arrow[r, "\alpha_0"] \arrow[drrr, equal] & E(\mathbf{W}g)_0 \arrow[r, "\mathbf{TFW}g_0"] & \mathbf{Map}(g)_0 \arrow[r] & Y_1 \arrow[d, "d_0"] \\
 & & & Y_0
\end{tikzcd}
$$

commutes because of Lemma 6.4.20.

The following commutes by definition of the constructions involved.

$$
\begin{tikzcd}
Y_0 \arrow[r, "\alpha_0"] \arrow[ddd, "r_0"'] & E(\mathbf{W}g)_0 \arrow[r, "\mathbf{TFW}g_0"] & \mathbf{Map}(g)_0 \arrow[dl] \arrow[dd] \\
 & A_0 \times Y_0 \arrow[d, "g_0 \times 1"] \arrow[ddl] & \\
 & Y_0 \times Y_0 \arrow[dr] & Y_1 \arrow[l] \arrow[d, "d_1"] \\
A_0 \arrow[rr, "g_0"'] & & Y_0
\end{tikzcd}
$$

The other conditions for natural isomorphism follow easily. $\square$

For an algebraic trivial cofibration (g, r, j, β), we can define a coalgebra structure on g by the maps which follow.

Note that in **Cat**$(\mathcal{E})$, a natural isomorphism $\beta : gr \Rightarrow 1$ corresponding to a morphism $\overline{\beta} : Y_0 \to Y_1$ in $\mathcal{E}$ can be written as a functor $\underline{\beta} : \mathbb{Y} \to \mathbb{Y}^{\mathcal{I}}$ with $\underline{\beta}_0 = \overline{\beta}$ and $\underline{\beta}_1$ defined as below:

$$\begin{tikzcd}
Y_1 \arrow[dr, dashed, "\underline{\beta}_1"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathbb{Y}^{\mathcal{I}}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Iso}(Y)_1 \times_{Y_0} Y_1 \arrow[d, "m"] \\
& Y_1 \times_{Y_0} \mathbf{Iso}(Y)_1 \arrow[r, "m"'] & Y_1
\end{tikzcd}$$

In which the two maps $Y_1 \to Y_2$ are given by $(\overline{\beta} \cdot d_1, 1_{Y_1})$ and $(d_0 \cdot g_1 r_1, \overline{\beta})$. The outside of the square commutes by the axioms for an internal natural isomorphism.

Without loss of generality, we can assume that on objects g is a coproduct inclusion $A_0 \hookrightarrow A_0 + C$. First we define the following maps.

$$\begin{tikzcd}
\mathbb{Y} \arrow[dr, dashed, "\hat{\beta}"] \arrow[drr, bend left, "\underline{\beta}"] \arrow[ddr, bend right, "{(r,1)}"'] & & \\
& \mathbf{Map}(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Y}^{\mathcal{I}} \arrow[d] \\
& \mathbb{A} \times \mathbb{Y} \arrow[r] & \mathbb{Y} \times \mathbb{Y}
\end{tikzcd}$$

Then we define $\tau : C \to A_0 + \mathbf{Map}(g)_0$ as the following composite:

$$C \hookrightarrow A_0 + C \xrightarrow{\hat{\beta}_0} \mathbf{Map}(g)_0 \hookrightarrow A_0 + \mathbf{Map}(g)_0.$$

We define $\alpha^* : \mathbb{Y} \to \mathbb{E}(\mathbf{W}g)$ on objects:

$$\begin{tikzcd}
& A_0 + \mathbf{Map}(g)_0 & \\
A_0 \arrow[ur, bend left, "\mathbf{TC}g_0"] \arrow[r, hook] & A_0 + C \arrow[u, dashed, "\alpha_0^*"] & C \arrow[l, hook'] \arrow[ul, bend right, "\tau"']
\end{tikzcd}$$

Lemma 6.4.22. *Let (g, r, j, β) be an algebraic trivial cofibration and consider the maps $\hat{\beta}$ and α_0^* as defined above. Then we have* $\mathbf{TFW}(g_0)\alpha_0^* = \hat{\beta}_0$

Proof. This is witnessed by the following diagrams, using the universal properties of the coproduct and the pullback.

$$\begin{tikzcd}
 & \mathbf{Map}(g)_0 \arrow[dr, equal, bend left] & \\
 & A_0 + \mathbf{Map}(g)_0 \arrow[u, "\mathbf{TFW}g_0"] & \mathbf{Map}(g)_0 \arrow[l, hook'] \\
A_0 \arrow[r, hook] \arrow[uur, bend left, "\mathbf{W}g"] \arrow[ur, hook, bend left] & A_0 + C \arrow[u, "\alpha^*"] & C \arrow[l, hook'] \arrow[u, "\hat{\beta}\iota_C"'] \arrow[ul, "\tau"]
\end{tikzcd}$$

in which the left side commutes due to the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "i"] \arrow[dr, "g"] \arrow[ddrr, bend right, "(1,g)"'] & & \mathbb{A}^{\mathcal{I}} \arrow[dr, "g"] & \\
 & \mathbb{Y} \arrow[dr, dashed, "\hat{\beta}"] \arrow[rr, bend left, "\underline{\beta}"] \arrow[ddr, bend right, "(r,1)"'] & & \mathbb{Y} \\
 & & \mathbf{Map}(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Y} \arrow[d] \\
 & & \mathbb{A} \times \mathbb{Y} \arrow[r] & \mathbb{Y} \times \mathbb{Y}
\end{tikzcd} \tag{6.5}$$

and the right side commutes due to the definition of τ.

$\square$

On morphisms, we define

$$\begin{tikzcd}
Y_1 \arrow[dr, dashed, "\alpha_1^*"] \arrow[drr, bend left, "\hat{\beta}_1"] \arrow[dd, "(d_1,d_0)"'] & & \\
 & E(\mathbf{W}g)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Map}(g)_1 \arrow[d] \\
Y_0 \times Y_0 \arrow[r, "\alpha_0^* \times \alpha_0^*"'] & E(\mathbf{W}g)_0 \times E(\mathbf{W}g)_0 \arrow[r] & \mathbf{Map}(g)_0 \times \mathbf{Map}(g)_0
\end{tikzcd}$$

in which the outside of the diagram commutes as witnessed by Equation 6.4.22 and the fact that $\hat{\beta}$ is an internal functor. It is not difficult to show that $\alpha^* := (\alpha_0^*, \alpha_1^*)$ assembles into an internal functor $\mathbb{Y} \to \mathbb{E}(\mathbf{W}g)$.

Lemma 6.4.23. *Let (g, r, j, β) be an algebraic trivial cofibration. Then (g, α^*) is a $\mathbb{TC}$-coalgebra.*

Proof. Without loss of generality, we can assume that on objects g is a coproduct inclusion $g_0 : A_0 \hookrightarrow A_0 + C$. We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathbf{TC}g"] \arrow[d, "g"'] & \mathbb{E}(\mathbf{W}g) \arrow[d, "\mathbf{F}g"] \\
\mathbb{Y} \arrow[r, equal] \arrow[ur, "\alpha^*"] & \mathbb{Y}.
\end{tikzcd}$$

On objects, the top triangle commutes by definition of α^* and the bottom diagram commutes by the following two diagrams.

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend right, "\mathbf{TC}g_0"'] \arrow[drr, bend right, "g"'] & A_0 + C \arrow[r, "\alpha_0^*"] & A_0 + \mathbf{Map}(g)_0 \arrow[d, "\mathbf{F}g_0"] \\
A_0 + C \arrow[rr, equal] & & A_0 + C.
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[dddd, hook] & A_0 + C \arrow[rrr, "\alpha_0^*"] \arrow[dr, "\hat{\beta}"] \arrow[ddrr, bend right, "{(r,1)}"'] & & & A_0 + \mathbf{Map}(g)_0 \arrow[dl, "\mathbf{TFW}g_0"] \arrow[dddd, "\mathbf{F}g_0"] \\
& & \mathbf{Map}(g)_0 \arrow[r, equal] \arrow[urr, hook] & \mathbf{Map}(g)_0 \arrow[d] & \\
& & & A_0 \times (A_0 + C) \arrow[ddr] & \\
& & & & \\
A_0 + C \arrow[rrrr, equal] & & & & A_0 + C.
\end{tikzcd}$$

On morphisms, the following diagram witnesses that $\alpha_1^* g_1 = \mathbf{TC}g$.

$$\begin{tikzcd}
A_1 \arrow[r, "g_1"] \arrow[d] \arrow[rrr, bend left, "\mathbf{W}g_1"] & Y_1 \arrow[r, "\alpha_1^*"] \arrow[d] \arrow[rr, bend left, "\hat{\beta}_1"] & E(\mathbf{W}g)_1 \arrow[r, "\mathbf{TFW}g_1"] \arrow[d] & \mathbf{Map}(g)_1 \arrow[d] \\
A_0 \times A_0 \arrow[r, "g_0 \times g_0"'] & Y_0 \times Y_0 \arrow[r, "\alpha_0^* \times \alpha_0^*"'] & E(\mathbf{W}g)_0 \times E(\mathbf{W}g)_0 \arrow[r, "\mathbf{TFW}g_0 \times \mathbf{TFW}g_0"'] & \mathbf{Map}(g)_0 \times \mathbf{Map}(g)_0
\end{tikzcd}$$

In this diagram, the top triangle commutes by Equation 6.4.22. The outer square for this diagram is the defining square for $\mathbf{TC}g_1$. Finally, the following diagram shows that $\mathbf{F}(g)_1 \alpha_1^* = 1_{Y_1}$.

$$\begin{tikzcd}
Y_1 \arrow[r, "\alpha_1^*"] \arrow[d, equal] \arrow[rr, bend right, "\hat{\beta}_1"'] & E(\mathbf{W}g)_1 \arrow[r] & \mathbf{Map}(g)_1 \arrow[r] & A_1 \times Y_1 \arrow[d] \\
Y_1 \arrow[rrr, equal] & & & Y_1
\end{tikzcd}$$

□

These processes are mutually inverse, which can be checked using the universal properties involved.

Proposition 6.4.24. *There is an isomorphism of categories* $\mathbb{TC}$-**Coalg** $\cong$ **AlgTrivCof**.

We have therefore shown that we can translate between $\mathbb{TC}$-coalgebras and algebraic trivial cofibrations.

Next, we turn our attention to fibrations.

Definition 6.4.25. Let $f : \mathbb{X} \to \mathbb{Y}$ be an internal functor in $\mathbf{Cat}(\mathcal{E})$ and $k : X_0 \times_{Y_0} \mathbf{Iso}(\mathbb{Y})_0 \to \mathbf{Iso}(\mathbb{X})$ a splitting of the map $(d_0, f_1) : \mathbf{Iso}(\mathbb{X})_0 \to X_0 \times_{Y_0} \mathbf{Iso}(Y)_0$. We call (f, k) a *cloven isofibration.*

Let $(f : \mathbb{X} \to \mathbb{Y}, k)$ and $(g : \mathbb{A} \to \mathbb{B}, l)$ be cloven isofibrations. We define a *morphism of cloven isofibrations* $(u, v) : (f, k) \to (g, l)$ to be a morphism $(u, v) : f \to g$ in $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ such that the following diagram commutes.

$$\begin{tikzcd}
X_0 \times_{Y_0} \mathbf{Iso}(\mathbb{Y})_0 \arrow[r, "u_0 \times_{v_0} \mathbf{Iso}(v)_0"] \arrow[d, "k"'] & A_0 \times_{B_0} \mathbf{Iso}(\mathbb{B})_0 \arrow[d, "l"] \\
\mathbf{Iso}(\mathbb{X})_0 \arrow[r, "\mathbf{Iso}(u)_0"'] & \mathbf{Iso}(\mathbb{A})_0 .
\end{tikzcd}$$

Denote the category of cloven isofibrations and morphisms of cloven isofibrations by **ClovenIsofib**.

Note that by Remark 6.4.3, $X_0 \times_{Y_0} \mathbf{Iso}(Y)_0 \cong \mathbf{Map}(f)_0$.

By Proposition 6.3.8, if (f, k) is a cloven isofibration, then f is an isofibration, with k providing a proof of this fact. We show that we can uniformly translate between $\mathbb{F}$-algebras and cloven isofibrations.

Let $(f : \mathbb{X} \to \mathbb{Y}, \phi : \mathbb{E}(\mathbf{WF}f) \to \mathbb{X})$ be an $\mathbb{F}$-algebra. Define the following maps:

$$h := (1_{X_0} \times d_1 \pi_{Y_1}, \pi_{Y_1}) : X_0 \times_{Y_0} Y_1 \to \mathbf{Map}(f)_1$$

$$b := (f_1 \cdot i_{\mathbb{X}} \times_{Y_0} 1_{Y_1}, f_1 \cdot i_{\mathbb{X}} \times_{Y_0} 1_{Y_1}) : X_0 \times_{Y_0} Y_1 \to Y_2 \times_{Y_1} Y_2$$

$$c := (i_{\mathbb{X}} \times \pi_{Y_1}, b) : X_0 \times_{Y_0} Y_1 \to \mathbf{Map}(f)_1$$

$$s := ((\iota_{X_0} \pi_{X_0}, h), c) : X_0 \times_{Y_0} Y_1 \to \mathbb{E}(\mathbf{WF}f)_1$$

$$k := \phi_1 s : X_0 \times_{Y_0} Y_1 \to X_1 .$$

Lemma 6.4.26. *Let* $(f : \mathbb{X} \to \mathbb{Y}, \phi : \mathbb{E}(\mathbf{WF}f) \to \mathbb{X})$ *be an* $\mathbb{F}$*-algebra. Then* (f, k) *is a cloven isofibration.*

Conversely, starting with $(f : \mathbb{X} \to \mathbb{Y}, k)$ a cloven isofibration, we can form the structure of an $\mathbb{F}$-algebra via the following maps.

Firstly, define $\phi_0 : (1_{X_0}, d_1 \cdot k) : E(\mathbf{W}f)_0 := X_0 + \mathbf{Map}(f)_0 \to X_0$. Next, we note that by Remark 6.4.3 in order to define a map $\phi_1 : E(\mathbf{W}f)_1 \to X_1$, it is enough to define maps $a : (X_0 \times X_0) \times_{\mathbf{Map}(f)_0 \times \mathbf{Map}(f)_0} \mathbf{Map}(f)_1 \to X_1$, $b : X_0 \times_{\mathbf{Map}(f)_0} \mathbf{Map}(f)_1 \to X_1$ and $c : \mathbf{Map}(f)_1 \to X_1$. We define these maps as follows:

$$a := \left(\; (X_0 \times X_0) \times_{\mathbf{Map}(f)_0 \times \mathbf{Map}(f)_0} \mathbf{Map}_1 \xrightarrow{\pi_{\mathbf{Map}(f)_1}} \mathbf{Map}(f)_1 \xrightarrow{\pi_{X_1}} X_1 \; \right),$$

$$\begin{tikzcd}
\mathbf{Map}(f)_1 \arrow[d, "d_0"'] \arrow[dr, dashed, "x"] \arrow[drr, bend left, "\pi_{X_1}"] & & \\
\mathbf{Map}(f)_0 \arrow[d, "k"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_1 \arrow[d, "d_0"] \\
\mathbf{Iso}(\mathbb{X})_1 \arrow[r, hook] & X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}$$

$$b := \left(X_0 \times_{\mathbf{Map}(f)_0} \mathbf{Map}(f)_1 \longrightarrow \mathbf{Map}(f)_1 \xrightarrow{x} X_2 \xrightarrow{m} X_1 \right),$$

$$c := \left(\mathbf{Map}(f)_1 \xrightarrow{(\pi_{X_1}, d_1, d_0)} X_1 \times \mathbf{Map}(f)_0 \times \mathbf{Map}(f)_0 \xrightarrow{(1, \mathrm{inv}\cdot k, k)} X_3 \xrightarrow{m^2} X_1 \right).$$

We define $\phi_1 := a + b + c : E(\mathbf{W}f)_1 \to X_1$. It is not hard to see that we have an internal functor $\phi := (\phi_0, \phi_1) : \mathbb{E}(\mathbf{W}f) \to \mathbb{X}$.

Lemma 6.4.27. *Let $(f : \mathbb{X} \to \mathbb{Y}, k)$ a cloven isofibration. Then (f, ϕ) is an $\mathbb{F}$-algebra.*

Proof. It is clear that on objects $\phi_0 \cdot \mathbf{TC}f_0 = 1_{X_0}$ and the following commutative diagram completes the proof that $f_0 \cdot \phi_0 = \mathbf{F}f_0$.

$$\begin{tikzcd}
\mathbf{Map}(f)_0 \arrow[r, "k"] \arrow[dr, equal] & \mathbf{Iso}(\mathbb{X})_0 \arrow[r, "d_1"] \arrow[d, "{(d_1, \mathbf{Iso}(f)_0)}"] & X_0 \arrow[dd, "f_0"] \\
& \mathbf{Map}(f)_0 \arrow[d] & \\
& X_0 \times Y_0 \arrow[r, "\pi_{Y_0}"'] & Y_0.
\end{tikzcd}$$

We note that $\mathbf{TC}f_1$ factors through $(X_0 \times X_0) \times_{\mathbf{Map}(f)_0 \times \mathbf{Map}(f)_0} \mathbf{Map}_1$ and $a \cdot \mathbf{TC}f = 1_{\mathbb{X}_1}$ by construction. It remains to show that $f_1 \cdot b = \pi_{Y_1} \pi_{\mathbf{Map}(f)_1}$ and $f_1 \cdot c = \pi_{Y_1}$ as these morphisms are $\mathbf{F}f_1|_{X_0 \times_{\mathbf{Map}(f)_0} \mathbf{Map}(f)_1}$ and $\mathbf{F}f_1|_{\mathbf{Map}(f)_1}$ respectively. This can be shown representably. $\square$

These processes are mutually inverse, which can again be checked using the universal properties involved.

Proposition 6.4.28. *There is an isomorphism of categories $\mathbb{F}$-***Alg** $\cong$ **ClovenIsofib**.

We also have equivalent characterisations of $\mathbb{C}$-coalgebras and $\mathbb{TF}$-algebras; for these, we omit the proof since it uses the same strategy as previously used and their construction will not be needed in the subsequent work. We record these below.

Definition 6.4.29. Let $f : \mathbb{X} \to \mathbb{Y}$ and an isomorphism $j : X_0 + C \to Y_0$ such that $j \cdot \iota_{X_0} = f_0$. We call (f, j) an *algebraic complemented inclusion on objects.*

Let $(f : \mathbb{X} \to \mathbb{Y}, j : Y_0 \to X_0 + C), (g : \mathbb{A} \to \mathbb{B}, t : Y_0 \to D)$ be a pair of algebraic complemented inclusion on objects. A morphism $(u, v) : f \to g$ is called a morphism of algebraic complemented inclusion on objects if the following diagram commutes.

$$\begin{tikzcd}
X_0 \arrow[r, "u_0"] \arrow[d, hook] & A_0 \arrow[d, hook] \\
X_0 + C \arrow[r, "t \cdot v_0 \cdot j^{-1}"'] & A_0 + D.
\end{tikzcd}$$

We denote the category of algebraic complemented inclusion on objects and the morphisms of these by **AlgCompIncObj**.

Proposition 6.4.30. *There is an isomorphism of categories $\mathbb{C}$-***Coalg** $\cong$ **AlgCompIncObj**.

Definition 6.4.31. Let $f : \mathbb{X} \to \mathbb{Y}$, $s : \mathbb{Y} \to \mathbb{X}$ be a splitting of f and $\beta : 1_{\mathbb{X}} \Rightarrow sf$ a natural isomorphism. We call (f, s, β) an *algebraic split epi equivalence*. We define a morphism of algebraic split epi equivalences to be commutative squares that preserve the choice of splitting. These form into a category **AlgSplitEpiEq**.

Proposition 6.4.32. *There is an isomorphism of categories* $\mathbb{TF}$-**Alg** $\cong$ **AlgSplitEpiEq**.

6.5 Type theoretic aspects

In this section, we restrict our attention to the setting of $\mathcal{E}$ a locally cartesian closed locos with coequalisers, which is a locally cartesian closed and lextensive category with parametrised list objects and coequalisers. Such an $\mathcal{E}$ gives us an example of a lextensive cartesian closed category with pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has left adjoint [Mai10]. We also restrict our attention from $\mathbf{Cat}(\mathcal{E})$ to $\mathbf{Gpd}(\mathcal{E})$— this is because one ingredient we need to model type theory is exponentiability of isofibrations. It is not true that all isofibrations in $\mathbf{Cat}(\mathcal{E})$ are exponentiable; however, this is true for isofibrations in $\mathbf{Gpd}(\mathcal{E})$. In fact, in Proposition 6.6.1 we show that that isofibrations in $\mathbf{Gpd}(\mathcal{E})$ are exponentiable if and only if $\mathcal{E}$ is locally cartesian closed, which shows that we must assume local cartesian closure. For $\mathcal{E}$ a locally cartesian closed locos with coequalisers, $\mathbf{Gpd}(\mathcal{E})$ has the structure of an algebraic model structure by Corollary 6.4.17.

The aim of this section is to prove that the $(\mathbb{TC}, \mathbb{F})$ algebraic weak factorisation system on $\mathbf{Gpd}(\mathcal{E})$ has the structure of a type theoretic algebraic weak factorisation system in the sense of [GL23, Definition 4.10] which we recall in Definition 6.5.8. By [GL23, Theorem 3.12], this proves that the cloven isofibrations in $\mathbf{Gpd}(\mathcal{E})$ give us an algebraic model of MLTT with strictly stable choices of Σ-, Π- and `Id`-types.

6.5.1 Exponentiability

Recall that a morphism $f : X \to Y$ in a category $\mathbb{C}$ is called exponentiable if the pullback functor $f^* : \mathbb{C}/Y \to \mathbb{C}/X$ has a right adjoint. We say that an algebraic weak factorisation system $(\mathbb{L}, \mathbb{R})$ satisfies the *exponentiability property* if any map in the image of the forgetful functor $\mathbb{R}$-**Alg** $\to \mathbb{C}^{\mathbf{2}}$ is exponentiable [GL23, Definition 3.4]. Exponentiability of isofibrations for internal groupoids is studied by Niefield-Pronk in [NP19], and immediately gives us the result of Proposition 6.5.1.

Proposition 6.5.1. $(\mathbb{TC}, \mathbb{F})$ *satisfies the exponentiability property.*

Proof. By Proposition 6.4.6, $\mathbb{F}$-algebras are precisely the isofibrations with a choice of lifting. Given that $\mathcal{E}$ is locally cartesian closed, we can apply Theorem 4.5 of [NP19]. □

6.5.2 Frobenius structure

The aim of this section is to construct a functorial Frobenius structure on $(\mathbb{TC}, \mathbb{F})$, as recalled in the following definition.

Definition 6.5.2. [GL23, Definition 3.5] Let $(\mathbb{L}, \mathbb{R})$ be an algebraic weak factorisation system on a category $\mathbb{C}$. A *functorial Frobenius structure* on $(\mathbb{L}, \mathbb{R})$ is given by the dotted map in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}\text{-}\mathbf{Alg} \times_{\mathbb{C}} \mathbb{L}\text{-}\mathbf{Coalg} \arrow[r, dashed, "\widehat{\mathbf{pb}}"] \arrow[d] & \mathbb{L}\text{-}\mathbf{CoAlg} \arrow[d] \\
\mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}} \arrow[r, "\mathbf{pb}"'] & \mathbb{C}^{\mathbf{2}}.
\end{tikzcd}$$

in which $\mathbf{pb} : \mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}}$ denotes pullback: $(f, g) \mapsto f^*(g)$.

To provide a functorial Frobenius structure in our setting, we must therefore construct a functor $\widehat{\mathbf{pb}} : \mathbb{F}\text{-}\mathbf{Alg} \times_{\mathbf{Cat}(\mathcal{E})} \mathbb{TC}\text{-}\mathbf{Coalg} \to \mathbb{TC}\text{-}\mathbf{Coalg}$. To do this, we show that algebraic trivial cofibrations are closed under pullback along cloven isofibrations in a uniform way. By Proposition 6.4.28 and Proposition 6.4.24, the result follows.

First, note that in an extensive category, complemented inclusions are stable under pullback along any map by definition [CLW93]. Given $i : A \to A + C$ and $f : X \to A + C$ we can deduce that $X \cong A \times_{A+C} X + C \times_{A+C} X$. Therefore, given an algebraic trivial cofibration (g, r, j, β) and a cloven isofibration (f, k), the map $f^*(g)$ has the structure of a complemented inclusion on objects functor, which we denote by j^*. Next, we show that retract equivalences are closed under pullback along cloven isofibrations in a canonical way.

Let $(g : \mathbb{A} \to \mathbb{Y}, r, \beta)$ be a retract equivalence and $(f : \mathbb{X} \to \mathbb{Y}, k : X_0 \times_{Y_0} Y_1 \to X_1)$ be a cloven isofibration. We define a map $r_0^* : X_0 \to A_0 \times_{Y_0} X_0$ by the following:

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[dr, dashed, "\beta'"] \arrow[ddr, bend right, equal] & Y_0 \arrow[dr, "\overline{\beta}"] & \\
& Y_1 \times_{Y_0} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "d_1"] \\
& X_0 \arrow[r, "f_0"'] & Y_0
\end{tikzcd}$$

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[dr, dashed, "r_0^*"] \arrow[ddr, bend right, "d_0 k \beta'"'] & Y_0 \arrow[dr, "r_0"] & \\
& X_0 \times_{Y_0} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \arrow[d, "g_0"] \\
& X_0 \arrow[r, "f_0"'] & Y_0
\end{tikzcd}$$

where the outside of the second diagram commutes by the following diagram.

$$\begin{tikzcd}
X_0 \arrow[r, "\beta'"] \arrow[dd, "f_0"'] & X_0 \times_{Y_0} Y_1 \arrow[r, "k"] \arrow[d, equal] & X_1 \arrow[r, "d_0"] \arrow[d, "f_1"] \arrow[dl, "l"] & X_0 \arrow[dd, "f_0"] \\
& X_0 \times_{Y_0} Y_1 \arrow[r] & Y_1 \arrow[dr, "d_0"] & \\
Y_0 \arrow[r, "r_0"'] \arrow[urr, "\overline{\beta}"] & A_0 \arrow[rr, "g_0"'] & & Y_0
\end{tikzcd}$$

Next, recall that the model structure on $\mathbf{Gpd}(\mathcal{E})$ is right proper since all objects are fibrant, and therefore weak equivalences are closed under pullback along fibrations [Hir02]. Therefore $f^*(g)$ is a weak equivalence and is in particular fully faithful. We construct a morphism $r_1^* : X_1 \to A_1 \times_{Y_1} X_1$ as follows:

$$
\begin{tikzcd}
X_1 \arrow[drr, bend left, equal] \arrow[dr, dashed, "r_1^*"] \arrow[dd] & & \\
& A_1 \times_{Y_1} X_1 \arrow[r, "f^*(g)_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\
X_0 \times X_0 \arrow[r, "r_0^* \times r_0^*"'] & A_0 \times_{Y_0} X_0 \times A_0 \times_{Y_0} X_0 \arrow[r, "f^*(g)_0 \times f^*(g)_0"'] & X_0 \times X_0.
\end{tikzcd}
$$

We define an internal functor by $r^* = (r_0^*, r_1^*)$.

Define a map $\overline{\beta^*} : X_0 \to X_1$ as the following composite:

$$
X_0 \xrightarrow{\beta'} X_0 \times_{Y_0} Y_1 \xrightarrow{k} X_1.
$$

Consider the following diagrams:

$$
\begin{tikzcd}
X_0 \arrow[r, "\beta'"] \arrow[dr, "\beta'"'] \arrow[ddrr, bend right, equal] & X_0 \times_{Y_0} Y_1 \arrow[r, "k"] \arrow[d, equal] & X_1 \arrow[dl, "l"] \arrow[dd, "d_1"] \\
& X_0 \times_{Y_0} Y_1 \arrow[dr] & \\
& & X_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_0 \arrow[r, "\beta'"] \arrow[d, "r_0^*"'] & X_0 \times_{Y_0} Y_1 \arrow[r, "k"] & X_1 \arrow[d, "d_0"] \\
X_0 \times_{Y_0} A_0 \arrow[rr, "f^*(g)_0"'] & & X_0
\end{tikzcd}
$$

in which the latter commutes by definition of β'.

It is not hard to prove that $\overline{\beta^*}$ defines a natural isomorphism $\beta^* : f^*(g)r^* \Rightarrow 1$. We conclude that algebraic trivial cofibrations are closed under pullback along isofibrations, the details of which are contained in the construction.

Lemma 6.5.3. *Let $(g : \mathbb{A} \to \mathbb{Y}, r, j, \beta)$ be an algebraic trivial cofibration and (f, k) be a cloven isofibration. Then $(f^*(g), r^*, j^*, \beta^*)$ is an algebraic trivial cofibration.*

We are now able to prove the following result.

Proposition 6.5.4. *There is a Frobenius structure on $(\mathbb{TC}, \mathbb{F})$.*

Proof. Let $(g : \mathbb{A} \to \mathbb{Y}, \alpha)$ be a $\mathbb{TC}$-coalgebra and $(f : \mathbb{X} \to \mathbb{Y}, s)$ be an $\mathbb{F}$-algebra. We uniformly construct a $\mathbb{TC}$-coalgebra structure on the pullback $f^*(g)$. By Lemma 6.4.21, we translate (g, α) into a algebraic trivial cofibration (g, r, j, β) and by Lemma 6.4.26, we translate (f, s) into a cloven isofibration (f, k). Therefore, by Lemma 6.5.3 $(f^*(g), r^*, j^*, \beta^*)$ is an algebraic trivial cofibration. Whence, we can apply Lemma 6.4.23 to obtain a $\mathbb{TC}$-coalgebra $(f^*(g), \alpha^*)$, as required. $\square$

6.5.3 Stable functorial choice of path objects

The aim of this section is to show that $(\mathbb{TC}, \mathbb{F})$ has a stable functorial choice of path objects. For $f : X \to Y$ in $\mathbb{C}$, denote the diagonal map by $\Delta_f : X \to X \times_Y X$. This extends to a functor $\Delta_{-} : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}}$. We recall the following definitions below from [GL23].

Definition 6.5.5. Let $(\mathbb{L}, \mathbb{R})$ be an algebraic weak factorisation system on $\mathbb{C}$. A *functorial factorisation of the diagonal* is a functor $\mathcal{P} : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}}$ such that

$$f : X \to Y \mapsto \quad X \xrightarrow{\lambda_f} \mathcal{P}X \xrightarrow{\rho_f} X \times_Y X$$

in which $\rho_f \cdot \lambda_f = \Delta_f$.

Such a functorial factorisation of the diagonal is called *stable* if $(h, k) : (f \to f') \in \mathbb{C}^{\mathbf{2}}$ is a pullback if and only if $\rho_{(h,k)} : \rho_f \to \rho_{f'}$ is a pullback.

A *stable functorial choice of path objects* consists of a lift of a stable functorial factorisation of the diagonal map, as shown in the following diagram.

$$\begin{tikzcd} \mathbb{R}\text{-}\mathbf{Alg} \arrow[r, dashed, "\widehat{\mathcal{P}}"] \arrow[d] & \mathbb{L}\text{-}\mathbf{Coalg} \times_{\mathbb{C}} \mathbb{R}\text{-}\mathbf{Alg} \arrow[d] \\ \mathbb{C}^{\mathbf{2}} \arrow[r, "\mathcal{P}"'] & \mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}}. \end{tikzcd}$$

In $\mathbf{Cat}(\mathcal{E})$, we claim that the functorial factorisation of the diagonal $(\mathbf{TC}(\Delta_{-}), \mathbf{F}(\Delta_{-})) : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}}$ gives us a stable functorial choice of path objects.

Proposition 6.5.6. *The functorial factorisation of the diagonal* $(\mathbb{TC}(\Delta_{-}), \mathbb{F}(\Delta_{-}))$ *is stable.*

Proof. Consider a cartesian square

$$\begin{tikzcd} \mathbb{A} \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{B} \arrow[r, "v"'] & \mathbb{Y}. \end{tikzcd} \tag{6.6}$$

Firstly, we argue that the following square is a pullback.

$$\begin{tikzcd} \mathbf{Map}(g) \arrow[r, "\mathbf{Map}(u,v)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Map}(f) \arrow[d] \\ \mathbb{B} \arrow[r, "v"'] & \mathbb{Y} \end{tikzcd} \tag{6.7}$$

It is enough to show that this is true in **Set** as $\mathbf{Map}(g)$ is a representable construction as it is built out of pullbacks, products and powers by categories, which are all representable notions. Therefore, this follows from [GL23, Proposition 4.4].

By the pullback lemma, given that Equation (6.7) and Equation (6.6) are pullbacks, it follows that the following square is a pullback:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "u"] \arrow[d, "\mathbf{C}g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X} \arrow[d, "\mathbf{C}f"] \\
\mathbf{Map}(g) \arrow[r, "\mathbf{Map}(u,v)"] & \mathbf{Map}(f).
\end{tikzcd}
\tag{6.8}
$$

Next, we show that given a pullback square as in Equation (6.6), the following square is a pullback:

$$
\begin{tikzcd}
\mathbb{E}(g) \arrow[r, "\mathbb{E}(u,v)"] \arrow[d, "\mathbf{TF}g"'] & \mathbb{E}(f) \arrow[d, "\mathbf{TF}f"] \\
\mathbb{B} \arrow[r, "v"'] & \mathbb{Y}
\end{tikzcd}
\tag{6.9}
$$

On objects, this is the square

$$
\begin{tikzcd}
A_0 + B_0 \arrow[r, "E(u,v)_0"] \arrow[d, "\mathbf{TF}g"'] & X_0 + Y_0 \arrow[d, "\mathbf{TF}f"] \\
B_0 \arrow[r, "v"'] & Y_0.
\end{tikzcd}
$$

which is a pullback due to distributivity of pullbacks over coproducts in $\mathcal{E}$ as seen in the following calculation:

$$
\begin{aligned}
(X_0 + Y_0) \times_{Y_0} B_0 &\cong (X_0 \times_{Y_0} B_0) + (Y_0 \times_{Y_0} B_0) \\
&\cong A_0 + B_0.
\end{aligned}
$$

On morphisms, consider the following diagram in $\mathcal{E}$:

$$
\begin{tikzcd}
E \arrow[dr, dashed, "t"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& E(g)_1 \arrow[r, "E(u,v)_1"] \arrow[d, "\mathbf{TF}g_1"'] & E(f)_1 \arrow[d, "\mathbf{TF}f_1"] \\
& B_1 \arrow[r, "v_1"'] & Y_1.
\end{tikzcd}
\tag{6.10}
$$

We describe below that given any diagram as in Equation (6.10), the dotted arrow $E \to E(g)_1$ exists; hence the above square is a pullback. First, we define a morphism $k : E \to A_0 \times A_0$ by the universal property of $A_0 \times A_0$ as the pullback over $v_0 \times v_0 : B_0 \times B_0 \to Y_0 \times Y_0$ and $f_0 \times f_0 : X_0 \times X_0 \to Y_0 \times Y_0$, given that the square from Equation (6.6) is a pullback:

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, dashed, "k"] & E(f)_1 \arrow[dr, "\mathbf{TF}f"] & \\
B_1 \arrow[dr, "{(d_1,d_0)}"'] & A_0 \times A_0 \arrow[r, "u_0 \times u_0"] \arrow[d, "g_0 \times g_0"'] \arrow[dr, phantom, "\lrcorner" very near start] & X_0 \times X_0 \arrow[d, "f_0 \times f_0"] \\
& B_0 \times B_0 \arrow[r, "v_0 \times v_0"'] & Y_0 \times Y_0 .
\end{tikzcd}$$

This in turn induces an arrow $t : E \to E(g)_1$ by the universal property of $E(g)_1$ as a pullback, given the commutativity of the outer square of the diagram below, which is by construction.

$$\begin{tikzcd}
E \arrow[drr, bend left, "q"] \arrow[dr, dashed, "t"] \arrow[ddr, bend right, "k"'] & & \\
& E(g)_1 \arrow[r, "\mathbf{TF}g_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B_1 \arrow[d, "{(d_1,d_0)}"] \\
& A_0 \times A_0 \arrow[r, "g_0 \times g_0"'] & B_0 \times B_0 .
\end{tikzcd}$$

It remains to show that $E(u,v)_1 t = p$, which will prove the commutativity of Equation (6.10), therefore showing that the inner square is a pullback. We show that $E(u,v)_1 t = p$ by using the universal property of $E(f)_1$ as a pullback: we show that the maps agree on both of its projections. This is shown in the diagrams below.

$$\begin{tikzcd}
E \arrow[rr, "t"] \arrow[dr, "q"] \arrow[dd, "p"'] & & E(g)_1 \arrow[dl, "\mathbf{TF}g_1"] \arrow[d, "{E(u,v)}"] \\
& B_1 \arrow[dr, "v_1"'] & E(f)_1 \arrow[d, "\mathbf{TF}f_1"] \\
E(f)_1 \arrow[rr, "\mathbf{TF}f_1"'] & & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, "t"] \arrow[dr, "k"] \arrow[dd, "p"'] & & E(g)_1 \arrow[dl, "{(d_1,d_0)}"] \arrow[d, "{E(u,v)}"] \\
& A_0 \times A_0 \arrow[dr, "u_0 \times u_0"'] & E(f)_1 \arrow[d, "{(d_1,d_0)}"] \\
E(f)_1 \arrow[rr, "{(d_1,d_0)}"'] & & X_0 \times X_0
\end{tikzcd}$$

Since pullbacks in $\mathbf{Cat}(\mathcal{E})$ are calculated pointwise, it follows that Equation (6.9) is a pullback square.

Hence, factorising the cartesian square in Equation (6.8), we obtain the cartesian square

$$\begin{tikzcd}
\mathbb{E}(\mathbf{W}g) \arrow[r, "{\mathbb{E}(u,v)}"] \arrow[d, "\mathbf{TFW}g"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{E}(\mathbf{W}f) \arrow[d, "\mathbf{TFW}f"] \\
\mathbf{Map}(g) \arrow[r, "v"'] & \mathbf{Map}(f)
\end{tikzcd} \tag{6.11}$$

By the pullback lemma, by pasting Equation (6.11) and Equation (6.7), we obtain the desired pullback square.

□

Proposition 6.5.7. *The factorisation of the awfs $(\mathbb{TC}, \mathbb{F})$ gives a stable functorial choice of path objects.*

Proof. It remains to show that we can lift the stable functorial factorisation $(\mathbf{TC}(\Delta_-), \mathbf{F}(\Delta_-)) : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}}$ to algebras— this follows from our definition of the stable functorial factorisation via the algebraic weak factorisation system; given an $\mathbb{F}$-algebra (f, α), $(\mathbf{TC}(\Delta_f), \sigma_{\Delta_f}) \in$ **TC-Coalg** and $(\mathbf{F}(\Delta_f), \kappa_{\Delta_f}) \in \mathbb{F}$-**Alg**. □

6.5.4 A type theoretic algebraic weak factorisation system

We are now able to state our main theorem. Recall the following definition.

Definition 6.5.8. [GL23, Definition 3.10] Let $\mathbb{C}$ be a category. A *type-theoretic algebraic weak factorisation system* consists of the following data:

1. An algebraic weak factorisation system $(\mathbb{L}, \mathbb{R})$ on $\mathbb{C}$ satisfying the exponentiability condition.

2. a functorial Frobenius structure on $(\mathbb{L}, \mathbb{R})$.

3. a stable, functorial choice of path objects on $(\mathbb{L}, \mathbb{R})$.

By [GL23, Theorem 3.12], any type theoretic algebraic weak factorisation system has an induced comprehension category which forms a model of MLTT with pseudo-stable choices of Σ-, Π- and $\mathtt{Id}$-types, and the right adjoint splitting of the induced comprehension category forms a model of MLTT with strictly stable choices of Σ-, Π- and $\mathtt{Id}$-types. We have therefore proven the following.

Theorem 6.5.9. *Let $\mathcal{E}$ be a locally cartesian closed locos with coequalisers. Then the awfs $(\mathbb{TC}, \mathbb{F})$ on the category* $\mathbf{Gpd}(\mathcal{E})$ *is equipped with the structure of a type theoretic awfs.*

Consequently, the right adjoint splitting of the comprehension category associated to $(\mathbb{TC}, \mathbb{F})$ is a model of MLTT with strictly stable choices of Σ-, Π- and $\mathtt{Id}$-types.

6.5.5 Using normal isofibrations

Theorem 6.5.9 provides a model of type theory in which the types are modelled by the cloven isofibrations. An alternative option would be a model in which types are modelled by *normal* isofibrations instead— these are isofibrations in which the chosen lift of an identity arrow is an identity. The type theory this models has an extra rule for compatibility with dependent refl terms; for $a : A \vdash b : B(a)$ the term $\mathrm{refl}_a : \mathtt{Id}_A(a, a)$ lifts to $\mathrm{refl}_b : \mathtt{Id}_{B(a)}(b, b)$. This approach is taken in [GL23] for the case that $\mathcal{E} = \mathbf{Set}$ and in [Agw25] for the case that $\mathcal{E}$ is a category of assemblies on a partial combinatory algebra.

In this work we study a cloven isofibration model was for two main reasons: firstly, to generalise Hofmann and Streicher's groupoid model, which uses cloven isofibrations. Secondly, to make this compatible with the natural model structure of $\mathbf{Cat}(\mathcal{E})$. However it is perfectly possible to find a model with normal fibrations too; in fact, the algebraic structure is simpler to define because of the one-step factorisation.

Let $f : \mathbb{X} \to \mathbb{Y}$. Define $\mathbf{N}(f)$ to be the morphism:

$$\mathbf{Map}(f) \to \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$$

and define $\mathbf{NTC}(f)$ to be the map $\mathbb{X} \to \mathbf{Map}(f)$. We define the point of $\mathbf{NF} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ denoted by $\omega : \mathbf{1} \Rightarrow \mathbf{NF}$ and the copoint of $\mathbf{NTC} : \mathbf{Cat}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ denoted by $\upsilon : \mathbf{NTC} \to 1$ to be the morphisms in $\mathbf{Cat}(\mathcal{E})^{\mathbf{2}}$ given component-wise by the commutative squares:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathbf{NTC}(f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\omega^f"] & \mathbf{Map}(f) \arrow[d, "\mathbf{NF}(f)"] \\ \mathbb{Y} \arrow[r, equal] & \mathbb{Y}. \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "\mathbf{NF}(f)"'] \arrow[dr, phantom, "\upsilon^f"] & \mathbb{X} \arrow[d, "f"] \\ \mathbf{Map}(f) \arrow[r, "\mathbf{NTC}(f)"'] & \mathbb{Y}. \end{tikzcd}$$

This give (co)pointed endofunctors $(\mathbf{NF}, \omega)$ and $(\mathbf{NTC}, \upsilon)$. In the following proposition, we describe how these extend to (co)monads.

Proposition 6.5.10. *Let $\mathcal{E}$ be an extensive category. In $\mathbf{Cat}(\mathcal{E})$, there is an algebraic weak factorisation system $(\mathbb{NTC}, \mathbb{NF})$ such that the $\mathbb{NF}$-algebras are precisely the normal isofibrations and $\mathbb{NTC}$-algebras are precisely the split monomorphic equivalences.*

Proof. We provide free algebra structure to $\mathbf{NF}(f)$ and free coalgebra structure to $\mathbf{NTC}(f)$.

We note that $\mathbf{Map}(\mathbf{NF}(f)) \cong (\mathbb{X} \times_Y \mathbb{Y}^{\mathcal{I}}) \times_{\mathbb{Y}} \mathbb{Y}^{\mathcal{I}}$. We induce a map $\mathbf{Map}(\mathbf{NF}(f)) \to \mathbf{Map}(f)$ which gives $\mathbf{NF}(f)$ free algebra structure; this is induced by the universal property of $\mathbf{Map}(f)$ as given below:

$$\begin{tikzcd} (\mathbb{X} \times_Y \mathbb{Y}^{\mathcal{I}}) \times_{\mathbb{Y}} \mathbb{Y}^{\mathcal{I}} \arrow[r] \arrow[dr, dashed, "\zeta^f"] \arrow[ddr, bend right] & \mathbb{Y}^{\mathcal{I}} \times_{\mathbb{Y}} \mathbb{Y}^{\mathcal{I}} \arrow[dr, "m"] & \\ & \mathbf{Map}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Y}^{\mathcal{I}} \arrow[d] \\ & \mathbb{X} \arrow[r, "f"] & \mathbb{Y}. \end{tikzcd}$$

It can be shown that $\zeta^f : \mathbf{NFNF}(f) \Rightarrow \mathbf{NF}(f)$ and as such $(\mathbf{NF}(f), \zeta^f)$ is an $\mathbf{NF}$-algebra and $\mathbf{NF}$ extends to a monad $\mathbb{NF} := (\mathbf{NF}, \omega, \zeta)$

On the other hand, $\mathbf{NTC}$-coalgebra structure is given to $\mathbf{NTC}(f)$ by the following map:

$$\begin{tikzcd} \mathbf{Map}(f) \arrow[drr, bend left, "\pi_{\mathbb{Y}^{\mathcal{I}}}"] \arrow[dr, dashed, "\lambda^f"] \arrow[ddr, bend right, "1\times\mathbf{NTC}(f)"'] & & \\ & \mathbf{Map}(\mathbf{NTC}(f)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Y}^{\mathcal{I}} \arrow[d] \\ & \mathbf{Map}(f) \times \mathbb{Y} \arrow[r] & \mathbb{Y} \times \mathbb{Y} \end{tikzcd}$$

This assembles into a map $\lambda^f : \mathbf{NTC}(f) \to \mathbf{NTCNTC}(f)$ and from this we can show that $\mathbb{NTC} := (\mathbf{NTC}, \upsilon, \lambda)$ is a comonad. $\square$

The proof of the following proceeds just like the proof of Theorem 6.5.9.

Theorem 6.5.11. *Let $\mathcal{E}$ be a locally cartesian closed extensive category. In* $\mathbf{Gpd}(\mathcal{E})$*, system* $(\mathbb{NTC}, \mathbb{NF})$ *is a type theoretic algebraic weak factorisation system.*

The benefits of this approach are that it as a simpler description of the algebraic structure required for modelling the type theory, and this makes it clear that fewer assumptions on $\mathcal{E}$ needed; in particular, no coequalisers or regularity/ exactness conditions are needed.

However, unlike our algebraic weak factorisation system, it is unclear whether this approach fits in with a model structure on $\mathbf{Cat}(\mathcal{E})$ in general. In [Agw25], an algebraic model structure is provided on $\mathbf{Gpd}(\mathbf{Asm}_A)$ in which this is one of the algebraic weak factorisation systems involved. In this model structure, the cofibrations are given by deformation retracts of complemented-inclusion on objects functors. It would be of interest to try and generalise this result to $\mathbf{Gpd}(\mathcal{E})$ for more general $\mathcal{E}$, although it should be noted that Agwu's proof makes use of a modified small object argument (which uses coequalisers) and that the category of assemblies over a partial combinatory algebra has enough projectives.

Proposition 6.5.10 clarifies a small error in [EKL05, Proposition 7.7], which claims that the trivial cofibrations of the natural model structure on $\mathbf{Cat}(\mathcal{E})$ are precisely the split monomorphic equivalences. However, in the absence of the Law of the Excluded Middle, it is not true that any split monomorphic equivalence is a complemented inclusion on objects, and so split monomorphic equivalences are not necessarily even cofibrations in this model structure; the trivial cofibrations must have the extra condition that they are complemented on objects (and hence also complemented on morphisms, and therefore also described as complemented equivalences). Proposition 6.5.10 shows us that the split monomorphic equivalences are instead the *normal* trivial cofibrations which lift against the normal fibrations.

6.6 2-categorical aspects

In this section, we exploit the work in Chapter 3 and [Bou10] to obtain a model of MLTT in elementary, $(2,1)$-categorical terms. We provide a list of necessary and sufficient axioms on a 2-category $\mathcal{K}$ that implies $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a locally cartesian closed locos with coequalisers, a result we believe is of independent interest. Then, we show that we can restrict this to $(2,1)$-categories to get a purely $(2,1)$-categorical model of MLTT.

We start by characterising a 2-dimensional version of local cartesian closure. Note that $\mathcal{E}$ being locally cartesian closed does not imply that $\mathbf{Cat}(\mathcal{E})$ is locally cartesian closed; this fails even in the case that $\mathcal{E} = \mathbf{Set}$. Instead of asking that all morphisms in $\mathcal{K}$ are exponentiable, we ask only that the groupoidal isofibrations are exponentiable. We also remark that an equivalent characterisation is that we ask for discrete opfibrations to be exponentiable, as is shown in Proposition 5.3.2.

Proposition 6.6.1. *Let $\mathcal{E}$ have finite limits and finite colimits. Then $\mathcal{E}$ is locally cartesian closed if and only isofibrations in* $\mathbf{Gpd}(\mathcal{E})$ *are exponentiable.*

Proof. Suppose $\mathcal{E}$ is a locally cartesian closed category with finite limits and coproducts. Then isofibrations in $\mathbf{Gpd}(\mathcal{E})$ are exponentiable by [NP19, Theorem 4.5].

Conversely, suppose isofibrations in $\mathbf{Gpd}(\mathcal{E})$ are exponentiable. Then for any internal functor $f : X \to Y$ in $\mathcal{E}$, we note that $\mathbf{disc}(f) : \mathbf{disc}(X) \to \mathbf{disc}(Y)$ is an isofibration in $\mathbf{Gpd}(\mathcal{E})$; this is easy to see representably. Hence, we have a right adjoint to the functor $\mathbf{disc}(f)^* : \mathbf{Gpd}(\mathcal{E})/\mathbf{disc}(Y) \to \mathbf{Gpd}(\mathcal{E})/\mathbf{disc}(X)$ which we denote by $\Pi_{\mathbf{disc}(f)} : \mathbf{Gpd}(\mathcal{E})/\mathbf{disc}(Y) \to \mathbf{Gpd}(\mathcal{E})/\mathbf{disc}(X)$. Construct the right adjoint to $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ by composing this series of right adjoints $(-)_0 \cdot \Pi_{\mathbf{disc}(f)} \cdot \mathbf{disc} : \mathcal{E}/X \to \mathcal{E}/Y$ which is left adjoint to $\Pi_0 \cdot \mathbf{disc}(f)^* \cdot \mathbf{disc} : \mathcal{E}/Y \to \mathcal{E}/X$ which we can easily verify is equal to $f^* : \mathcal{E}/Y \to \mathcal{E}/X$. For Π_0 to exist, we needed to assume that we had coequalisers. □

We can state this property of isofibrations representably for any 2-category $\mathcal{K}$.

Definition 6.6.2. Let $\mathcal{K}$ be a 2-category. We say a morphism $F : X \to Y$ in $\mathcal{K}$ is a groupoidal isofibration if for all $A \in \mathcal{K}$ the functor $\mathcal{K}(A, F) : \mathcal{K}(A, X) \to \mathcal{K}(A, Y)$ is an isofibration between groupoids.

For $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, this notion recovers exactly isofibrations in $\mathbf{Gpd}(\mathcal{E})$.

We are now able to give a complete characterisation of locally cartesian closed locoses in purely 2-categorical terms. Note that this is very similar to Theorem 5.4.2.

Theorem 6.6.3. *Let $\mathcal{E}$ be a locally cartesian closed locos with coequalisers. Then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ satisfies the conditions listed below, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$, in which $\mathcal{E}$ is a locally cartesian closed locos with coequalisers.*

1. *$\mathcal{K}$ has pullbacks and powers by* $\mathbf{2}$.
2. *$\mathcal{K}$ has codescent objects of categories internal to $\mathcal{K}$ whose source and target maps form a two-sided discrete fibration, and these are effective in the sense that they are isomorphic to the higher kernel of the codescent morphism.*
3. *Codescent morphisms are effective in $\mathcal{K}$.*
4. *Discrete objects in $\mathcal{K}$ are BO-projective, in the sense of Definition 2.4.18.*
5. *For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*
6. *$\mathcal{K}$ is lextensive.*
7. *Groupoidal isofibrations in $\mathcal{K}$ are exponentiable.*
8. *$\mathcal{K}$ has finite 2-colimits.*

Proof. Suppose $\mathcal{E}$ is a locally cartesian closed locos with coequalisers. Then $\mathbf{Cat}(\mathcal{E})$ satisfies (1)-(5) by [Bou10, Theorem 4.18], (6) by Lemma 4.4.2, (7) by [NP19, Theorem 4.5] and (8) by Theorem 3.5.2.

Conversely, suppose $\mathcal{K}$ is a 2-category satisfying (1)-(8). Again, (1)-(5) shows that $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$, (6) shows that $\mathcal{E}$ is lextensive by Lemma 4.4.2. (7) implies that $\mathcal{E}$ is locally cartesian closed by Proposition 6.6.1. (8) implies that $\mathcal{K}$ has a natural numbers object by Corollary 3.6.3. Moreover, it has coequalisers by Lemma 3.6.1. In a local cartesian closed category, having a natural numbers object is equivalent to having a parametrised list objects by [Joh02b, Theorem 2.5.17]; hence $\mathcal{E}$ is a locally cartesian closed locos with coequalisers. □

Below, we state a $(2,1)$ version of [Bou10, Theorem 4.18].

Theorem 6.6.4. *Let $\mathcal{E}$ be a category with pullbacks. Then the $(2,1)$-category $\mathbf{Gpd}(\mathcal{E})$ satisfies the conditions listed below. Conversely, if $\mathcal{K}$ is a $(2,1)$-category satisfying the conditions listed below, then there is a $(2,1)$-equivalence $\mathcal{K} \simeq \mathbf{Gpd}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$.*

1. *$\mathcal{K}$ has pullbacks and powers by $\mathcal{I}$.*
2. *$\mathcal{K}$ has codescent objects of groupoids internal to $\mathcal{K}$ whose source and target maps form a two-sided discrete fibration, and these are effective in the sense that they are isomorphic to the higher kernel of the codescent morphism.*
3. *Codescent morphisms are effective in $\mathcal{K}$.*
4. *Discrete objects in $\mathcal{K}$ are BO-projective, in the sense of Definition 2.4.18.*
5. *For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*

Proof. It is not hard to see that for any $\mathcal{E}$ with pullbacks, the $(2,1)$-category $\mathbf{Gpd}(\mathcal{E})$ satisfies the required conditions.

Conversely, suppose that $\mathcal{K}$ is a $(2,1)$-category satisfying the conditions of the Theorem. Note that conditions (1)-(5) are the conditions for [Bou10, Theorem 4.18] with the exceptions of (1) and (2), which we show are $(2,1)$-versions. For (1), it is not hard to see that in a $(2,1)$-category, having powers by $\mathcal{I}$ is equivalent to having powers by **2** when considered as a 2-category. Moreover, considering $\mathcal{K}$ as a $(2,1)$-category, natural transformations between functors $\mathbf{2} \to \mathcal{K}(X,Y)$ are all invertible. Therefore, if we show that categories internal to a $(2,1)$-category are groupoids, we can apply [Bou10, Theorem 4.18] and conclude the result as desired. For $\mathbb{X}$ a category internal to $\mathcal{K}$, we have a 2-cell $d_1 \Rightarrow d_0 : X_1 \to \mathbb{X}$ in $\mathcal{K}$ by the universal property of the catead; since $\mathcal{K}$ is a $(2,1)$-category, this is an isomorphism. Thus, by definition of internal natural isomorphism, we obtain a morphism $(-)^{-1} : X_1 \to X_1$ which satisfies the equations for $\mathbb{X}$ to be a not just a category but a groupoid internal to $\mathcal{K}$ (Definition 2.2.11. $\square$

We can therefore apply the work in the rest of this section abstractly to a $(2,1)$-category.

Theorem 6.6.5. *Let $\mathcal{K}$ be a $(2,1)$-category satisfying the axioms listed below. Then we have a $(2,1)$-equivalence $\mathcal{K} \simeq$ $\mathbf{Gpd}(\mathcal{E})$ with $\mathcal{E}$ a locally cartesian closed locos with coequalisers; consequently, there is a description of a type theoretic algebraic weak factorisation system on $\mathcal{K}$, and the right adjoint splitting of the comprehension category associated to this is a model of MLTT with strictly stable choices of Σ-, Π- and `Id`-types.*

1. *$\mathcal{K}$ has pullbacks and powers by $\mathcal{I}$.*
2. *$\mathcal{K}$ has codescent objects of groupoids internal to $\mathcal{K}$ whose source and target maps form a two-sided discrete fibration.*
3. *Codescent morphisms are effective in $\mathcal{K}$.*
4. *Discrete objects in $\mathcal{K}$ are BO-projective, in the sense of Definition 2.4.18.*
5. *For every object $A \in \mathcal{K}$, there is a BO-projective object $P \in \mathcal{K}$ and a codescent morphism $c : P \to A$.*

6. $\mathcal{K}$ *is lextensive.*

7. *Isofibrations in* $\mathcal{K}$ *are exponentiable.*

8. $\mathcal{K}$ *has finite* 2*-colimits.*

Proof. Let $\mathcal{K}$ be a $(2,1)$-category satisfying the above conditions. Conditions (1)-(5) imply that Theorem 6.6.4 can be applied and so $\mathcal{K} \simeq \mathbf{Gpd}(\mathcal{E})$. Including (6)-(8) allows us to apply Theorem 6.6.3 and conclude that $\mathcal{E}$ is a locally cartesian closed locos with coequalisers. By Theorem 6.5.9, we obtain a type theoretic algebraic weak factorisation system on $\mathcal{K}$ with the required properties. □

6.7 Examples and future directions

We conclude by spelling out what Theorem 6.3.15 and Theorem 6.5.9 means for certain examples. To the author's knowledge, internal groupoid models of MLTT have not been considered previously, with the exception of some upcoming work by Awodey and Emmenegger, and Agwu.

6.7.1 Realisability models

Fix A a partial combinatory algebra (PCA) and consider the category of assemblies $\mathbf{Asm}_A$ over this PCA (see [RR90] for a formal definition, or [Spe23] for a comprehensive review). This is a locally cartesian closed locos with coequalisers, and so $\mathbf{Gpd}(\mathbf{Asm}_A)$ gives a model of MLTT where types are realised by elements of the PCA. This gives a realisability model of MLTT, which is different from the approaches to 2-dimensional effective considered by both Speight, and Awodey and Emmenegger (see [AE25] for Awodey and Emmenegger's approach).

Moreover, the full subcategory of *modest* assemblies is also a locally cartesian closed locos with coequalisers, and so $\mathbf{Gpd}(\mathbf{Mod}_A)$ gives us a modest realisability model of MLTT.

The inclusion of $\mathbf{Gpd}(\mathbf{Mod}_A) \hookrightarrow \mathbf{Gpd}(\mathbf{Asm}_A)$ will be studied together with Sam Speight in future work which relates to the work of Chapter 7; we can treat the modest assemblies as being small relative to the assemblies, and so the model of type theory we obtain includes a univalent universe of modest 0-types.

It is interesting to note that the effective topos [Hyl82] is the exact completion of the category of assemblies over Kleene's first algebra. Moreover, there is a sense in which the map $\mathcal{E} \mapsto \mathbf{Gpd}(\mathcal{E})$ is a kind of $(2,1)$-exact completion of a 1-category [BG14, Corollary 61]. Hence $\mathbf{Gpd}(\mathbf{Asm}_{K_1})$ could be described as some kind of effective $(2,1)$-topos. Moreover, this will not be a Grothendieck $(2,1)$-topos as its 1-category of 0-truncated objects (which is the exact completion of $\mathbf{Asm}_{K_1}$) is not a Grothendieck 1-topos.

We could also apply this approach to the effective topos itself. The effective topos $\mathbf{Eff}$ [Hyl82] is a suitable environment for higher-order recursion theory, and is of interest to logicians interested in issues in computability. It is an elementary topos with natural numbers object. Therefore, we obtain an algebraic model structure on $\mathbf{Cat}(\mathbf{Eff})$. The associated internal model of MLTT given by $\mathbf{Gpd}(\mathbf{Eff})$ gives a setting in which higher-order recursion is baked into the types in some way.

6.7.2 Arithmetic Π-pretoposes

Arithmetic Π-pretoposes were constructed to be the syntactic categories for set-level MLTT [Str93]. They are univalent universes of dependent type theories that satisfy axiom K and are closed under the empty type, unit type, sum types, dependent sum types, propositional truncations, quotient sets, and parametrised natural numbers type. Therefore, the categorical constructions of this chapter could equivalently be formulated in the language of set-level MLTT, which is the internal language on an arithmetic Π-pretopos— see [Mai10, §3] for details. As a result, the statement and proof of Theorem 6.5.9 could be written using set-level MLTT. As a result, if we use set-level MLTT as our foundation for mathematics and construct the category of groupoids internal to this foundation, we still obtain a groupoid model of groupoid-level MLTT. This is a relative consistency result, akin to results in classical set theory which state that the consistency of ZF can be proven using ZF with a universe. Our result proves the consistency of groupoid-level MLTT, given the consistency of set-level MLTT.

One example of interest is taking $\mathcal{E}$ to be a model of Palmgren's constructive elementary theory of the category of sets [Pal12]. Such a thing is a constructive categorical model of Bishop's set theory. In such a setting, we get a constructive, intuitionistic version of Hofmann and Streicher's original groupoid model of MLTT. A concrete example of such a model of CETCS is given by the category of setoids [EP20].

6.7.3 Elementary toposes with a natural numbers object

Let $\mathcal{E}$ be an elementary topos with a natural numbers object. Any elementary topos with a natural numbers object is in particular a locally cartesian closed locos with coequalisers, and so both Theorem 6.4.16 and Theorem 6.5.9 apply and so we obtain an algebraic model structure on $\mathbf{Cat}(\mathcal{E})$ and also an algebraic model of MLTT given by the cloven isofibrations in $\mathbf{Gpd}(\mathcal{E})$. Below, we give multiple examples of categories that are elementary toposes with a natural numbers object.

Constructive versions of classical results

An example of an elementary topos with a natural numbers object is given by $\mathcal{E} = \mathbf{Set}$. In this case, we obtain a constructive version of the classical model structure on **Cat**; in fact, the Axiom of Choice is equivalent to the existence of the classical model structure on **Cat**— that it is a model structure in which the fibrations are isofibrations and the cofibrations are injective on objects functors. If we do not assume the Axiom of Choice, then complemented inclusion on object functors are injective on object functors, but without assuming the law of the excluded middle injective on object functors are not complemented inclusion on object functors. Moreover, the weak equivalences of this model structure are functors which are fully faithful and essentially surjective on objects, which are actual equivalences of categories if and only if the Axiom of Choice holds [FS90]. Therefore, assuming the Axiom of Choice, the (algebraic) model structure considered in this chapter agrees with the classical model structure on **Cat**, and so constructively this model structure is the correct one to consider.

In this case, Theorem 6.5.9 provides an algebraic version of Hofmann and Streicher's groupoid model [HS98]; forgetting the algebraic structure, the display maps are given by isofibrations. Another algebraic version of Hofmann and Streicher's model is given by [GL23, Theorem 4.5]. The algebraic weak factorisation system considered there has right algebras given by *normal* isofibrations, which are cloven isofibration in which identities lift to identities. Assuming the Axiom of Choice, this agrees with our case; lifts can be chosen to be normal. However, constructively, there is a difference in these models of MLTT.

In fact, isofibrations are equivalent to normal isofibrations in $\mathbf{Cat}(\mathcal{E})$ if and only if injective morphisms in $\mathcal{E}$ are complemented inclusions [BL24, Corollary 3.12]; therefore whenever $\mathcal{E}$ satisfies the internal Law of the Excluded Middle, this is a generalisation of the model given by [GL23, Theorem 4.5].

For a presheaf category

Let $\mathbb{C}$ be a small category. For $\mathcal{E} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, we have $\mathbf{Cat}(\mathcal{E}) \cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}]$. To see this, note that $\mathbb{X} \in \mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$ is

$$
\begin{tikzcd}
\ldots \arrow[r] & X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "i" description]
\end{tikzcd}
$$

with $X_0, X_1 \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. Due to the coherence conditions for sources, targets, composition that means that for each $c \in \mathbb{C}$, we have a small category:

$$
\begin{tikzcd}
\ldots \arrow[r] & X_1(c) \times_{X_0(c)} X_1(c) \arrow[r, shift left=2, "p_1(c)"] \arrow[r, "m(c)" description] \arrow[r, shift right=2, "p_2(c)"'] & X_1(c) \arrow[r, shift left=2, "d_1(c)"] \arrow[r, shift right=2, "d_0(c)"'] & X_0(c) \arrow[l, "i" description]
\end{tikzcd}
$$

which assembles into a functor $\mathbb{X} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$.

An internal functor $f : \mathbb{X} \to \mathbb{Y}$ is an internal isofibration if and only if

$$\mathbb{X}(\mathcal{I}) \to \mathbb{Y}(\mathcal{I}) \times_{Y_0} X_0$$

is a split epi in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. A map in a presheaf category is split epi if and only if it is levelwise split epi, so f is an internal isofibration if and only if for every $c \in \mathbb{C}^{\mathrm{op}}$

$$\mathbb{X}(\mathcal{I})(c) \to \mathbb{Y}(\mathcal{I})(c) \times_{Y_0(c)} X_0(c)$$

is a split epi in **Set**, where the equivalence $\mathbb{Y}(\mathcal{I} \times_{Y_0} X_0)(c) \cong \mathbb{Y}(\mathcal{I})(c) \times_{Y_0(c)} X_0(c)$ is because limits are computed pointwise in presheaf categories. This happens if and only if $f(c) : \mathbb{X}(c) \to \mathbb{Y}(c)$ is an isofibration in **Cat**. These are precisely the fibrations in the projective model structure on $[\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}]$, as described in [Hir02].

Given $F : \mathbb{X} \to \mathbb{Y}$ an equivalence of internal categories in $\mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$, then for every $c \in \mathbb{C}$, it is indeed true that $F(c) : \mathbb{X}(c) \to \mathbb{Y}(c)$ is an equivalence of categories, which are the weak equivalences in the projective model structure on $[\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}]$. However, the it is not always true that the projective weak equivalences are equivalences of internal categories in $\mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$. Let $F : \mathbb{X} \to \mathbb{Y}$ be such that for all $c \in \mathbb{C}$ we have that $F(c) : \mathbb{X}(c) \to \mathbb{Y}(c)$ is an equivalence of categories. Hence, for all $c \in \mathbb{C}$, we have $(F(c))^{-1} : \mathbb{Y}(c) \to \mathbb{X}(c)$. Whilst it is possible to extend this to an assignment on morphisms $f : c \to c'$ in $\mathbb{C}$, it does not seem possible to make this assignment strictly functorial; only up to isomorphism. Therefore, despite having the same fibrations, the model structure of this work is not the same as the projective model structure on $[\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}]$.

For similar reasons, the cofibrations are also not always simply pointwise. Given a cofibration in $\mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$, this is indeed pointwise a cofibration in **Cat**. Conversely, assuming the Law of the Excluded Middle in **Set**, $F : \mathbb{X} \to \mathbb{Y}$ is levelwise injective-on-objects if and only if for all $c \in \mathbb{C}$, we have that $F(c) : \mathbb{X}(c) \to \mathbb{Y}(c)$ is injective-on-objects in **Cat**, which happens if and only if for all $c \in \mathbb{C}$ the map $F_0(c) : X_0(c) \to Y_0(c)$ is an injection in **Set**, which happens if and only if $F_0 : X_0 \to Y_0$ is a monomorphism in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, which happens if and only if $F : \mathbb{X} \to \mathbb{Y}$ is a monomorphism-on-objects in $\mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$. The cofibrations of the model structure that we study are the complemented-inclusion-on-objects internal functors. Therefore cofibrations are equally described as pointwise cofibrations if and only if the internal Law of the Excluded Middle holds in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. We note that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ satisfies LEM if and only if $\mathbb{C}$ is a groupoid. So if $\mathbb{C}$ is a groupoid, the fibrations and the cofibrations are both calculated pointwise.

However, in the algebraic setting, this reliance on LEM disappears. The problem of making strictly functorial choices is handled by the algebraic data. Given an ordinary weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathbb{M}$, the classes of pointwise $\mathcal{L}$-maps and pointwise $\mathcal{R}$-maps do not usually form a weak factorisation system on $[\mathbb{C}, \mathbb{M}]$; this is due to the same problem that we encounter above; the lifts can be defined but not chosen naturally. However, given an *algebraic* weak factorisation system $(\mathbb{L}, \mathbb{R})$ on a category $\mathbb{M}$, there is a monad whose algebras are precisely the pointwise $\mathbb{R}$-algebras and a comonad whose coalgebras are precisely the pointwise $\mathbb{L}$-algebras, and these do indeed form an algebraic weak factorisation system on $[\mathbb{C}, \mathbb{M}]$ [Rie11, §4.3]. [Rie11, Example 4.4] explains that the algebraic weak factorisation systems of the algebraic version of Lack's trivial model structure are "pointwise awfs". Note that this is not the same as the algebraic projective model structure on $[\mathbb{C}, \mathbb{M}]$ described in [Rie11, §4.4], whose underlying model structure of this is the projective model structure, and the weak factorisation systems of the algebraic projective model structure are not the algebraically pointwise awfs.

Therefore the algebraic model structure considered in this thesis on $\mathbf{Cat}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$ is calculated pointwise in an algebraic sense— explicitly, a map $f : \mathbb{X} \to \mathbb{Y}$ can be equipped with the structure of being an algebraic fibration (resp. algebraic trivial fibration, algebraic cofibration, algebraic trivial cofibration) if and only if for all $c \in \mathbb{C}$ $f(c) : \mathbb{X}(c) \to \mathbb{Y}(c)$ can be equipped with the structure of being an algebraic fibration (resp. algebraic trivial fibration, algebraic cofibration, algebraic trivial cofibration) in **Cat**.

Any presheaf category has a natural numbers object given by the constant sheaf on the natural numbers. It is also an elementary topos, and therefore a locally cartesian closed locos with coequalisers, and so Theorem 6.5.9 can be applied and so $\mathbf{Gpd}([\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]) \cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Gpd}]$ gives us an indexed model of MLTT in which a type is actually a family of types coherently indexed by the elements and morphisms of $\mathbb{C}$. This could be of interest for further study.

For a Grothendieck topos

A Grothendieck topos

$$\mathcal{E} \underset{\longrightarrow}{\overset{\longleftarrow}{}} [\mathbb{C}^{op}, \mathbf{Set}]$$

can be described as a reflective full subcategory of a presheaf [MM94] and therefore has a natural numbers object given by the sheafification of the natural numbers object in $[\mathbb{C}^{op}, \mathbf{Set}]$.

In this case, our model structure does not usually agree with Joyal and Tierny's model structure on strong stacks [JT06]; it agrees if and only if $\mathcal{E}$ satisfies the external Axiom of Choice; this occurs if and only if fully faithful and essentially surjective on objects functors (their trivial fibrations) are equivalences of categories— in that case, they are equivalent to split epimorphic-on-objects and fully faithful functors— see Theorem 5.5.6 part (4). They do not give an explicit description of their fibrations, rather characterising them by their lifting property with respect to trivial fibrations. Hence, in the case that $\mathcal{E}$ satisfies the external Axiom of Choice, the work of this chapter gives an explicit description of the fibrations in their model structure. If the Axiom of Choice does not hold, then their weak equivalences are strictly contained in ours. In this case, our cofibrations are strictly contained in theirs as their cofibrations are injective on objects. Consequently, their fibrations are strictly contained in ours.

For $\mathcal{E}$ a Grothendieck topos, the associated model of MLTT given by $\mathbf{Gpd}(\mathcal{E})$ can be thought of as having geometric types that are glued together from more simple indexed types.

For any Grothendieck topos satisfying the Axiom of Choice we have proven that strong stacks give a model of MLTT. Moreover, since Diaconescu's Theorem can be formulated in any topos, such a Grothendieck topos also satisfies the law of the excluded middle. Therefore, by [BL24, Corollary 3.12], cloven isofibrations are equivalently described as normal isofibrations.

6.7.4 n-fold Categories and internal n-fold Categories

For $\mathcal{E} = \mathbf{Cat}$, we recover the model structure on **DblCat** studied in [FPP08, Section 8.2]. As such, we obtain an algebraic version of this.

We note that the work of this chapter extends these results; if $\mathcal{E}$ is lextensive, locally finitely presentable and cartesian closed, then $\mathbf{Cat}(\mathcal{E})$ is lextensive, locally finitely presentable and cartesian closed. Hence, by Remark 6.2.1, we obtain an algebraic model structure on $\mathbf{DblCat}(\mathcal{E}) := \mathbf{Cat}(\mathbf{Cat}(\mathcal{E}))$. This is a novel observation. Moreover, we see that $\mathbf{DblCat}(\mathcal{E})$ is lextensive, locally finitely presentable and cartesian closed, so iterating this result gives us an algebraic model structure on $n\mathbf{FoldCat}(\mathcal{E})$ for any $n \in \mathbb{N}$. In particular, we obtain (algebraic) model structures on $n\mathbf{FoldCat}$ for each n. This is different to the model structure on $n\mathbf{FoldCat}$ described in [FP10], as it is different even when $n = 2$, as noted in the introduction of [FP10]. Thus these examples are novel and of possible interest for further study.

However, it should be noted that by the association of our model structure with Lack's trivial model structure on a 2-category, this in some sense takes a 2-dimensional view of $\mathbf{nFoldCat}(\mathcal{E})$, which perhaps neglects the higher-dimensional structure.

We cannot use this recursion technique to get higher dimensional models of MLTT, since $\mathcal{E}$ locally cartesian closed does not imply $\mathbf{Cat}(\mathcal{E})$ is locally cartesian closed. Indeed, for $\mathcal{E} = \mathbf{Set}$, it is well-known that **Cat** is not locally cartesian closed— see Example 5.3.1.

Chapter 7

Class $(2,1)$-categories

7.1 Introduction

In [Law70; Law73] Lawvere suggested the existence of a generalised logic valued in the 2-category **Cat** with, for example, quantification given by Kan extension. A notable attempt at giving a first order axiomatisation of the 2-categorical properties of the category of categories was given in [Web07], which presents a categorification of the notion of an elementary topos and defines an elementary 2-topos to be a cartesian closed 2-category with finite 2-limits, a duality involution and a discrete opfibration classifier.

One conceptual problem with this approach is that it is not clear what situation is being axiomatised; indeed, they suggest that we require cartesian closure, but whether the category of locally small categories or the very large category of large categories is cartesian closed depends on the choice of how to handle large objects in set-theoretic foundations. Indeed, it is well known that if C and D are two categories then we have the following properties: if C and D are both small then $[C, D]$ is small too. If C is small but D is not, then $[C, D]$ exists but is not necessarily small. If both C and D of are large, then the existence of this functor category requires additional assumptions in our metatheory such as Grothendieck universes.

On the other hand, if we wanted not to consider issues of size, we might consider **Cat**, the 2-category of small categories. This *is* cartesian closed, but does not have the desired discrete opfibration classifier; it has a classifier for discrete opfibrations with finite fibres, but finiteness is not a first-order property, so there is not much structural meaning to studying such a classifier. Axiomatisations of the 2-category **Cat** are explored Chapters 4 and 5 in both classical and constructive logic.

As such, it makes sense to explore the prospect of a non-cartesian closed elementary 2-topos; having a discrete opfibration classifier is inherently interlinked with size issues. In the one-dimensional case, a similar problem arises when considering axiomatising the category of classes as opposed to sets. This was originally explored in [JM95], and subsequently developed by many others in an area of study known as *Algebraic Set Theory*— see [Awo08] for a comprehensive overview. By specifying a class of "small maps", it is possible to have a non-cartesian closed category with cartesian closure for small objects. This captures the behaviour of forming function classes in the logical foundations of von Neumann-Gödel-Bernays set theory, in which we have the axiom of Bounded Separation as opposed to Full Separation; consequently it is possible to form the class of functions between a set and a class and the set of functions between two sets, but not the class of functions between two proper classes.

The development of Homotopy Type Theory as well as higher category theory has made it clear that the higher dimensional analogue of sets should be *groupoids* rather than categories, and categories are instead the higher dimensional analogue of posets, as suggested by Voevodsky [Voe14]. Other explorations of higher-dimensional logics such as the recent work of Helfer [Hel24] and the programme of Cisinski et. al. [Cis+25] have noted the importance of groupoid objects in their explorations. Moreover, it seems that the consideration of non-groupoid objects significantly complicates the theory— see for example attempts at studying directed type theory in contrast to its undirected version [LH11].

As such, the approach presented here is to consider a strict $(2,1)$-dimensional categorification of a category with a class of small maps of [JM95] with the goal of capturing the $(2,1)$-categorical properties of the large $(2,1)$-category of groupoids in von Neumann-Gödel-Bernays set theory. In such a $(2,1)$-category, which we call a *class* $(2,1)$*-category*, we can form the $(2,1)$-category of small objects. We prove that we have cartesian closure for small objects (Proposition 7.5.19), and that we can form the large object of maps between a small object and a large one (Theorem 7.5.20).

Other notable approaches to axiomatising the 2-category of categories are given by the notion of an elementary cosmos [Str80], which consists of a 2-category $\mathcal{K}$ together with a way to associate a presheaf object $\mathbf{Psh}(\mathbb{X}) \in \mathcal{K}$ to each object $\mathbb{X} \in \mathcal{K}$. Relatedly, there is the notion of a Yoneda structure on a 2-category [SW78], which in a similar spirit to that of a class $(2,1)$-category isolates a class of "admissible" maps, which are to be thought of as locally small maps and show that the important universal properties of the Yoneda embedding can be reconstructed for any Yoneda structure. We note that elementary cosmoses suffer from the same problems of size that were noted for Weber's elementary 2-toposes. The issues of size are handled by a Yoneda structure's distinguished class of admissible maps; however it seems that they lack the ability to express an internal logic. We show that every class $(2,1)$-category has an associated Yoneda structure (Theorem 7.5.26) and therefore we have a working notion of presheaf object (Corollary 7.5.21). It does not seem to be the case that any Yoneda structure forms a class $(2,1)$-category.

One of our main results in this work is Theorem 7.5.28 which shows that the $(2,1)$-category of small objects models 1-dimensional Martin-Löf type theory (MLTT) with pseudo-strict Σ-, Π- and $\mathtt{Id}$-types, a higher dimensional analogue of the result that in a 1-category with a class of small maps, the small objects forms an arithmetic Π-pretopos, which gives a model of a weak version of Zermelo-Fraenkel set theory and a model of 0-dimensional MLTT. We prove this by showing that the sub-$(2,1)$-category of small objects in a class $(2,1)$-category is $(2,1)$-equivalent to a $(2,1)$-category of internal groupoids, by showing that it satisfies the conditions required to apply Theorems 6.6.4 and 6.6.5. By the explicit description of the model of type theory that we constructed in Chapter 6, we are able to prove that the 0-types are modelled by the discrete opfibrations; as such, the small discrete opfibration classifier which is part of the structure of a class $(2,1)$-category provides a (univalent) universe (in the sense of Hofmann and Streicher) for small 0-types in this model (Theorem 7.5.32). Interestingly, this mirrors the situation constructed in the HoTT-Lean program [Hua+26], which formalises the meta-theory of Homotopy Type Theory in the proof assistant Lean.

One of the key ideas for categorification is that we replace "small maps" with "small discrete opfibrations" and we replace "regular epimorphisms" with "codescent morphisms", which is just one way in which we can categorify the notion of a regular epimorphism to the strict two dimensional case— see the discussion of two dimensional exactness in Section 2.4. We use the theory of BO-regularity and exactness to prove results about class $(2,1)$-categories, and as such contribute to the theory of two dimensional exactness. One result we prove which we believe to be of independent interest is a BO-regular categorification of the so-called "reverse pullback lemma" of [Gra21, Lemma 1.15]. This is Lemma 7.5.23, and was proven together with Adrian Miranda and Raffael Stenzel.

7.1.1 Outline

We begin in Section 7.2 by reviewing the axioms of von Neumann-Bernays-Gödel set theory, which is a standard foundation of mathematics that is used to handle classes as well as sets. We use this to formally prove some categorical properties of the category of classes. This allows us, in Section 7.3 to formally prove some results about the $(2, 1)$-category of large groupoids. We prove results which allow us to exhibit the $(2, 1)$-category of large groupoids as the archetypal example of a class $(2, 1)$-category.

We define the notion of a class $(2, 1)$-category in Section 7.4, and move on in Section 7.5 to prove some results about them. Our main result is that the full sub-$(2, 1)$-category of small objects for a class category gives a model of intensional Martin-Löf type theory, which we prove in Theorem 7.5.28.

In Section 7.7, we investigate to what extent the axioms are stable under slicing. We first prove some stability results for strict slicing, and then move on to the more complicated but more natural setting of isofibrational slicing, showing that all but one of these axioms are stable in this case.

We finish in Section 7.8 by providing examples. In one dimensions, the key examples that are not the category of classes are the category of presheaves on a 1-category, the category of sheaves on a 1-category with respect to a Grothendieck topology, and the effective topos. In [BM08; Awo+09; BM12], it is also shown that these concepts can be internalised to any class category, giving more examples.

Interesting examples of class $(2, 1)$-categories are given by groupoids internal to a class 1-category and the $(2, 1)$-category of prestacks on a 1-category. In addition, we show that any $(2, 1)$-category of semi-strict stacks satisfies all the axioms except for one of them. This requires for us to develop some basic theory for $(2, 1)$-categories of stacks, proving that they have certain colimits and satisfy some exactness properties, results we believe are of independent interest.

7.2 Categorical properties of the category of classes

In this section, we prove properties of the category of classes. Whilst most of this is probably known, we present these results explicitly using the axioms of von Neumann-Bernays-Gödel set theory for completeness. Moreover, in this way, we can make clear which of these argument work in other metatheories— for example in more constructive or intuitionistic metatheories. We also note that if we take different metatheories, we can prove different results about the category of classes: whether one uses classical reasoning, intuitionistic reasoning or constructive reasoning allows one to determine whether the category of classes is exact or merely regular, for instance. As such, we emphasise those facts which are true in all of these theories, so that we can have a somewhat agnostic approach. Being able to handle the breadth of these metatheories is one of the impressive aspects of Algebraic Set Theory. We note that investigations into the categorical properties of the category of classes in NBG were considered in [For04], and categories of classes in more general foundations have been pursued in [AFW06], both in the context of Algebraic Set Theory.

We make precise what the notion of a class means in von Neumann-Bernays-Gödel set theory (NBG) and explain how working in this framework addresses issues of paradox. We prove categorical properties about the category of classes; of particular interest to us will be that it is regular and not cartesian closed.

7.2.1 von Neumann-Bernays-Gödel Set Theory

In order to study the categorical properties of the category of class defined in von Neumann-Bernays-Gödel set theory (NBG), we must work in an extended version of NBG which we call NBG+ and define below. For NBG, we take as a standard reference [Men09]. NBG consists of a first order theory with a binary predicate $\in$. The objects of NBG are defined by first-order formulas and are called *classes*. We call a class x a *set* if there exists a class A with $x \in A$. Classes which are not sets are called proper classes. Below, we denote sets with lower-case letters and general classes by upper-case letters. We postulate the axioms (NBG1)-(NBG13) below.

(NBG1) (Extensionality) Two classes A and B are equal if and only if $x \in A \iff x \in B$, i.e. that they have the same elements.

(NBG2) (Pairing) If x and y are sets then there is a class $\{x, y\}$ with $a \in \{x, y\} \implies (a = x) \vee (a = y)$.

Using these axioms, we can define the notion of an ordered pair, which we write $\langle x, y\rangle$.

(NBG3) (Null set) $(\exists\emptyset)(\forall X)X \notin \emptyset$.

(NBG4) ($\in$-relation) There exists a class **I** such that for any sets X, Y we have $X \in Y \iff \langle X, Y\rangle \in \mathbf{I}$.

(NBG5) (Intersection) If A is a class and B is a class then there exists a class $A \cap B$ for which $x \in A \cap B \implies (x \in A) \wedge (x \in B)$.

(NBG6) (Complements) If A is a class then there exists a class A° such that for all a we have $a \in A^\circ \iff a \notin A$.

(NBG7) (Domain) If A is a class, there exists a class $\mathbf{Dom}(A)$ such that for all u we have $u \in \mathbf{Dom}(A)$ if and only if $\exists v$ with $\langle u, v\rangle \in A$.

(NBG8) If A is a class, there exists a class Z such that $\langle a, b\rangle \in Z \iff a \in A$.

(NBG9) If A is a class then there exists classes Y and Z such that $\langle a, b, c\rangle \in A \iff \langle b, c, a\rangle \in Y \iff \langle a, c, b\rangle \in Z$.

From (NBG1)-(NBG9) we can prove the following, which tells us that classes are built out of first-order formulae.

Theorem 7.2.1 (Class Existence Theorem)**.** *For any first-order formula ϕ built up only out of $\wedge, \vee, \implies, \neg$ and quantifiers, there is a class A in which $x \in A$ if and only if $\phi(x)$ holds.*

Proof. This is [Men09, Proposition 4.4]. □

In particular, we can define ϕ to be the formula $x = x$ and we obtain a universe V of all sets.

Thus, from these first axioms, we can define the cartesian product.

Definition 7.2.2. We define $A \times B := \{x : \phi(A, B, x)\}$ in which

$$\phi(A, B, x) := (\exists a \in A)(\exists b \in B)x = \{a, b\}.$$

First we note that the class defined above exists by (NBG2) and Class Existence Theorem (Theorem 7.2.1). The formula precisely says that elements of $A \times B$ are pairs (a, b) with $a \in A$ and $b \in B$. From this, we can define the notion of function between classes.

Definition 7.2.3. Let $F \subseteq A \times B$ in which for all $a \in A$ there is a unique $b \in B$ for which $(a, b) \in F$. We call F a *class function* (or simply a *function*) and write $F : A \to B$. We define a bijection to be a class function which is injective and surjective.

Remark 7.2.4. In some formal treatments of NBG (and similarly ZFC), the notion of a function is slightly different; a function is defined to be a class F which consist of ordered pairs and such that $\langle x, y\rangle \in F \wedge \langle x, z\rangle \in F \implies y = z$. In particular, it does not have specified domain and codomain. By the Domain Axiom (NBG7), we can show that there exists some domain for the function, but there is still no need to specify a codomain. This becomes important in stating the Axiom of Replacement (NBG13) below. We choose to present the definition of a function as having specified domain and codomain partly because it is philosophically natural to the category theorist, and partly because it allows us to easily define the category of classes. We make the following definition to emphasise this distinction.

Definition 7.2.5. A *definable mapping* is a class F which consist of ordered pairs and such that

$$\langle x, y\rangle \in F \wedge \langle x, z\rangle \in F \implies y = z.$$

(NBG10) (Sum sets) If x is a set then there exists a set $\bigcup x$ such that $b \in \bigcup x \iff \exists a \in x$ with $b \in a$.

(NBG11) (Power sets) If x is a set, then there exists a set $\mathbf{Pow}(x)$ for which $b \in \mathbf{Pow}(x) \iff b \subseteq x$.

(NBG12) (Infinity) There exists a set $\mathbb{N}$ such that $\emptyset \in \mathbb{N}$ and $x \in \mathbb{N} \implies x \cup \{x\} \in \mathbb{N}$.

(NBG13) (Replacement) For any definable mapping F, we have

$$\forall x \exists w \forall u (u \in w) \iff (\exists v)(\langle v, u\rangle \in F \wedge v \in x)$$

Remark 7.2.6. Note that the Axiom of Sum Sets (NBG10) also implies that the disjoint union over a set of sets is also a set, since disjoint unions can be reformulated as unions of ordered pairs.

Remark 7.2.7. Note that the Axiom of Replacement regards definable mappings, which are more general than class functions since the objects of its consideration need not have specified domain and codomain. By applying this to actual functions, a consequence of this axiom is that the image of a set under a class function is itself a set.

These axioms imply that the subset of a set is a set, an axiom which is often present in similar axiomatisations of set theory.

Lemma 7.2.8. *(Subsets) Let x be a set and let $A \subseteq x$. Then A is a set.*

Proof. This is [Men09, Exercise 4.18, Corollary 4.6 (b)]. □

Corollary 7.2.9. *(Bounded Separation) Let ϕ be a first order formula and let x be a set. Then*

$$\{y \in x : \phi(y)\}$$

is a set.

We have presented a finite list of axioms; if the reader is unconcerned with a finite axiomatisation (NBG3)-(NBG8) can be replaced by assuming the Class Existence Theorem (Theorem 7.2.1).

Creating a distinction between sets and proper classes prevents paradoxes, as our axioms allow for less manipulation of proper classes. In particular, note that proper classes cannot be elements of a class, otherwise it is a set by definition. In this way, we escape Russell's paradox: let $Y := \{x \mid x \notin x\}$ which exists by the Class Existence Theorem (Theorem 7.2.1). Russell's paradox goes on to show that $Y \in Y \iff Y \notin Y$, therefore proving a contradiction. However, in NBG this contradiction only proves that Y is not a set but rather a proper class; in the previous line, for $Y \in Y$ to be considered, we are tacitly assuming that Y is a set and so the contradiction simply disproves this assumption. Other set theoretic paradoxes such as the ones of Cantor and Burali-Forti are avoided in a similar fashion.

NBG is a conservative extension of ZF which is finitely axiomatisable. A full and more formal account of it and its properties is given in [Men09, Chapter 4].

Most mathematics occurs using sets, however occasionally it goes outside of this world and so it is incredibly useful to have a logic which can deal with proper classes as well. It gives us a set of rules for safely manipulating proper classes. This is of great importance in Category Theory, since most of the categories we often consider are not small (e.g. **Set**, **Cat**, **Group**, **Vect**...) and so exists outside of ZF.

7.2.2 The category of classes

We wish to form a category **Class** whose objects are classes and whose morphisms are class functions. In ZF, in order to study the category of sets we must work in some metatheory with a concept of classes (for example in NBG itself, or alternatively in ZF with a Grothendieck universe). Similarly, in NBG, in order to study the category of classes, we must work in a metatheory in which we can have an object of classes. There are various ways to do this, but in this chapter, we will work in a conservative extension of NBG which is a first order theory with a binary predicate $\in$. The objects of are called *collections*. As in NBG, we call a collection X a class if there exists a collection $\mathbf{X}$ such that $X \in \mathbf{X}$. In this theory, which we denote NBG+, a collection x is a set if there exists a class X and therefore a collection $\mathbf{X}$ such that $x \in X \in \mathbf{X}$. We require collections to satisfy even fewer axioms than classes— in fact, we will only need axioms (NBG1)-(NBG9). This is so that we can define the notion of collection function in the same way we do class function.

We define a category to have a collection of objects and a collection of morphisms together with the appropriate maps which satisfy the usual axioms of a category.

We define the category **Class** to have as its objects, the collection of classes and as its morphisms the collection of class morphisms. We note that the category **Class** is really very large; given two classes X, Y, there is not even a class of morphisms between them— indeed, this is one of the key categorical aspects of the theory, which we shall explore below.

7.2.3 Categorical properties

In this subsection, we explore the categorical properties of **Class** in order to prove 2-categorical properties of **CAT**. Some of the arguments of this section work similarly to the corresponding proof for the category of sets in ZF, but occasionally have to be careful that things well-defined. We find out that **Class** is not an elementary topos like the category of sets in ZFC; however, it shares many of the nice properties of an elementary topos.

Lemma 7.2.10. **Class** *has an initial and a terminal object.*

Proof. The empty class exists by (NBG3), and this gives the initial object; it is not hard to see that $\emptyset \times A \cong \emptyset$ for all classes A and so there is trivially a unique class function $! : \emptyset \to A$. By (NBG12), we know that $\emptyset$ is a set. Therefore, by (NBG9), we can form **Pow**$(\emptyset)$; this has precisely one element. It is again easy to check that for any non-empty A, $A \times$ **Pow**$(\emptyset) \cong A$, and so there is a unique choice of map from any class into **Pow**$(\emptyset)$; this is therefore the terminal object of **Class** and we denote it **1**. $\square$

Proposition 7.2.11. **Class** *has finite limits.*

Proof. We have already shown that **Class** has products; a small modification to the formula defining products gives the existence of pullbacks. Let $f : A \to C \leftarrow B : g$ be a cospan. Define $\psi(A, B, C, f, g, x) := (\exists a \in A)(\exists b \in B)x = \{a, b\} \wedge fa = gb)$. Define $A \times_C B := \{x : \psi(A, B, C, f, g, x)\}$, which exists by comprehension (Theorem 7.2.1). It is easy to check that this defines the pullback of f and g; by Lemma 7.2.10, we are done. $\square$

Proposition 7.2.12. **Class** *has coproducts.*

Proof. Consider the formula $\phi(A, B, x) := (x = a \in A) \vee (x = b \in B) \wedge \neg((x = a \in A) \wedge (x = b \in B))$. Then define $A + B := \{x : \phi(A, B, x)\}$. This has the categorical property of the coproduct. $\square$

Proposition 7.2.13. **Class** *is extensive in the sense of Definition 4.4.1.*

Proof. We have shown that all pullbacks exists; we show that pullbacks of coproducts injections are disjoint and stable under pullback. First, let A, B be classes and ι_A, ι_B be the evident coproduct injections into $A + B$, we note that their pullback is given by those x which satisfy the formula

$$\psi(A, B, A + B, \iota_A, \iota_B) := (\exists a \in A)(\exists b \in B)(x = \{a, b\} \wedge \iota_A(a) = \iota_B(b))$$

but there are no such classes which satisfy this formula as if they did, then it would imply that there is an $x \in A + B$ with $x = a \in A \wedge x = b \in B$, contradicting the definition of $A + B$.

Next, consider the diagram below:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "r"] & Y \arrow[l, "g"'] \arrow[d, "t"] \arrow[dl, phantom, "\llcorner", very near start] \\
A \arrow[r, hook, "\iota_A"'] & A + B & B \arrow[l, hook', "\iota_B"]
\end{tikzcd}$$

We want to show that $Z \cong X + Y$. Note that there is a unique map $X + Y \to Z$ by the universal property of the coproduct. Define $Z \to X + Y$ as follows. If $z \in Z$ has the property that $h(z) = a \in A$, then there exists a unique $x \in X$ with $f(x) = z$ and $r(x) = a$; if instead $h(z) = b \in B$ then there exists a unique $y \in Y$ with $f(y) = z$ and $t(y) = b$. This uses the universal property of the pullback. This assignment define a class function $Z \to X + Y$ which is well-defined as coproducts are disjoint. This composes with the universal map $X + Y \to Z$ to make the identity on both sides; by Lemma 2.5.4, this proves the claim. □

Proposition 7.2.14. **Class** *is regular.*

Proof. Let $F : X \to Y$ in **Class**. Define $e(F) : X \to F$ by the rule $x \mapsto \langle x, Fx\rangle$ and $m(F) : F \to Y$ by sending $\langle x, y\rangle \mapsto y$. It is easy to prove that $m(F)$ is a monomorphism. We claim that $e(F)$ is a regular epimorphism. Define R to be the equivalence relation $xRx' \iff fx = fx'$. Then, for any $G : X \to Z$ with $Gx = Gx' \iff Fx = Fx'$, we can define a unique class function $F \to Z$ by the assignment $\langle x, Fx\rangle \mapsto Gx$, which is well-defined since F and G are class functions, and R guarantees that

$$\langle\langle x, Fx\rangle, Gx\rangle = \langle\langle x, Fx\rangle, Gx'\rangle \implies Gx = Gx'.$$

Next, we show that these factorisations are preserved under pullback. Consider the cospan $F : X \to Y \leftarrow A : G$. We must show that $G^*(F) \cong G^*(m(F))$, but this is evident when writing out these sets explicitly:

$$G^*(F) := \{(a, x) : Ga = Fx\}$$

$$G^*(m(F)) := \{(a, \langle x, Fx\rangle : Ga = Fx\}.$$

□

Corollary 7.2.15. **Class** *has coequalisers of kernel pairs.*

Remark 7.2.16. The question of whether **Class** is exact depends on whether we assume the Axiom of Foundation in NBG or not— which says that for all non-empty classes X there exists a set $y \in x$ such that $x \cap y = \emptyset$. Indeed, the Axiom of Foundation is independent of the other axioms given [Men09, §4.5]. As explained in [Acz78; Acz88], this axiom is undesirable from a constructive point of view since it implies many instances of the Law of the Excluded middle. As such, we have left the Axiom of Foundation out of our formulation.

If we assume the Axiom of Foundation, we can define the notion of a rank of a set (formally as a class function $\mathbf{rank} : V \to \mathbf{Ord}$), and from this we can build a cumulative hierarchy of sets $V_\alpha := \{x : \mathbf{rank}(x) < \alpha\}$. This allows us to construct coequalisers of equivalence relations. Given a class X and an equivalence relation R on it, we can form equivalence classes $[x] := \{x' : xRx'\}$. The naive approach to forming the quotient X/R would be to form the class consisting of these equivalence classes— indeed, this works if X is itself a set. However, if X is a proper class, then $[x]$ may itself be a proper class, and therefore cannot be the element of a class. When assuming the Axiom of Foundation, a trick due to Dana Scott [Sco55] allows us to replace $[x]$ by just those elements of the equivalence classes with least rank.

$$\widetilde{[x]} := \{x' : xRx' \wedge \textbf{rank}(x') \leq \textbf{rank}(y)(\forall y)xRy\}$$

By Foundation and Replacement, we can deduce that this is itself a set, and hence $X/R := \{\widetilde{[x]} : x \in X\}$ gives the quotient. Therefore, in the presence of the Axiom of Foundation, **Class** is exact.

Proposition 7.2.17. **Class** *is well-pointed.*

Proof. This follows from (NBG1), as the statement of being well-pointed in **Class** is equivalent to the extensionality. □

Proposition 7.2.18. **Class** *is not cartesian closed; however, for* $X, Y \in$ **Class** *such that* X *is a set,* Y^X *exists. Moreover, if* Y *is a set,* Y^X *is a set.*

The problem with cartesian closure amounts to the fact that a class function $F : A \to B$ is a sub*class* of the product $F \subseteq A \times B$, and so if this is a proper class, it cannot be the element of a class of functions from A to B; in fact, the cardinality of F is equal to the cardinality of A, and so by the definition of class function, F is a set if and only if A is a set. This means that if A is a set, any class function $A \to B$ can be the element of a class, and so we can form the function class B^A. Note that B need not be a set.

Proof. We prove this by providing a class function which cannot be the element of a class, since it is class sized. For this, we take the identity class function on V, which corresponds to the diagonal monomorphism $V \subseteq V \times V$ whence $\mathrm{id}_V \cong V$ and so cannot be the element of a class.

Define the formula

$$\phi(X, Y, F) := (F \subseteq X \times Y) \wedge ((\forall x \in X)\langle x, y\rangle = \langle x, y'\rangle \implies y = y').$$

By the Class Existence Theorem (Theorem 7.2.1), there is a class $Y^X := \{F : \phi(X, Y, F)\}$; this the class of set-functions $F : X \to Y$— that is sub*sets* of the class $X \times Y$ as opposed to subclasses. Note, however, that for X a class, this does not form the exponentiable, as it misses out all possible class functions— those sub-proper classes of $X \times Y$. If X is a set, then every class-function is a set-function as we can prove that for f a class-function then $f \cong X$. It is trivial to show that this is the exponentiable object in the presence of well-pointedness (Proposition 7.2.17), as the class of class-functions is equal to the class of set-functions.

If Y is also a set, then we can form the sum set $\bigcup(X \times Y)$ by the axiom of sum sets (NBG10) and it is clear that $Y^X \subseteq \bigcup(X \times Y)$ and so it follows from Lemma 7.2.8 that Y^X is itself a set too. □

Define **Set** to be the subcategory of **Class** whose objects are sets. Note that this is slightly different from the category of sets internal to class, whose class of objects is V, which we use in Proposition 7.3.8.

Proposition 7.2.19. *The map* $\mathbf{1} \to \mathbf{1} + \mathbf{1}$ *is a subobject classifier in* §.

Proof. From (NBG6), it follows that every set has a complement in **Set** which implies that **Set** is Boolean and thus 2-valued. The result then follows given that the terminal and $\mathbf{1} + \mathbf{1}$ are sets, which follows from the Axiom of Infinity (NBG12). $\square$

Note that this means that $\S$ satisfies the Law of the Excluded Middle. However, it is not assumed (in this paper, at least) to satisfy the Axiom of Choice. In fact, the Axiom of Choice is independent of the axioms (NBG1)-(NBG13).

Corollary 7.2.20. **Set** *is an elementary topos with natural numbers object.*

Proof. The category **Set** has finite limits by Proposition 7.2.11 and by noting that if x and y are sets, then $x \times y \subseteq \bigcup x \cup \bigcup y$ and so by Lemma 7.2.8, it follows that $x \times y$ is a set; the pullback is a subset of this and the terminal is a set by the Axiom of Infinity (NBG12), and so it follows that the finite limit of a diagram of sets exists and is itself a set. It is cartesian closed by Proposition 7.2.18 and has a subobject classifier by Proposition 7.2.19. $\square$

7.3 2-categorical properties of the category of large categories

We move on to investigating 2-categorical properties of the 2-category of categories. Let $\mathbf{CAT} := \mathbf{Cat}(\mathbf{Class})$ be the 2-category of categories internal to the category of classes. This means that our categories can have a class of objects and a class of morphisms, so we will also refer to these as large categories. This section will describe 2-categorical properties of **CAT**; of particular interest will be that it is not cartesian closed. However, some important exponentiables do exist.

Whilst we prove 2-categorical properties about **CAT**, these properties also hold in $\mathbf{GPD} \hookrightarrow \mathbf{CAT}$, the full subcategory of those large categories which have only invertible morphisms, which is the $(2,1)$-category of interest for the rest of this paper; we talk about **CAT** in this section as it is more widely applicable and we hope that is will be of more general interest.

Proposition 7.3.1. **CAT** *has finite* 2*-limits.*

Proof. By Proposition 4.2.2 if $\mathcal{E}$ has finite limits, then $\mathbf{Cat}(\mathcal{E})$ has finite 2-limits. The result therefore follows, since **Class** has finite limits (Proposition 7.2.11). $\square$

Proposition 7.3.2. **CAT** *is BO-regular and BO-exact.*

Proof. It is shown in [BG14] that $\mathbf{Cat}(\mathcal{E})$ is always BO-exact (and therefore BO-regular). $\square$

Proposition 7.3.3. **CAT** *is lextensive.*

Proof. Proposition 4.4.2 tells us that if $\mathcal{E}$ is extensive, then $\mathbf{Cat}(\mathcal{E})$ is extensive. The result then follows by Proposition 7.3.1. $\square$

Proposition 7.3.4. **CAT** *has* 2*-colimits of diagrams of small categories.*

Proof. **Cat** has 2-colimits by [BBP99]. Alternatively, if we want to be precise about the theory we are working in, we can apply Theorem 3.5.2 to **Cat**(**Set**) which is applicable by Proposition 7.2.20. □

Definition 7.3.5. In **CAT**, we call a functor $F : \mathbb{X} \to \mathbb{Y}$ a *discrete opfibration* if for any $x \in \mathbb{X}$ and $g : Fx \to y$ in $\mathbb{Y}$, there exists a unique $f : x \to x'$ in $\mathbb{X}$ such that $Ff = g$.

Let $F : \mathbb{X} \to \mathbb{Y}$ be a discrete opfibration. For any $y \in \mathbb{Y}$, we obtain a class $F^{-1}(x)$ containing those $x \in \mathbb{X}$ with $Fx = y$. If for every $y \in \mathbb{Y}$ this is a *set* instead of a class, we call F a *small* discrete opfibration.

Using this definition, we can define the notion of a discrete opfibration in an arbitrary 2-category.

Definition 7.3.6. For a 2-category $\mathcal{K}$, a functor $F : \mathbb{X} \to \mathbb{Y}$ is called a *discrete (op)fibration* if, for all $\mathbb{A} \in \mathcal{K}$, the functor

$$\mathcal{K}(\mathbb{A}, F) : \mathcal{K}(\mathbb{A}, \mathbb{X}) \to \mathcal{K}(\mathbb{A}, \mathbb{Y})$$

is a discrete (op)fibration in **CAT**.

Note that we can give a more explicit description without reference to **CAT**, as given in Remark 2.1.2. One of the key ingredients for a class $(2,1)$-category will be the notion of a discrete opfibration classifier. For $\mathbb{X} \in \mathcal{K}$, we denote by **Dopfib**$(\mathbb{X})$ the category whose objects are discrete opfibrations over $\mathbb{X}$ and given $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ discrete opfibrations, a morphism $\hat{H} : F \to G$ consists of a morphism $H : \mathbb{A} \to \mathbb{B}$ in $\mathcal{K}$ such that the triangle below commutes.

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{B} \arrow[dl, "G"] \\ & \mathbb{X} & \end{tikzcd}$$

Definition 7.3.7. [Web07] Let $\mathcal{K}$ be a 2-category with pullbacks along discrete opfibrations.

A *discrete opfibration classifier* is a discrete opfibration $p : \mathbb{S}_* \rightarrowtail \mathbb{S}$ such that for all $\mathbb{X} \in \mathcal{K}$ the pullback functor:

$$\mathcal{K}(\mathbb{X}, \mathbb{S}) \xrightarrow{p^*} \mathbf{DopFib}(\mathbb{X})$$

is fully faithful.

In the following, we write $\underline{\mathbf{Set}}$ to mean the category internal to **Class** whose object of objects is V and whose class of morphisms the class of set-functions which can be defined using the Class Existence Theorem (Theorem 7.2.1). We write $\underline{\mathbf{Set}}_*$ to be the class of pointed sets, i.e. a pair (X, x) with $x \in X$; this can similarly defined by the Class Existence Theorem.

Proposition 7.3.8. *In* **CAT***, the forgetful functor* $p : \underline{\mathbf{Set}}_* \to \underline{\mathbf{Set}}$ *sending a pointed set to its underlying set provides a discrete opfibration classifier. The discrete opfibrations in the essential image of the pullback functor*

$$\mathbf{CAT}(\mathbb{X}, \underline{\mathbf{Set}}) \xrightarrow{p^*} \mathbf{DopFib}(\mathbb{X})$$

are exactly the small discrete opfibrations.

Proof. This is first claimed in [Web07, Example 4.2]. We briefly describe the process here for completeness. For any $\mathbb{X} \in \mathbf{CAT}$, pulling back along p gives us a functor

$$p^*_{\mathbb{X}} : \mathbf{CAT}(\mathbb{X}, \underline{\mathbf{Set}}) \to \mathbf{Dopfib}/_{\mathbb{X}}(p^*(F) : F \times_{\underline{\mathbf{Set}}} p \to \mathbb{X}, p^*(G) : G \times_{\underline{\mathbf{Set}}} \mathbb{X} \to \mathbb{X}).$$

For $F, G : \mathbb{X} \to \underline{\mathbf{Set}}$, we show that this induces a bijection

$$\mathbf{CAT}(\mathbb{X}, \underline{\mathbf{Set}})(F, G) \cong \mathbf{Dopfib}/_{\mathbb{X}}(p^*(F), p^*(G)).$$

Given $\alpha : F \Rightarrow G$, pulling back gives us a functor $\hat{\alpha} : F \times_{\underline{\mathbf{Set}}} p \to G \times_{\underline{\mathbf{Set}}} p$ in $\mathbf{Dopfib}/_{\mathbb{X}}(p^*(F), p^*(G))$ given by $(x, (Fx, a \in Fx)) \mapsto (x, (Gx, \alpha_x(a) \in Gx))$ and on morphisms by naturality of α.

Conversely, any $\hat{\alpha} : F \times_{\underline{\mathbf{Set}}} p \to G \times_{\underline{\mathbf{Set}}} p$ in in $\mathbf{Dopfib}/_{\mathbb{X}}(p^*(F), p^*(G))$ is on objects an assignment $(x, (Fx, a \in Fx)) \mapsto (x, (Gx, b \in Gx))$ which amounts to, for each $x \in \mathbb{X}$, a map $\alpha_x : Fx \to Gx$ which we define for each $a \in Fx$ by the b given in $\hat{\alpha}(x, (Fx, a \in Fx))$. Naturality follows easily from the definition of $\hat{\alpha}$ on morphisms in $F \times_{\underline{\mathbf{Set}}} p$, giving us a natural transformation $\alpha : F \Rightarrow G$. These processes are mutually inverse, supplying the desired bijection. ☐

Remark 7.3.9. Note that the previous proposition restricts to **GPD** by taking the core of $p : \underline{\mathbf{Set}}_* \to \underline{\mathbf{Set}}$; we form an groupoid internal to **Class** whose object of objects is V and whose object of morphisms is the class of *invertible* set-functions, which can be defined by the Class Existence Theorem (Theorem 7.2.1). To avoid overladen notation, we denote this internal groupoid by $\underline{\mathbf{Set}}$ as well, leaving the context to make it clear whether we are speaking about the internal category or internal groupoid.

7.3.1 Properties of small discrete opfibrations

Proposition 7.3.10. *Any isomorphism in* **CAT** *is a small discrete opfibration.*

Proof. Let $F : \mathbb{X} \to \mathbb{Y}$ be an isomorphism with inverse $F^{-1} : \mathbb{Y} \to \mathbb{X}$. Then for any $g : Fx \to y$ there exists a unique $F^{-1}(g) : x \to F^{-1}(y)$ with $F(F^{-1})(g) = g$. Moreover, for every $y \in \mathbb{Y}$, $F^{-1}(y)$ is a singleton, which is necessarily a set. ☐

Proposition 7.3.11. *Small discrete opfibrations are closed under composition in* **CAT**.

Proof. Let $F : X \to Y$ and $G : Y \to Z$ be small discrete opfibrations. We show that $GF : X \to Z$ is too. Note that $(GF)^{-1}(z) = \coprod_{y \in G^{-1}(z)} F^{-1}(y)$. Now, since G is a small discrete opfibration, by the Axiom of Sum Sets (NBG10), the class $\coprod_{y \in G^{-1}(z)} y$ is a set.

Define a formula $\phi(F, x) := (\exists y)(x = \langle y, z \rangle \wedge u \in z \iff Fu = y)$ and define a class $\{x : \phi(F, x)\}$ which exists by the Class Existence Theorem (Theorem 7.2.1) and by the assumption that F is a small discrete opfibration, which allows us to form z in this formula. This is a definable mapping, and so we can apply the Axiom of Replacement (NBG13) to the domain $\coprod_{y \in G^{-1}(z)} y$ and conclude that its image is a set; the image of this is exactly given by $(GF)^{-1}(z) = \coprod_{y \in G^{-1}(z)} F^{-1}(y)$. ☐

In fact, the conclusion of the above proposition is *equivalent* to the Axiom of Replacement in the presence of Class Existence Theorem (Theorem 7.2.1) and the Axiom of Power Sets (NBG11), as we prove below. In a 2-category with a discrete opfibration classifier we can define an internal Axiom of Replacement by stating that the discrete opfibrations classified are closed under composition; in the presence of some 2-dimensional axioms similar to (NBG1)-(NBG9) and power set axiom, we can run an argument in the internal language which proves an internal version of the Axiom of Replacement (NBG13). Interestingly, closure of small discrete opfibrations under composition is one of the axioms that is considered in [Hel24] without reference to the Axiom of Replacement.

Proposition 7.3.12. *Suppose we work in NBG set theory without assuming the Axiom of Replacement (NBG13), and suppose instead that small discrete opfibrations in* **CAT** *are closed under composition. Then the Axiom of Replacement holds.*

Proof. The idea of the proof is that for any definable mapping as in the statement of the Axiom of Replacement, given any set that will serve as its domain, we can construct a pair of small discrete opfibrations whose composed fibre is the image. By assumption that discrete opfibrations are closed under composition, it follows that the image is a set, as required.

Let F be a definable mapping. Let $\phi(F, X, y) := (\exists v \in X)(\langle v, y\rangle \in F$. Then, by the Class Existence Theorem (Theorem 7.2.1), there exists a class $\text{img}_X(F) := \{y : \phi(F, X, y)\}$. Note that this could be empty. We show that if X is actually a set x, this class is actually a set, which verifies the statement of Replacement (NBG13).

Let x be a set. First, construct the class function $F^{-1} : \text{img}_x(F) \to \mathbf{Pow}(x)$ which acts by the rule

$$y \mapsto \{a \in x : Fa = y\}$$

which is well-defined by the Power Set Axiom (NBG11). Note that the fibres of this map are either $y \in \text{img}_x(F)$ or $\emptyset$ and so this is a small discrete opfibration.

By the Power Set Axiom (NBG11), the map $! : \mathbf{Pow}(x) \to \mathbf{1}$ is a small discrete opfibration too; the fibre above the unique element $* \in \mathbf{1}$ is $!^{-1}(*) = \mathbf{Pow}(x)$, which is a set. The composite of these maps is $!F^{-1} : \text{img}_x F \to \mathbf{1}$ has fibre above the unique element $(!F^{-1})^{-1}(*) = \text{img}_x(F)$. Since small discrete opfibrations are closed under composition, this is a set, as required. $\square$

Proposition 7.3.13. *Small discrete opfibrations are closed under pullback in* **CAT**.

Proof. Let $G : \mathbb{A} \to \mathbb{Y} \leftarrow \mathbb{X} : F$ be a span in which $F : \mathbb{X} \to \mathbb{Y}$ is a small discrete opfibration, and take the pullback $F^*(G) : F^*(\mathbb{A}) \to \mathbb{A}$. Discrete opfibrations are closed under pullbacks since they are the right class of an orthogonal factorisation system [SW73]. Therefore, it remains to show that the resulting discrete opfibration is small. For $a \in \mathbb{A}$, the description of the pulled-back fibre is

$$F^*(G)^{-1}(a) = \{(x, a) : Fx = Ga\} \cong F^{-1}(Ga)$$

which is a set since F is a small discrete opfibration. $\square$

Proposition 7.3.14. *Small discrete opfibrations are closed under coproduct in* **CAT**.

Proof. Let $F : X \to \mathbb{Y}$ and $G : \mathbb{A} \to \mathbb{B}$ be small discrete opfibrations. Form $F + G : \mathbb{X} + \mathbb{A} \to \mathbb{Y} + \mathbb{B}$. Firstly, this is a discrete opfibration since coproducts in **CAT** do not add any new morphisms and so for any $c \in \mathbb{Y} + \mathbb{B}$ we have

$$(F+G)^{-1}(c) = \begin{cases} F^{-1}(c) & \text{if } c \in \mathbb{Y}, \\ G^{-1}(c) & \text{if } c \in \mathbb{B}. \end{cases}$$

In either case, this is a set since both F and G are small. □

In the following, we instantiate the method described in [Str01], which shows that Conduché fibrations in **Cat** are exponentiable, adapting it to the setting of small discrete opfibrations in **CAT**. We note that whilst discrete opfibrations are Conduché fibrations, the method in loc. cit. does not readily extend to **CAT** in NBG set theory, as there must be some size restriction in order for the constructions to be well-defined. We note that the below proof works for small discrete opfibrations but it is not true for non-small discrete opfibrations; indeed, by considering the proper class V, we obtain a non-small discrete opfibration $V \to \mathbf{1}$ which is not exponentiable by the proof of Proposition 7.2.18. In subsequent proof, we make clear when we are using the smallness condition.

Proposition 7.3.15. *Small discrete opfibrations are exponentiable in* **CAT**.

Proof. Let $p : \mathbb{E} \to \mathbb{B}$ be a small discrete opfibration. Since the forgetful functor $U : \mathbf{CAT}/_{\mathbb{E}} \to \mathbf{CAT}$ is monadic (with right adjoint $- \times \mathbb{E} \to \mathbb{E}$), in order to show that pullback along p has a right adjoint, it is enough to show that $Up^* : \mathbf{CAT}/_{\mathbb{B}} \to \mathbf{CAT}$ has a right adjoint [Dub06].

Let $\mathbb{X}$ be a category. We define a category $\Pi_p(\mathbb{X})$ over $\mathbb{B}$ as follows: its objects are pairs (b, s) in which $b \in B_0$ and $s : p^{-1}(b) \to X_0$; note that the collection of such pairs is a class since it is a subclass of $B_0 \times X_0^{p^{-1}(b)}$ which is a well-defined class since $p^{-1}(b)$ is a set. For any $\beta : b \to b'$, we define the set-function $m_E(\beta) : p^{-1}(b) \to p^{-1}(b')$ by sending $e \in p^{-1}(b)$ to the unique lift of b' that comes from $\beta : p(e) \to b'$ given by the definition of discrete opfibration. Note that by the uniqueness in the definition of a discrete opfibration, composition interacts well with these functions; given a composable pair $\beta : b \to b'$ and $\beta' : b' \to b''$ the functions $M_E(\beta) \circ M_E(\beta')$ and $M_E(\beta \circ \beta')$ are necessarily equal. A morphism $(b, s) \to (b', s')$ in $\Pi_p(\mathbb{X})$ is defined to be a pair (β, σ), in which $\sigma : s' M_E(\beta) \Rightarrow s$ is specified for each $e \in p^{-1}(b)$; note that this forms a class of morphisms since such pairs are a subclass of $X_0 \times X_1^{p^{-1}(\beta)}$, which is a well-defined class since p is a small discrete opfibration. Composition of $\Pi_p(\mathbb{X})$ is well-defined by the definition of these natural transformations, given the composition of $M_E(-)$ is on-the-nose. Note that this avoids some of the complication in the argument constructed in [Str01]. We have a functor $h : \Pi_p(\mathbb{X}) \to \mathbb{B}$ defined on objects by $(b, s) \mapsto b$ and on morphisms by $(\beta, \sigma) \mapsto \beta$. By construction $(\Pi_p(\mathbb{X}), h) \in \mathbf{CAT}/_{\mathbb{B}}$, and we claim that it provides a right adjoint to Up^*.

Let $u : \mathbb{A} \to \mathbb{B}$ be functor. We show that

$$\mathbf{CAT}/_{\mathbb{B}}((\mathbb{A}, u), (\Pi_p(\mathbb{X}), h)) \cong \mathbf{CAT}(\mathbb{A} \times_{\mathbb{B}} \mathbb{E}, \mathbb{X}).$$

Given $\phi : (\mathbb{A}, u) \to (\Pi_p(\mathbb{X}), h)$, we write $\phi(a) = (\phi_1(a), \phi_2(a))$, noting that this means $\phi_2(a) : p^{-1}(\phi_1(a)) \to X_0$ and $\phi_1(a) = h(\phi(a))$. We define a functor $\phi^\sharp : \mathbb{A} \times_{\mathbb{B}} \mathbb{E} \to \mathbb{X}$ by $(a, e) \mapsto \phi_2(a)(e) \in X_0$ which is well-defined because $e \in p^{-1}(u(a))$ by definition of the pullback and $u(a) = h(\phi(a)) = \phi_1(a)$ by definition of a morphism in $\mathbf{CAT}/_{\mathbb{B}}$ and the definition of h. Let $\alpha : \phi \Rightarrow \psi : \mathbb{A} \to \Pi_p(\mathbb{X})$. For all $a \in A_0$, this gives $\alpha_a : \phi(a) \to \psi(a)$ which, unwinding the definitions above, is the same as giving $\beta : u(a) \to u(a)$ for which unwinding the definitions given for each $e \in p^{-1}(u(a))$ above gives us a morphism $\sigma_e : \phi_2(a)(M_E(\beta))(e) \to \psi_2(a)(e)$ and $\sigma : \phi_2(a)M_E(\beta) \to \psi_2(a)$. We note that by the definition of discrete opfibration, it follows that for $e \in p^{-1}(u(a))$ we have $M_E(\beta)(e) = e$, and so giving $\alpha_a : \phi_a \to \psi_a$ amounts to giving a function $\sigma_e : \phi(a)(e) \to (\psi(a)(e))$ for each $e \in p^{-1}(u(a))$. Define $\alpha^\sharp : \phi^\sharp \to \psi^\sharp$ by $\alpha(a,e) = \sigma_e : \phi_2(a)(e) \to \psi_2(a)(e)$. This gives a functor $\mathbf{CAT}/_{\mathbb{B}}((\mathbb{A},u),(\Pi_p(\mathbb{X}),h)) \to \mathbf{CAT}(\mathbb{A},\mathbb{X})$.

Conversely, given $\theta : \mathbb{A} \times_{\mathbb{B}} \mathbb{E} \to \mathbb{X}$, we define $\theta^\flat : \mathbb{A} \to \Pi_p(\mathbb{X})$ by sending $a \in A_0$ to the pair $(u(a), s)$ where $s : p^{-1}(u(a)) \to X_0$ is the function that sends $e \in p^{-1}(u(a))$ to $\theta(a,e)$. On natural transformation, the argument in the other direction is reversible, and so we obtain a functor

$$(-)^\flat : \mathbf{CAT}(\mathbb{A} \times_{\mathbb{B}} \mathcal{E}, \mathbb{X}) \to \mathbf{CAT}/_{\mathbb{B}}((\mathbb{A},u),(\Pi_p(\mathbb{X}),h)).$$

It is not hard to show that $(-)^\sharp$ and $(-)^\flat$ are mutually inverse, proving the claim.

$\square$

Proposition 7.3.16. *(Cancellation)*

Consider the following commutative diagram in which F, G and hence GF are discrete opfibrations.

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "GF"'] & & \mathbb{Y} \arrow[dl, "G"] \\ & \mathbb{Z} & \end{tikzcd}$$

If GF is a small discrete opfibration, then so is F.

Proof. Let $y \in \mathbb{Y}$. Note that $(GF)^{-1}(Gy)$ is a set since GF is a small discrete opfibration, and $F^{-1}(y) \subseteq (GF)^{-1}(Gy)$ since $x \in F^{-1}(y)$ means that $Fx = y$ and so $GFx = Gy$. By Lemma 7.2.8, F is a small discrete opfibration. $\square$

Lemma 7.3.17. *Every discrete opfibration that is a monomorphism is a small discrete opfibration.*

Proof. This follows from the Empty Set Axiom (NBG3) and the Power Set Axiom (NBG11), which proves that the one element class (which is the power set of the empty set) is a set and the fact that a monomorphism is a map whose fibres are either empty or has 1 element. $\square$

7.4 Axioms for a class $(2,1)$-category

In this section, we give the central definition of this chapter.

Definition 7.4.1. Let $\mathcal{K}$ be a BO-regular and 2-extensive $(2,1)$-category and let $\mathcal{S}$ be a class of discrete opfibrations which includes all isomorphisms. We call $(\mathcal{K},\mathcal{S})$ a pre-class $(2,1)$-category. We call the morphisms in $\mathcal{S}$ *small maps*. For $\mathbb{X} \in \mathcal{K}$, we write $\mathbf{Dopfib}/^{\mathcal{S}}_{\mathbb{X}}$ for the category whose objects are morphisms of $\mathcal{S}$ that have $\mathbb{X}$ as their target and morphisms are commutative triangles

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{B} \arrow[dl, "G"] \\ & \mathbb{X} & \end{tikzcd}$$

with $F, G \in \mathcal{S}$.

In Lemma 2.1.3, we prove that an object $X \in \mathcal{K}$ is discrete if and only if $! : X \to \mathbf{1}$ is a discrete opfibration. In the 1-dimensional case, an object in a class category is called small if the unique map to the terminal is small. Due to Lemma 2.1.3, this only allows us to define a notion of smallness for discrete objects.

Definition 7.4.2. Let $(\mathcal{K},\mathcal{S})$ be a pre-class $(2,1)$-category. We call $X \in \mathbf{Disc}(\mathcal{K})$ *small* if the morphism $! : X \to \mathbf{1}$ is in $\mathcal{S}$.

From this, we can give a notion of smallness for general objects.

Definition 7.4.3. Let $(\mathcal{K},\mathcal{S})$ be a pre-class $(2,1)$-category. We call $\mathbb{X} \in \mathcal{K}$ *small* if there exists a small discrete object X and codescent morphism $q : X \twoheadrightarrow \mathbb{X}$ such that $(d_1, d_0) : q \downarrow q \to X \times X$ is in $\mathcal{S}$.

We denote the full sub-$(2,1)$-category of small objects by $\mathcal{K}_\sigma$.

Note that since any identity is a codescent morphism and $\mathcal{S}$ includes all isomorphisms and hence identities, the above definitions agree on discrete objects to give a general definition of smallness on all objects.

Consider the following list of axioms:

(S1) (Replacement) Small maps are closed under composition.

(S2) (Stability) In any 2-pullback square

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "G"'] & \mathbb{X} \arrow[d, "F"] \\ \mathbb{B} \arrow[r, "Q"'] & \mathbb{Y} \end{tikzcd} \tag{7.1}$$

if F belongs to $\mathcal{S}$ then G belongs to $\mathcal{S}$.

(S3) (Finiteness) The maps $0 \to \mathbf{1}$ and $\mathbf{1} + \mathbf{1} \to \mathbf{1}$ belong to $\mathcal{S}$.

(S4) (Sums) If $\mathbb{X} \to \mathbb{Y}$ and $\mathbb{X}' \to \mathbb{Y}'$ belong to $\mathcal{S}$ then so does $\mathbb{X} + \mathbb{X}' \to \mathbb{Y} + \mathbb{Y}'$.

(S5) (Exponentiability) Every map in $\mathcal{S}$ is exponentiable.

(S6) (Representability) There exists a morphism $p : \mathbb{S}_* \to \mathbb{S}$ in $\mathcal{K}$ such that for any $\mathbb{X} \in \mathcal{K}$, there is an equivalence of categories $\mathcal{K}(\mathbb{X}, \mathbb{S}) \simeq \mathbf{Dopfib}^{\mathcal{S}}/\mathbb{X}$.

In particular, axiom (S6) says that for any map $f : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{S}$, there exists a morphism $\chi_f : \mathbb{Y} \to S$ such that the following is a $(2,1)$-pullback square.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \arrow[d, "p"] \\ Y \arrow[r, "\chi_f"'] & \mathbb{S}. \end{tikzcd}$$

Note that the morphism χ_f is not unique, but unique up to unique isomorphism. This axiom says that we have a discrete opfibration classifier in the sense of Definition 7.3.7 and $\mathcal{S}$ is the essential image of the associated pullback functor.

(S7) (Colimits) $\mathcal{K}$ has small 2-colimits of diagrams of small objects.

(S8) (Cancellation) Let $F : \mathbb{X} \to \mathbb{Y}$ and $G : \mathbb{Y} \to \mathbb{Z}$ be discrete opfibrations. If $GF \in \mathcal{S}$ then $F \in \mathcal{S}$.

(S9) (Small exactness) Small cateads in $\mathcal{K}$ are effective.

(Ex) (Discrete projectivity) Every small discrete object in $\mathcal{K}$ is BO-projective.

Definition 7.4.4. We call a pre-class $(2,1)$-category that satisfies (S1)-(S9) and (Ex) a *class* $(2,1)$*-category*.

Note that everything in these definitions is closed under strict $(2,1)$-equivalence. Let $(\mathcal{K}, \mathcal{S})$ and $(\mathcal{K}', \mathcal{S}')$ be class categories with classifiers $p : \mathbb{S}_* \to \mathbb{S}$ and $p' : \mathbb{S}'_* \to \mathbb{S}'$ respectively. A *class* $(2,1)$*-functor* is a $(2,1)$-functor $F : \mathcal{K} \to \mathcal{K}'$ such that if $f \in \mathcal{S}$ then $Ff \in \mathcal{S}'$ and $Fp = p'$.

Remark 7.4.5. The axiom (Ex) is a bit different from the others. Whilst the others are simply $(2,1)$-dimensional versions of the axioms that appear in parts of the literature on algebraic set theory, axiom (Ex) is new. It is indeed independent of the other axioms, since there are examples of categories which satisfy (S1)-(S9) but do not satisfy (Ex)— indeed, we prove that the isofibrational slice $(2,1)$-category $\mathbf{Gpd}/^{\cong}_{\mathcal{I}}$ satisfies (S1)-(S9) but not (Ex) in Proposition 7.7.18, showing that it is independent of the other axioms. Indeed, axioms (S1)-(S9) are shown to be stable under (isofibrational) slicing in Section 7.7, while (Ex) is not stable.

Axiom (Ex) tells us that this is what we might call a 1-localic theory (in the sense of [Lur09, §6.4.5]) as it enforces that our small objects are generated by the discrete objects. We include it because it is this axiom which allows us to prove that the small objects are equivalent to a $(2,1)$-pretopos of groupoids internal to a 1-pretopos. Without this axiom, we are not able to prove that our small objects form a model of type theory.

In the following, we develop most of the theory without reference to it, being very clear when we use this axiom. The hope is that in the future we can remove it to get a more genuinely 2-dimensional theory.

Remark 7.4.6. A lot of the theory that follows also works for 2-categories as opposed to $(2,1)$-categories. The reason we restrict to $(2,1)$-categories is twofold. Firstly, our main theorem is that the sub-$(2,1)$-category of small objects models Martin-Löf type theory. If we were to consider 2-categories instead, we would instead want to prove that the sub-$(2,1)$-category of small objects models *directed* type theory. The required prerequisite literature has not yet been developed, and there are many competing definitions of what a directed type theory should be. In future work with Fernando Chu, we plan to look at models of a particular kind of directed type theory and show that for suitably nice $\mathcal{E}$, the

2-category $\mathbf{Cat}(\mathcal{E})$ models directed type theory. The second reason is that for 2-categories, in order to phrase our definitions correctly, we would need a duality involution, otherwise our definition of small does not make sense; the map $(d_1, d_0) : q \downarrow q \to X \times X$ is not normally a discrete (op)fibration in **Cat**, but rather $q \downarrow q \to X^{\mathrm{op}} \times X$ is instead. There are various examples of categories that we would want to be examples of elementary 2-toposes, but fail to have a duality involution— for example the 2-category of prestacks on the walking monad. By restricting to $(2, 1)$-categories, which always have a duality involution, we avoid getting caught in the crossfire of these discussions. We finish this remark by noting that one way you could formulate these conditions without needing a duality involution is to consider a 2-category with a class of 2-sided discrete opfibrations. This is closer in spirit to Street's elementary cosmoi [Str80], and solves the aforementioned problems; indeed, the span $(d_1, d_0) : X \leftarrow q \downarrow q \to X$ is always a 2-sided discrete fibration in **Cat**. We do not present this formulation here mostly because of the first reason.

7.5 Properties of Class $(2, 1)$-categories

7.5.1 Properties of small discrete objects

We start by establishing some basic properties of the discrete objects in a class $(2, 1)$-category. In particular, we prove that $\mathbf{Disc}(\mathcal{K}_\sigma)$ is a locally cartesian closed, lextensive category with a natural numbers object and coequalisers. This is useful as these results will allow us to prove properties of not-necessarily discrete objects in the next subsection, and also to prove that the discrete objects in a class $(2, 1)$-category can be sensibly equipped with the structure of a class 1-category in Section 7.8.2, which allows us to relate the 1-dimensional case to our case. Note that the terminal object in $\mathcal{K}$ is necessarily discrete by its universal property. Throughout this section, we assume that $(\mathcal{K}, \mathcal{S})$ is a pre-class-$(2, 1)$-category and assume the additional axioms as and when we need them. We remark that we do not need (Ex) for this section.

Lemma 7.5.1. *The terminal object $\mathbf{1} \in \mathcal{K}$ is small.*

Proof. This follows from the fact that all isomorphisms are assumed to be in $\mathcal{S}$, since $\mathbf{1}$ is discrete and $\mathbf{1} \to \mathbf{1}$ is an equality, and hence isomorphism. □

Lemma 7.5.2. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S8). Let $Y \in \mathbf{Disc}(\mathcal{K})$ with Y small. Then any morphism $f : X \to Y$ in $\mathbf{Disc}(\mathcal{K})$ is in $\mathcal{S}$.*

Proof. This follows from Cancellation (S8), given that $Y \to \mathbf{1}$ is in $\mathcal{S}$ and $X \to \mathbf{1}$ is a discrete opfibration, by Lemma 2.1.3. □

Lemma 7.5.3. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S8) and (S2). Then $\mathbf{Disc}(\mathcal{K}_\sigma)$ has finite limits.*

Proof. Since the terminal object is small by Lemma 7.5.1, it remains to show that $\mathbf{Disc}(\mathcal{K}_\sigma)$ is closed under pullbacks. Given a pullback diagram as in Diagram (7.1) in which X, Y and B are in $\mathbf{Disc}(\mathcal{K}_\sigma)$, then it follows by Lemma 7.5.2 that F is in $\mathcal{S}$. By (S2), it therefore follows that G is in $\mathcal{S}$ too. But then $A \to \mathbf{1}$ which is the composite of G and $B \to \mathbf{1}$ is in $\mathcal{S}$ by (S1), as required. □

Lemma 7.5.4. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3) and (S4). Then* **Disc**$(\mathcal{K}_\sigma)$ *has coproducts.*

Proof. Let $X, Y \in$ **Disc**$(\mathcal{K}_\sigma)$. By (S4), $X + Y \to \mathbf{1} + \mathbf{1}$ is in $\mathcal{S}$ and by (S3), $\mathbf{1} + \mathbf{1} \to \mathbf{1}$ is in $\mathcal{S}$, so by (S1), $X + Y \to \mathbf{1}$ is in $\mathcal{S}$ too. From this, it follows that $X + Y$ is small and discrete. □

Lemma 7.5.5. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S8),(S2), (S3) and (S4). Then* **Disc**$(\mathcal{K}_\sigma)$ *is extensive.*

Proof. We have shown that **Disc**$(\mathcal{K}_\sigma)$ has pullbacks, so that pullbacks of finite coproduct injections exist. By Lemma 2.5.10, 2-extensivity of $\mathcal{K}$ implies extensivity of **Disc**$(\mathcal{K})$; therefore the result follows by closure of **Disc**$(\mathcal{K}_\sigma)$ under coproducts. □

Lemma 7.5.6. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S8), (S2) and (S5). Then* **Disc**$(\mathcal{K}_\sigma)$ *is locally cartesian closed.*

Proof. Let $f : X \to Y$ be in **Disc**$(\mathcal{K}_\sigma)$. We construct a right adjoint to $f^* :$ **Disc**$(\mathcal{K}_\sigma)/Y \to$ **Disc**$(\mathcal{K}_\sigma)/X$. Note that $i :$ **Disc**$(\mathcal{K}_\sigma) \to \mathcal{K}_\sigma$ is fully faithful. Now, $i(f)^* : \mathcal{K}_\sigma/Y \to \mathcal{K}_\sigma/X$ has a right adjoint because $i(f) \in \mathcal{S}$ by Lemma 7.5.2 and so is exponentiable by (S5); denote this by Π_f. We observe that for any $b : B \to X$ in **Disc**$(\mathcal{K}_\sigma)$, $\Pi_f(b)$ is itself discrete since for any $a : \mathbb{A} \to Y$ in $\mathcal{K}_\sigma$ we have $\mathcal{K}_\sigma/Y(a, \Pi_f(b)) \cong \mathcal{K}_\sigma/X(i(f)^*(a), b)$ but this is a set as b is a discrete object in $\mathcal{K}_\sigma/X$. Thus, for any $a : A \to Y$ in **Disc**$(\mathcal{K}_\sigma)/X$, we have the following string of isomorphisms:

$$\begin{aligned}
\mathbf{Disc}(\mathcal{K}_\sigma)/X(f^*(a), b) &\cong \mathcal{K}_\sigma/X(i(f)^*(a), i(b)) && \text{by fully faithfulness,}\\
&\cong \mathcal{K}_\sigma/Y(a, \Pi_f(b)) && \text{by adjointness,}\\
&\cong \mathbf{Disc}(\mathcal{K}_\sigma)/Y(a, \Pi_f(b))
\end{aligned}$$

witnessing the fact that Π_f restricts to a right adjoint to f^*. □

Lemma 7.5.7. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S7). Then* **Disc**$(\mathcal{K}_\sigma)$ *has coequalisers.*

Proof. Let $f, g : X \to Y$ be a parallel pair in **Disc**$(\mathcal{K}_\sigma)$. By (S7), the coequaliser of this exists in $\mathcal{K}_\sigma$ and since this is a 1-colimit, the coequaliser is discrete. □

Therefore we have shown the following:

Theorem 7.5.8. *Let $(\mathcal{K}, \mathcal{S})$ be a class $(2, 1)$-category. Then* **Disc**$(\mathcal{K}_\sigma)$ *is a locally cartesian closed, lextensive category with coequalisers.*

7.5.2 Properties of general objects

In this section, we start to prove some basic results about class $(2, 1)$-categories, being careful with which assumptions we are using. In particular, the properties in this section allow us to prove that $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K}_\sigma)$) in Proposition 7.5.18.

Lemma 7.5.9. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2) and (S8). Then $\mathcal{K}_\sigma$ is closed under finite* 1*-limits.*

Proof. We first show that $\mathcal{K}_\sigma$ is closed under binary products. Let $\mathbb{X}, \mathbb{Y}$ be in $\mathcal{K}_\sigma$ with $x : X_0 \twoheadrightarrow \mathbb{X}$ and $y : Y_0 \twoheadrightarrow \mathbb{Y}$ in which $X_0, Y_0 \in \mathbf{Disc}(\mathcal{K}_\sigma)$. We note that for any $\mathbb{A} \in \mathcal{K}$ the functors $\mathbb{A} \times -$ and $- \times \mathbb{A}$ preserve codescent morphisms because they act on them by pulling back, and codescent morphisms are stable under pullbacks in an SO-regular $(2,1)$-category. Hence, it follows that we have a codescent morphism $x \times y : X_0 \times Y_0 \twoheadrightarrow X_0 \times \mathbb{Y} \twoheadrightarrow \mathbb{X} \times \mathbb{Y}$. By Lemma 7.5.3, $X_0 \times Y_0$ is small and discrete. By commutativity of pullbacks and products and stability of maps in $\mathcal{S}$ under pullbacks, it follows that $(x \times y) \downarrow (x \times y) \cong (x \downarrow x) \times (y \downarrow y)$ is in $\mathcal{S}$, and so by definition $\mathbb{X} \times \mathbb{Y}$ is a small object.

The proof of equalisers is similar, again using that small discrete objects are closed under equalisers and that codescent morphisms are closed under pullbacks in an SO-regular $(2,1)$-category. □

Lemma 7.5.10. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3) and (S4). Then $\mathcal{K}_\sigma$ has coproducts.*

Proof. Let $\mathbb{X}, \mathbb{Y}$ be in $\mathcal{K}_\sigma$ with $x : X_0 \twoheadrightarrow \mathbb{X}$ and $y : Y_0 \twoheadrightarrow \mathbb{Y}$ in which $X_0, Y_0 \in \mathbf{Disc}(\mathcal{K}_\sigma)$. As codescent morphisms are the left class of a factorisation system (Proposition 2.4.17), codescent morphisms are closed under coproduct and so we have a codescent morphism $x+y : X_0+Y_0 \twoheadrightarrow \mathbb{X}+\mathbb{Y}$. It follows from 2-dimensional extensivity that $(x+y) \downarrow (x+y) \cong (x \downarrow x) + (y \downarrow y)$; this can be shown using properties that

$$(X_0 + Y_0) \times (X_0 \times Y_0) \cong (X_0 \times X_0) + (X_0 \times Y_0) + (Y_0 \times X_0) + (Y_0 \times Y_0)$$

, and by the disjointness of coproducts pulling back the spans $(X_0 \times Y_0) \to (\mathbb{X} + \mathbb{Y}) \times (\mathbb{X} + \mathbb{Y}) \leftarrow (\mathbb{X} + \mathbb{Y})^{\mathbf{2}}$ is the initial object. From this, the joint source and target map is simply the coproduct of the source and target maps for $\mathbb{X}$ and $\mathbb{Y}$ respectively, and so it is in $\mathcal{S}$ by (S4), as required. □

Lemma 7.5.11. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3), (S4). Then the copower $\mathbf{2} \odot \mathbf{1} \in \mathcal{K}$ exists and is small.*

Proof. We construct an object which we denote $\mathbf{2} \odot \mathbf{1} \in \mathcal{K}$ and argue that it has the universal property of the copower.

First, we note that the objects $\mathbf{1}+\mathbf{1}$, $\mathbf{1}+\mathbf{1}+\mathbf{1}$ and $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ are small by (S3) and Lemma 7.5.4. We form the diagram in $\mathcal{K}$

$$\begin{tikzcd} \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1} \arrow[r, shift left=2, "\pi_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] & \mathbf{1}+\mathbf{1}+\mathbf{1} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbf{1}+\mathbf{1} \arrow[l, "i" description] \end{tikzcd}$$

This is a catead: to check that the source and target map form a 2-sided discrete opfibration, we show that for all $A \in \mathcal{K}$, $\mathcal{K}(A, (d_1, d_0)) : \mathcal{K}(A, \mathbf{1}+\mathbf{1}+\mathbf{1}) \to \mathcal{K}(A, \mathbf{1}+\mathbf{1})$ is a discrete opfibration which is easy to see as all the objects being mapped into are discrete and therefore it is a function between sets, which is trivially a discrete opfibration. We define $\mathbf{2} \odot \mathbf{1} \in \mathcal{K}$ as the codescent object of this catead, which exists by assumption that $\mathcal{K}$ is BO-regular. This is small by construction.

This has the required properties; unwinding the universal property of the codescent morphism gives exactly the universal product of the copower of $\mathbf{2}$ so that for any $\mathbb{X} \in \mathcal{K}$

$$\mathcal{K}(\mathbf{2} \odot \mathbf{1}, \mathbb{X}) \simeq \mathbf{Cat}(\mathbf{2}, \mathcal{K}(\mathbf{1}, \mathbb{X})).$$

□

Corollary 7.5.12. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3), (S4). Then $\mathcal{K}$ is powered by* **2**.

Proof. Since $\mathcal{K}$ is BO-regular it has finite limits, and so for any $\mathbb{A} \in \mathcal{K}$, the power $\mathbb{A}^{\mathbf{2}\odot\mathbf{1}}$ exists in $\mathcal{K}$. This has the universal property of being powered by **2**, since $\mathbf{2} \odot \mathbf{1}$ has the universal property of being the copower of **2**. □

We prove in addition to this that if $\mathbb{A}$ is small, then $\mathbb{A}^{\mathbf{2}}$ is also small, meaning that $\mathcal{K}_\sigma$ is also powered by **2**. First, we prove the following.

Lemma 7.5.13. *Let the following be a comma object in a 2-category $\mathcal{K}$:*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \arrow[dl, Rightarrow] \\ Y \arrow[r] & \mathbb{Z} \end{tikzcd}$$

If X and Y are discrete objects in $\mathcal{K}$, then so is W.

Proof. Since comma objects and discrete objects are representable notions, it is enough to prove this in **Cat**, where this is straightforward from the definitions. □

Proposition 7.5.14. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3), (S4). Then given any $\mathbb{A}$ that is small, $\mathbb{A}^{\mathbf{2}}$ is also small. Hence $\mathcal{K}_\sigma$ is powered by* **2**.

Proof. Let $\mathbb{A} \in \mathcal{K}$ be small. By Corollary 7.5.12, $\mathbb{A}^{\mathbf{2}}$ exists; we show that it is small. Since $\mathbb{A}$ is small, there exists a discrete A and $a : A \to \mathbb{A}$ such that $(d_1, d_0) : a \downarrow a \to A \times A$ is in $\mathcal{S}$. First note that by construction, we have the following pullback square defining $a \downarrow a$:

$$\begin{tikzcd} a \downarrow a \arrow[r, "\hat{a}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^{\mathbf{2}} \arrow[d] \\ A \times A \arrow[r, "a \times a"'] & \mathbb{A} \times \mathbb{A}. \end{tikzcd}$$

Since codescent morphisms are part of a weak factorisation system (Proposition 2.4.17), they are closed under products, and so $a \times a$ is a codescent morphism and also they are closed under pullbacks, and so $\hat{a} : a \downarrow a \to \mathbb{A}^{\mathbf{2}}$ is a codescent morphism. By Lemma 7.5.13, it follows that both $a \downarrow a$ and $a \downarrow a \downarrow a$ are discrete; it remains to show that the maps $a \downarrow a \to \mathbf{1}$ and $(p, q) : a \downarrow a \downarrow a \to a \downarrow a \times a \downarrow a$ are in $\mathcal{S}$.

Firstly, since small maps are closed under pullbacks (Lemma 7.5.9), $A \times A \to \mathbf{1}$ is in $\mathcal{S}$. By the assumption that $\mathbb{A}$ is small, $(s, t) : a \downarrow a \to A \times A$ is in $\mathcal{S}$ and so by (S1), it follows that their composite $a \downarrow a \to \mathbf{1}$ is in $\mathcal{S}$.

Next, we note that since $a \downarrow a \to \mathbf{1}$ is in $\mathcal{S}$, it follows by Cancellation (S8) that any discrete opfibration out of $a \downarrow a$ is in $\mathcal{S}$ too; hence the maps $d_1, d_0 : a \downarrow a \to A$ are in $\mathcal{S}$. Therefore, since $p, q : a \downarrow a \downarrow a \to a \downarrow a$ are pullbacks of d_1, d_0, it follows by (S2) that p and q are in $\mathcal{S}$ too.

By Cancellation (S8), it follows that $(p, q) : a \downarrow a \downarrow a \to a \downarrow a \times a \downarrow a$ is in $\mathcal{S}$, since $\pi_{a\downarrow a} \cdot (p, q) = p$ and both p and $\pi_{a\downarrow a}$ are in $\mathcal{S}$, as required.

□

Corollary 7.5.15. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S3), (S4) and (S8). Then $\mathcal{K}_\sigma$ has finite 2-limits.*

Proof. It is well-known that finite 2-limits can be constructed by a terminal object, pullback and powers by **2** (see, for example [Str76]). The result then follows by Lemma 7.5.9 and Proposition 7.5.14. □

Proposition 7.5.16. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5) and (S8). Then $\mathcal{K}_\sigma$ is BO-regular.*

Proof. We have already proven in Corollary 7.5.15 that $\mathcal{K}_\sigma$ is finitely complete. Let $F : X \to Y$ be a codescent morphism in $\mathcal{K}_\sigma$. Then by regularity of $\mathcal{K}$ it is the codescent morphism of its higher kernel in $\mathcal{K}$; by Corollary 7.5.15, this higher kernel is actually in $\mathcal{K}_\sigma$. The fact that codescent morphisms in $\mathcal{K}_\sigma$ are closed under pullback then follows from the fact that they are closed under pullbacks in $\mathcal{K}$. Codescent morphisms are closed under forming the diagonal in $\mathcal{K}$, and since small objects are closed under pullbacks, this also descends to $\mathcal{K}_\sigma$. □

Proposition 7.5.17. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5), (S8) and (S9). Then $\mathcal{K}_\sigma$ is BO-exact.*

Proof. A BO-exact category is precisely a BO-regular category with effective cateads; this is exactly the assumption given in (S9). □

Note that the next proposition is the first time we are using the axiom (Ex).

Proposition 7.5.18. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5),(S7), (S8), (S9) and (Ex). Then we have a $(2, 1)$-equivalence $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K}_\sigma)$). Moreover, **Disc**$(\mathcal{K}_\sigma)$ is a locally cartesian closed, lextensive category a natural numbers object and coequalisers.*

Proof. We check that $\mathcal{K}_\sigma$ satisfies the conditions to be able to apply Proposition 6.6.4. Satisfaction of condition (1) of the theorem is proven in Proposition 7.5.14; conditions (2) and (3) are proven to be satisfied by Proposition 7.5.17; condition (4) is assumption (Ex) and condition (5) is satisfied by the fact that small discrete objects are BO-projective and the definition of small objects. Hence we can apply the theorem and deduce that $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K}_\sigma)$); the extra properties of **Disc**$(\mathcal{K}_\sigma)$ are proven in Theorem 7.5.8 with the exception of a natural numbers object. However, since we have assumed finite 2-colimits in $\mathcal{K}_\sigma$ (S7), and **Disc**$(\mathcal{K}_\sigma)$ is locally cartesian closed, we satisfy the additional hypothesis for Theorem 3.5.2 and so we can deduce that we have free monoids in **Disc**$(\mathcal{K}_\sigma)$, from which [Joh02b, Remark D5.3.4] shows that under the hypothesis our category satisfies, this is equivalent to having a natural numbers object.

□

7.5.3 Groupoid theory

This section explores the consequences of the small objects in a class $(2,1)$-category forming a $(2,1)$-category of internal groupoids. We develop some important aspects of groupoid theory, so that we can justify the suggestion of using class $(2,1)$-categories as a foundation for mathematics. In particular, we show that there is a sensible notion of Yoneda Lemma in a class $(2,1)$-category and that we can internally give a version of Hofmann and Streicher's groupoid model of intensional type theory [HS98].

The notion of Yoneda Structure [SW78] provides a formal categorical approach to developing a suitable notion of Yoneda Lemma abstractly in a 2-category. It formalises the important aspects of the presheaf construction and the Yoneda embedding. We show that our object $\mathbb{S}$ allows us to provide a good notion of presheaf construction.

Proposition 7.5.19. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5),(S7), (S8), (S9) and (Ex). Then $\mathcal{K}_\sigma$ is cartesian closed. In particular, Given small $\mathbb{A}$ and $\mathbb{B}$, the exponential $\mathbb{B}^{\mathbb{A}}$ is again small.*

Proof. By Proposition 7.5.18, it is enough to notice these properties about $\mathbf{Gpd}(\mathcal{E})$ for $\mathcal{E} = \mathbf{Disc}(\mathcal{K}_\sigma)$. But $\mathbf{Disc}(\mathcal{K}_\sigma)$ is locally cartesian closed, in particular cartesian closed. It follows that $\mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ is cartesian closed. □

Note that our theory allows us to not only construct the exponential of two small objects, but more generally as described in the following.

Theorem 7.5.20. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4),(S5),(S7), (S8), (S9) and (Ex). Let $\mathbb{A}, \mathbb{B} \in \mathcal{K}$ with $\mathbb{A}$ a small object. Then the exponential $\mathbb{B}^{\mathbb{A}}$ exists in $\mathcal{K}$.*

Proof. We need to show that $\mathbb{A} \to \mathbf{1}$ is exponentiable. Note that this is not a discrete opfibration in general, but it is an isofibration. Isofibrations in $\mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ are exponentiable precisely because $\mathbf{Disc}(\mathcal{K}_\sigma)$ is locally cartesian closed; this is shown in [NP19]. □

Note that the following corollary actually starts to make use of the Representability Axiom (S6).

Corollary 7.5.21. *Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category with small map classifier $p : \mathbb{S}_* \to \mathbb{S}$. Let $\mathbb{A}$ be a small object. Then the presheaf object $\mathbb{S}^{\mathbb{A}}$ exists.*

This means that we can reconstruct the arguments in [Web07] and get a good Yoneda structure on our category, which we outline as below.

Firstly, given any $f : \mathbb{A} \to \mathbb{B}$ in $\mathcal{K}$, we denote the comma of f against $1_{\mathbb{B}}$ by by $f \downarrow \mathbb{B}$:

$$\begin{tikzcd} f \downarrow \mathbb{B} \arrow[r, "s"] \arrow[d, "t"'] & \mathbb{A} \arrow[d, "f"] \arrow[dl, Rightarrow] \\ \mathbb{B} \arrow[r, equal] & \mathbb{B}. \end{tikzcd}$$

This induces a morphism $(s,t) : f \downarrow \mathbb{B} \to \mathbb{A} \times \mathbb{B}$ by the universal property of $\mathbb{A} \times \mathbb{B}$. This is always a discrete opfibration in a $(2,1)$-category $\mathcal{K}$, since the span $\mathbb{B} \leftarrow f \downarrow \mathbb{B} \to \mathbb{A}$ is always a 2-sided discrete fibration.

Definition 7.5.22. Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category. We call a morphism $f : \mathbb{A} \to \mathbb{B}$ in $\mathcal{K}$ *locally small* if the map $(s,t) : f \downarrow \mathbb{B} \to \mathbb{A} \times \mathbb{B}$ is in $\mathcal{S}$. We call an object $\mathbb{A} \in \mathcal{K}$ *locally small* if $1_{\mathbb{A}}$ is locally small.

We can show that small objects are locally small. To prove this, we use a result which was proven together with Adrian Miranda and Raffael Stenzel, which proves a 2-dimensional version of [Gra21, Lemma 1.15] that is known as the reverse pullback lemma. The proof can be generalised for any kind of descent data; in particular, it can give a new proof of the 1-dimensional result [Gra21, Lemma 1.15] as well as being generalised to an $(\infty,1)$-categorical version of the result. In this thesis, we state it for BO-regular 2-categories.

Lemma 7.5.23. *Let $\mathcal{K}$ be a BO-regular 2-category and consider the pair of commutative squares displayed below, in which g is a codescent morphism.*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{C} \arrow[r, "h"] \arrow[d, "q"] & \mathbb{E} \arrow[d, "r"] \\ \mathbb{B} \arrow[r, "g"'] & \mathbb{D} \arrow[r, "k"'] & \mathbb{F} \end{tikzcd}$$

If the left and outer squares are pullbacks then the right square is also a pullback.

Proof. Recall that for a morphism $G : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$, we denote by $\mathbf{K}(G)$ its higher kernel and $Q\mathbf{K}(G)$ its codescent object (Definition 2.4.11). Since $\mathcal{K}$ is BO-regular, codescent morphisms are stable under pullback, so f is a codescent morphism. Since codescent morphisms are effective, it follows that $\mathbb{D} = Q\mathbf{K}(g)$ and $\mathbb{C} = Q\mathbf{K}(f)$. By noting that comma objects and pullbacks are themselves stable under pullback, it follows that higher kernels are stable under pullback, so that $q^*\mathbf{K}(g) \cong \mathbf{K}(q^*(g))$; since the left square is a pullback, this is equal to $\mathbf{K}(f)$. Hence we have the following calculation:

$$\begin{aligned} \mathbb{C} &= Q\mathbf{K}(f) \\ &= Q\mathbf{K}(q^*(g) \\ &= Q(q^*\mathbf{K}(g)) \\ &= q^*Q\mathbf{K}(g) \\ &= q^*(\mathbb{D}) \end{aligned}$$

showing that the right hand square is a pullback, as required. □

Lemma 7.5.24. *Let $\mathbb{A}$ be a small object in $(\mathcal{K}, \mathcal{S})$. Then $\mathbb{A}$ is locally small.*

Proof. First note that $1_{\mathbb{A}} \downarrow \mathbb{A} = \mathbb{A}^{\mathbf{2}}$. Since $\mathbb{A}$ is small, there is by definition a small discrete object and codescent morphism $a : A_0 \twoheadrightarrow \mathbb{A}$ and, as noted in Proposition 7.5.14, a pullback square:

$$
\begin{tikzcd}
a \downarrow a \arrow[r, two heads, "\hat{a}"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^{\mathbf{2}} \arrow[d] \\
A_0 \times A_0 \arrow[r, two heads, "a \times a"'] & \mathbb{A} \times \mathbb{A}.
\end{tikzcd}
$$

Moreover, $(d_1, d_0) : q \downarrow q \to A_0 \times A_0$ is in $\mathcal{S}$, so there is a pullback square:

$$
\begin{tikzcd}
a \downarrow a \arrow[r] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \arrow[d] \\
A \times A \arrow[r, "\chi_{(d_1,d_0)}"'] & \mathbb{S}.
\end{tikzcd}
\tag{7.2}
$$

By the universal property of the codescent morphism and the definitions we have given, this induces a factorisation of the commutative pullback square in Equation (7.2) into the following:

$$
\begin{tikzcd}
a \downarrow a \arrow[r, two heads, "\hat{a}"] \arrow[d, "{(d_1,d_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^{\mathbf{2}} \arrow[r, dashed, "\exists!"] \arrow[d] & \mathbb{S}_* \arrow[d, "p"] \\
A \times A \arrow[r, two heads, "a \times a"] & \mathbb{A} \times \mathbb{A} \arrow[r, dashed, "\exists!"'] & \mathbb{S}
\end{tikzcd}
$$

Hence, by Lemma 7.5.23, the right hand square is a pullback, and so $\mathbb{A}$ is locally small. □

Remark 7.5.25. Classically, there is a characterisation of small categories as those that are locally small and such that their presheaf category is locally small [FS95]. However, the proof is not valid intuitionistically— see [Hel24, Remark B.11] for more details, and as such we do not have such a theorem in our setting.

Theorem 7.5.26. *Let $(\mathcal{K}, \mathcal{S})$ be a class $(2, 1)$-category. Then there is a Yoneda structure on $\mathcal{K}$ in which the admissible objects are given by the small objects.*

Proof. We follow the proof in [Web07, §5], noting that there, cartesian closedness is only used to construct a presheaf object $\mathbb{S}^{\mathbb{A}}$; by Corollary 7.5.21, for $\mathbb{A}$ small, the presheaf object exists, and by Lemma 7.5.24, small objects are locally small, and so are admissible in the Yoneda structure given in [Web07]. □

Remark 7.5.27. It is possible that we can extend the theorem above to include all locally small objects— indeed, this is what the notion of admissible object was meant to capture originally. To prove this, we would need to show that for any locally small object $\mathbb{A} \in \mathcal{K}$, the isofibration $\mathbb{A} \to \mathbf{1}$ is exponentiable, given that the map $\mathbb{A}^{\mathcal{I}} \to \mathbb{A} \times \mathbb{A}$ is exponentiable.

We move onto showing that we can internally give a model of intensional MLTT in a class $(2, 1)$-category. This builds upon the work of Chapter 6 which provides a model of intensional MLTT for categories of internal groupoids.

Theorem 7.5.28. *Assume $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5),(S7), (S8), (S9) and (Ex). Then, there is a description of a type theoretic algebraic weak factorisation system on $\mathcal{K}_\sigma$, and the right adjoint splitting of the comprehension category associated to this is a model of MLTT with strictly stable choices of Σ-, Π- and $\mathtt{Id}$-types. Moreover, this model of type theory has a univalent universe for 0-types given by the classifier $p : \mathbb{S}_* \to \mathbb{S}$.*

Proof. By Proposition 7.5.18, $\mathcal{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ with $\mathbf{Disc}(\mathcal{K}_\sigma)$ is a locally cartesian closed, lextensive category with a natural numbers object and coequalisers. Theorem 6.5.9 can be applied and give a model of MLTT with the desired properties.

$\square$

Our proof did not require the use of the classifier for small discrete opfibrations. However, if it additionally satisfies (S6) (so that we are working in a class $(2,1)$-category), then it is furthermore a model of intensional MLTT with a univalent universe of small 0-types. The rest of this section is dedicated to proving this fact.

Definition 7.5.29. In MLTT, a dependent type $y : \mathbb{Y} \vdash \mathbb{X}(y)$ is called a 0-*type* if given $y : \mathbb{Y} \vdash x, x' : \mathbb{X}(y)$, the identity type $\mathtt{Id}_{\mathbb{X}(y)}(x, x')$ has either 0 or 1 terms.

Chapter 6 explains the explicit construction of this model of type theory. Recall that For any morphism $F : \mathbb{X} \to \mathbb{Y}$, we denote the diagonal by $\Delta_F : \mathbb{X} \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ as given in the diagram below.

$$\begin{tikzcd}
\mathbb{X} \arrow[dr, dashed, "\Delta_F"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \mathbb{X} \times_{\mathbb{Y}} \mathbb{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X} \arrow[d, "F"] \\
& \mathbb{X} \arrow[r, "F"] & \mathbb{Y}
\end{tikzcd}$$

In our model of type theory, the identity type of a type $y : \mathbb{Y} \vdash \mathbb{X}(y)$, with associated isofibration $F : \mathbb{X} \to \mathbb{Y}$ is given by the second map in the factorisation of the diagonal $\Delta_F : \mathbb{X} \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ as a trivial cofibration followed by a fibration:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathbf{TC}(\Delta_F)"] \arrow[rr, bend right, "\Delta_F"'] & \mathbb{E}(\mathbf{W}\Delta_F) \arrow[r, "\mathbf{F}(\Delta_F)"] & \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}.
\end{tikzcd}$$

Recall also that by construction the map $\mathbf{F}(\Delta_F)$ factorises as

$$\begin{tikzcd}
\mathbb{E}(\mathbf{W}\Delta_F) \arrow[r, "\mathbf{TF}(\mathbf{W}\Delta_F)"] \arrow[rr, bend right, "\mathbf{F}(\Delta_F)"'] & \mathbf{Map}(\Delta_F) \arrow[r, "\mathbf{M}\Delta_F"] & \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}
\end{tikzcd}$$

in which we define $\mathbf{M}\Delta_F : p_{\mathbb{X} \times_{\mathbb{Y}} \mathbb{X}} \circ p_{\mathbb{X} \times (\mathbb{X} \times_{\mathbb{Y}} \mathbb{X})}$, using the notation of Section 6.4.1.

In $\mathbf{Gpd}$, for $(x, x', y) \in \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$, we have $\mathtt{Id}_{\mathbb{X}(y)}(x, x') := \mathbf{F}(\Delta_F)^{-1}(x, x', y)$, and so $y : \mathbb{Y} \vdash \mathbb{X}(y)$ is a 0-type if and only if the fibres of $\mathbf{F}(\Delta_F)^{-1}$ are either empty or have one element— that is if and only if the map $\mathbf{F}(\Delta) : \mathbb{E}(\mathbf{W}\Delta) \to \mathbb{X} \times \mathbb{X}$ is a monomorphism. The following shows that discrete opfibrations are 0-types.

Proposition 7.5.30. *Let $F : \mathbb{X} \to \mathbb{Y}$ be a discrete opfibration. Then $F(\Delta_F) : \mathbb{E}(\mathbf{W}\Delta) \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ is a monomorphism.*

Proof. Firstly, we show that $M\Delta_F : \mathbf{Map}(\Delta_F) \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ is a monomorphism. Since the mapping space construction and monomorphisms are representable notions, it is enough to prove this in **Gpd**. In **Gpd**, we have the explicit description $\mathbf{Map}(\Delta_F) = \{(y, x_1, x_2, h : x_1 \to x_2) : y \in \mathbb{Y}, x_1, x_2 \in F^{-1}(y)\}$. Since F is a discrete opfibration, $F^{-1}(y)$ is a set and so it follows that h must be an equality, and $x_1 = x_2$; it follows that $M\Delta_F$ is a monomorphism.

We note that $\mathbf{TF}(\mathbf{W_F}) : \mathbb{E}(\mathbf{W}\Delta_F) \to \mathbf{Map}(\Delta_F)$ is a trivial fibration, and therefore an equivalence of internal groupoids. The result therefore follows, since the composition of an equivalence with a monomorphism is again a monomorphism.

□

The converse, that 0-types give precisely the discrete opfibrations, is also true.

Proposition 7.5.31. *Let $F : \mathbb{X} \to \mathbb{Y}$ be an isofibration in* $\mathbf{Gpd}(\mathcal{E})$ *such that* $\mathbf{F}(\Delta_F) : \mathbb{E}(\mathbf{W}\Delta_F) \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ *is a monomorphism. Then F is a discrete opfibration.*

Proof. Consider the factorisation of $\mathbf{F}(\Delta_F)$ as $\mathbf{TF}(\mathbf{W}\Delta_F)$ followed by $\mathbf{M}(\Delta_F)$. By the left-cancellation property of monomorphisms, it follows that $\mathbf{TF}(\mathbf{W}\Delta_F)$ is a monomorphism; since it is a trivial fibration, it is also a split epimorphism on objects and fully faithful functor, and is hence an isomorphism. Therefore, it follows that the map $\mathbf{M}(\Delta_F) : \mathbf{Map}(\Delta_F) \to \mathbb{X} \times \mathbb{X}$ is a monomorphism. Since $\mathbf{Map}(\Delta_F)$, monomorphisms and discrete opfibrations are all defined representably, it is therefore enough to prove the claim that if $\mathbf{M}(\Delta_F) : \mathbf{Map}(\Delta_F) \to \mathbb{X} \times_{\mathbb{Y}} \mathbb{X}$ is a monomorphism in **Gpd**, then $F : \mathbb{X} \to \mathbb{Y}$ is a discrete opfibration.

Since F is an isofibration, given any $h : Fx \to y$ in $\mathbb{Y}$, there exists a $f : x \to x'$ in $\mathbb{X}$ with $Ff = h$. We show that this is unique, proving that F is actually a discrete opfibration; this follows from the fact that $\mathbf{M}(\Delta_F)$ is a monomorphism. □

Hence the 0-types in the model are exactly the discrete opfibrations.

Theorem 7.5.32. *Assume $(\mathcal{K}, \mathcal{S})$ is a class $(2,1)$-category. Then, there is a description of a type theoretic algebraic weak factorisation system on $\mathcal{K}_\sigma$, and the right adjoint splitting of the comprehension category associated to this is a model of MLTT with strictly stable choices of Σ-, Π- and* $\mathtt{Id}$*-types. Moreover, this model of type theory has a univalent universe for 0-types given by the classifier $p : \mathbb{S}_* \to \mathbb{S}$.*

Proof. By Theorem 7.5.28, it remains to show the claim that $\mathbb{S}$ is a univalent universe for 0-types. Note that in this case, the adjective univalent can be dropped since two 0-types are equivalent if and only if they are isomorphic. Propositions 7.5.30 and 7.5.31 prove that the 0-types are exactly the small discrete opfibrations; these are classified by axiom (S6) for a class $(2,1)$-category. □

7.6 Adding extra axioms

In this section we add extra axioms into the theory to get stronger results about the internal logic of the small objects.

Recall the definition of full subobject classifier (Definition 4.6.1). In a class $(2,1)$-category, we call a full subobject classifier $\mathbf{1} \to \Omega$ *small* if Ω is a small object in $(\mathcal{K}, \mathcal{S})$.

Proposition 7.6.1. *Suppose $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5), (S7), (S8) and (Ex) and moreover has a small full subobject classifier. Then $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K})$) with **Disc**$(\mathcal{K})$ an elementary topos with natural numbers object.*

Proof. By Proposition 7.5.18, $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K}_\sigma)$) with **Disc**$(\mathcal{K}_\sigma)$ a locally cartesian closed, lextensive category with a natural numbers object and coequalisers. By additionally assuming it has a small full-subobject classifier, it follows from Proposition 4.6.5 that **Disc**$(\mathcal{K}_\sigma)$ has a subobject classifier. Now, **Disc**$(\mathcal{K}_\sigma)$ is a cartesian closed category with finite limits, a subobject classifier and a natural numbers object, and so is an elementary topos with natural numbers object. $\square$

Remark 7.6.2. Assuming that we have a full subobject classifier is equivalent in this context to assuming that we have a discrete monomorphism classifier— that is a discrete monomorphism that classifies all discrete monomorphisms. Thus this extra assumption is in line with Lurie's higher toposes [Lur09], which assume a classifier for every cardinal.

Corollary 7.6.3. *Suppose $(\mathcal{K}, \mathcal{S})$ satisfies (S1), (S2), (S3), (S4), (S5), (S7), (S8) and (Ex) and moreover has a small full subobject classifier. Then $\mathcal{K}_\sigma$ is SO-exact.*

Proof. The previous proposition allows us to deduce that **Disc**$(\mathcal{K})$ is an elementary topos, and therefore exact. In Proposition 5.3.5, it is shown that **Gpd**$(\mathcal{E})$ is SO-exact if and only if $\mathcal{E}$ is exact, and so the result follows. $\square$

By adding some extra axioms, we can show that our theory has a very strong internal set theory given by the small discrete objects by using the work of Chapter 4.

Definition 7.6.4. Let $(\mathcal{K}, \mathcal{S})$ be a class category that is 2-well pointed (Definition 4.4.11), satisfies the Categorified Axiom of Choice (Definition 4.7.12) and moreover has a small full subobject classifier (Definition 4.6.1). We call $(\mathcal{K}, \mathcal{S})$ a model of the *elementary theory of the* $(2,1)$*-category of groupoids*.

Theorem 7.6.5. *Let $(\mathcal{K}, \mathcal{S})$ be a model of the elementary theory of the $(2,1)$-category of groupoids. Then we have a $(2,1)$-equivalence $\mathcal{K}_\sigma \simeq$ **Gpd**(**Disc**$(\mathcal{K})$) with **Disc**$(\mathcal{K})$ a model of the elementary theory of the category sets. Therefore, the internal groupoid theory of such a $(2,1)$-category provides a conservative extension of ZFC.*

Proof. By adding in the extra axioms, $\mathcal{K}_\sigma$ satisfies the elementary theory of the 2-category of small categories (Definition 4.8.1), of which there is a Morita biequivalence between the category of models of such things and the category of models of the elementary theory of the category of sets (Theorem 4.8.13). As such, it is equiconsistent with ZFC without the Axiom of Replacement, by [Shu19, Corollary 9.5]. But (S1) adds the internal, Categorified Axiom of Replacement into the theory, showing that the small objects prove the same theorems as ZFC. $\square$

Definition 7.6.6. Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category. We say that $\mathcal{K}$ satisfies *Strong Separation* if every discrete opfibration that is a monomorphism is small.

In the 1-dimensional theory, the assumption that every monomorphism is small is quite strong and implies full separation of subsets and also implies some of the other axioms for a class category— see [Awo08, Remark 4]. Note that in NBG, **GPD** satisfies Strong Separation by Lemma 7.3.17.

In [Hel24], the importance of the smallness of the monomorphism fibration $\mathrm{Mon}(\mathbb{S}) \hookrightarrow \mathbb{S}$ is pointed out. This is a full monomorphism that is also a setoidal opfibration. For the proof of the following, we assume knowledge of the axioms for a Helfer 2-topos, given in [Hel24, §4].

Theorem 7.6.7. *Suppose $(\mathcal{K}, \mathcal{S})$ is a class $(2,1)$-category and suppose that $\mathcal{K}$ satisfies strong separation and that the monomorphism fibration Mon$(\mathbb{S}) \hookrightarrow \mathbb{S}$ is a small setoidal opfibration. Then $\mathbb{S}$ is an internal 1-topos in $\mathcal{K}$.*

Proof. We show that $\mathcal{K}$ is a Helfer 2-topos. Firstly, $\mathcal{K}$ has finite 2-limits, so in particular has pita limits. As $\mathcal{K}$ is a $(2,1)$-category, $\mathcal{K}$ has strict cores and these are all taken to be the identity. The axioms about complete congruences follow from BO-regularity of $\mathcal{K}$. Finally, the classifier is plentiful; by (S1), the composite of small discrete opfibrations are small and the extra axioms in the theorem give the other axioms for a plentiful generic discrete opfibration. The result therefore follows from [Hel24, Theorem 5.18]. □

7.7 Stability under slicing

In this section, we investigate the stability of our axioms under various kinds of slicing.

7.7.1 Stability under strict slicing

Let $\mathcal{K}$ be a $(2,1)$-category, and let $\mathbb{X} \in \mathcal{K}$. We denote by $\mathcal{K}/_{\mathbb{X}}$ the $(2,1)$-category whose objects are morphisms $F : \mathbb{A} \to \mathbb{X}$, and given $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$, a morphism $\hat{H} : F \to G$ consists of a morphism $H : \mathbb{A} \to \mathbb{B}$ in $\mathcal{K}$ such that the triangle below commutes.

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{B} \arrow[dl, "G"] \\ & \mathbb{X} & \end{tikzcd}$$

We call H the *underlying morphism* of $\hat{H} : F \to G$. Given $\hat{H}, \hat{H}' : F \to G$ in $\mathcal{K}/_{\mathbb{X}}$ which have underlying morphisms $H, H' : \mathbb{A} \to \mathbb{B}$, a 2-cell $\hat{\alpha} : \hat{H} \Rightarrow \hat{H}'$ is simply a 2-cell in $\alpha : H \Rightarrow H'$.

The following is easy to see.

Lemma 7.7.1. *Let $\mathcal{K}$ be a $(2,1)$-category. Then $\mathcal{K}/_{\mathbb{X}}$ has all limits and colimits that $\mathcal{K}$ does.*

Proof. The proof for 1-dimensional (co)limits follows from the corresponding well-known fact for 1-categories. We prove that we have powers and copowers by $\mathbf{2}$. For $F : \mathbb{A} \to \mathbb{X}$, we define $F^{\mathbf{2}} : \mathbb{A}^{\mathbf{2}} \to \mathbb{X}$ by the composite of F with $s : \mathbb{A}^{\mathbf{2}} \to \mathbb{A}$; this has the universal property of the power by $\mathbf{2}$. We define $\mathbf{2} \odot F : \mathbf{2} \odot \mathbb{A} \to \mathbb{X}$ by the composite

$$\mathbf{2} \odot \mathbb{A} \xrightarrow{\cong} (\mathbf{2} \odot \mathbf{1}) \times \mathbb{A} \xrightarrow{\pi_{\mathbb{A}}} \mathbb{A} \xrightarrow{F} \mathbb{X}.$$

□

Similarly, the following is straightforward to see.

Lemma 7.7.2. *Let $\mathcal{K}$ be a $(2,1)$-category, $\mathbb{X} \in \mathcal{K}$ and $F : \mathbb{A} \to \mathbb{X}$. Then*

$$\mathcal{K}/_{\mathbb{X}}/_F \cong \mathcal{K}/\mathbb{A}.$$

It follows easily from the definition given in Definition 4.4.1 that extensivity of $\mathcal{K}$ descends to the slice.

Lemma 7.7.3. *Let $\mathcal{K}$ be an extensive $(2,1)$-category. Then for any $\mathbb{X} \in \mathcal{K}$, $\mathcal{K}/_{\mathbb{X}}$ is extensive.*

Proof. Let $\mathbb{X} \in \mathcal{K}$ and $F : \mathbb{A} \to \mathbb{X}$, $G : \mathbb{B} \to \mathbb{X}$. Then we have the following string of equations, which follow from the definition of extensivity and Lemma 7.7.2, proving the claim.

$$\mathcal{K}/_{\mathbb{X}}/_F \times \mathcal{K}/_{\mathbb{X}}/_G \cong \mathcal{K}_{\mathbb{A}} \times \mathcal{K}_{\mathbb{B}} \simeq \mathcal{K}/_{\mathbb{A}+\mathbb{B}} \cong \mathcal{K}/_{\mathbb{X}}/_{F+G}$$

□

Lemma 7.7.4. *Let $\mathcal{K}$ be a BO-regular (resp. BO-exact) $(2,1)$-category. Then for any $\mathbb{X} \in \mathcal{K}$, $\mathcal{K}/_{\mathbb{X}}$ is BO-regular (resp. BO-exact).*

Proof. This follows from Lemma 7.7.1, since limits and colimits are calculated as in $\mathcal{K}$. □

Let $(\mathcal{K}, \mathcal{S})$ be a pre-class $(2,1)$-category and denote by $\mathcal{S}/_{\mathbb{X}}$ the class of morphisms in $\mathcal{K}/_{\mathbb{X}}$ whose underlying morphism is in $\mathcal{S}$. Then by Lemmas 7.7.4 and 7.7.3, it follows that $(\mathcal{K}/_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$ is a pre-class $(2,1)$-category.

It is straightforward from its definition and the lemmas above that if $(\mathcal{K}, \mathcal{S})$ is a class $(2,1)$-category then $(\mathcal{K}/_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$ satisfies (S1), (S2), (S3), (S4), (S5), (S7), (S8) and (S9).

Let $p : \mathbb{S}_* \to \mathbb{S}$ be a classifier for $(\mathcal{K}, \mathcal{S})$. We note that $p \times \mathbb{X} : \mathbb{S}_* \times \mathbb{X} \to \mathbb{S} \times \mathbb{X}$ is the underlying morphism of a morphism $\widehat{p \times \mathbb{X}} : (\pi_{\mathbb{X}} : \mathbb{S}_* \times \mathbb{X} \to \mathbb{X}) \to (\pi_{\mathbb{X}} : \mathbb{S} \times \mathbb{X} \to \mathbb{X})$.

Proposition 7.7.5. *Let $(\mathcal{K}, \mathcal{S})$ be a pre-class $(2,1)$-category satisfying the Representability axiom (S6) with classifier $p : \mathbb{S}_* \to \mathbb{S}$. Then $\widehat{p \times \mathbb{X}}$ is a classifier for $(\mathcal{K}/_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$.*

Proof. Let $F : \mathbb{A} \to \mathbb{X}$ $G : \mathbb{B} \to \mathbb{X}$ and $\hat{H} : F \to G$ in $\mathcal{S}/_{\mathbb{X}}$ with underlying morphism $H : \mathbb{A} \to \mathbb{B}$ in $\mathcal{S}$. Then there is a pullback square in $\mathcal{K}$

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "H"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \arrow[d, "p"] \\ \mathbb{B} \arrow[r, "\chi_H"'] & \mathbb{S}. \end{tikzcd} \tag{7.3}$$

Since $(\pi_{\mathbb{X}} : - \times \mathbb{X} \to \mathbb{X}) : \mathcal{K} \to \mathcal{K}/_{\mathbb{X}}$ is right adjoint to the forgetful $\Sigma : \mathcal{K}/_{\mathbb{X}} \to \mathcal{K}$, it preserves limits. The following is also a pullback square, which can be shown representably in **Gpd**.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "(\mathrm{id}_{\mathbb{A}}, F)"] \arrow[d, "H"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A} \times \mathbb{X} \arrow[d, "H \times \mathbb{X}"] \\ \mathbb{B} \arrow[r, "(\mathrm{id}_{\mathbb{B}}, G)"'] & \mathbb{B} \times \mathbb{X} \end{tikzcd}$$

We paste the above pullback squares together to get a pullback square

$$
\begin{tikzcd}
\mathbb{A} \arrow[r] \arrow[d, "H"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \times \mathbb{X} \arrow[d, "p"] \\
\mathbb{B} \arrow[r, "\chi_H"'] & \mathbb{S} \times \mathbb{X}.
\end{tikzcd}
\tag{7.4}
$$

Showing that $\hat{H}$ is classified in $\mathcal{K}/_{\mathbb{X}}$.

□

Hence $(\mathcal{K}/_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$ satisfies (S6) too.

Theorem 7.7.6. *Let* $(\mathcal{K}, \mathcal{S})$ *be a pre-class* $(2,1)$*-category satisfying (S1)-(S9). Then* $(\mathcal{K}/_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$ *is a pre-class* $(2,1)$*-category satisfying (S1)-(S9).*

7.7.2 Stability under isofibrational slicing

In the 2-dimensional setting, often the strict slice is not well-behaved, and it is instead a different notion of slice which is important to use. Important variation include the (co)lax slice, the (op)fibrational slice and the isofibrational slice. For a $(2,1)$-category, all these notions coincide but are not usually the same as the strict slice unless $\mathcal{K}$ is a locally discrete $(2,1)$-category. In this section, we prove that all the axioms are stable under isofibrational slicing except for Discrete Projectivity (Ex).

Recall the definition of isofibration given in Definition 2.1.4 as being functors $F : \mathbb{A} \to \mathbb{X}$ with chosen lifts for any 2-cell $\alpha : H \Rightarrow G : \mathbb{B} \to \mathbb{X}$.. Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category, and let $\mathbb{X} \in \mathcal{K}$. Let $\mathcal{K}/^{\cong}_{\mathbb{X}}$ denote the isofibrational slice of $\mathcal{K}$ over $\mathbb{X}$; that is: the objects of this $(2,1)$-category are isofibrations over $\mathbb{X}$ and given $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ isofibrations, a morphism $\hat{H} : F \to G$ consists of a morphism $H : \mathbb{A} \to \mathbb{B}$ in $\mathcal{K}$ such that the triangle below commutes and that sends the chosen lifts of F to the chosen lifts of G.

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{B} \arrow[dl, "G"] \\
& \mathbb{X} &
\end{tikzcd}
$$

We call H the *underlying morphism* of $\hat{H} : F \to G$.

Given $\hat{H}, \hat{H}' : F \to G$ in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ which have underlying morphisms $H, H' : \mathbb{A} \to \mathbb{B}$, a 2-cell $\hat{\alpha} : \hat{H} \Rightarrow \hat{H}'$ is simply a 2-cell in $\alpha : H \Rightarrow H'$. Note that this means that since $\mathcal{K}$ is a $(2,1)$-category, so is $\mathcal{K}/^{\cong}_{\mathbb{X}}$. Since we impose the condition of underlying morphisms preserving the chosen lifts, this is not simply a full sub-$(2,1)$-category of $\mathcal{K}/_{\mathbb{X}}$.

Define $\mathcal{S}/^{\cong}_{\mathbb{X}}$ be be the class of morphisms in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ whose underlying morphism is in $\mathcal{S}$.

We show that $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies all the axioms to be a class $(2,1)$-category except for the axiom (Ex).

Lemma 7.7.7. *Let* $\mathcal{K}$ *be a* $(2,1)$*-category. Then* $\mathcal{K}/^{\cong}_{\mathbb{X}}$ *has any limits that* $\mathcal{K}$ *does, calculated in* $\mathcal{K}$*.*

Proof. Firstly, $\mathcal{K}/^{\cong}_{\mathbb{X}}$ always has a terminal object, given by $1_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$, which is naturally a isofibration. If $\mathcal{K}$ has an initial object $\mathbf{0}$, then for any $\mathbb{X} \in \mathcal{K}$, $\mathbf{0} \to \mathbb{X}$ is a isofibration and is the initial object in $\mathcal{K}/^{\cong}_{\mathbb{X}}$. If $F : \mathbb{A} \to \mathbb{X}$ and

$G : \mathbb{B} \to \mathbb{X}$ are isofibrations, then it is easy to verify that $F \times_{\mathbb{X}} G : \mathbb{A} \times_{\mathbb{X}} \mathbb{B} \to \mathbb{X}$ is a isofibration with the universal property of the product.

Next, we treat equalisers. Given $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ and a parallel pair $\hat{H}, \hat{H}' : F \to G$, we claim that the equaliser of this is given by first taking the equaliser $Q : \mathcal{E} \to \mathbb{A}$ of the underlying morphisms $H, H' : \mathbb{A} \to \mathbb{B}$; the equaliser of $\hat{H}, \hat{H}'$ is given by $P := FQ : \mathcal{E} \to \mathbb{X}$. We must prove that this is an isofibration. Since limits and isofibrations are representable, it is enough to prove this in **Gpd**. Let $e \in \mathcal{E}$ and $\phi : Pe \cong x$ in $\mathbb{X}$. Since $Pe = F(Qe)$ and F is a isofibration, there exists a $\psi : Qe \cong a$ in $\mathbb{A}$ with $F(\psi) = \phi$. Since $Pe = F(Qe) = G(HQe)$ and G is an isofibration, there is a lift $\psi' : HQe \cong b$ in $\mathbb{B}$ such that $G(\psi') = \phi$. Similarly, $Pe = F(Qe) = G(H'Qe)$, and so there exists $\psi'' : H'Qe \cong b'$ such that $G(\psi'') = \phi$; since $H'Qe = HQe$ as $e \in \mathcal{E}$, and H and H' are morphisms of fibrations and so preserve chosen lifts, it follows that $\psi' = \psi''$, and so it follows that there exists an isomorphism $\xi : e \cong e'$ in $\mathcal{E}$ with $Q(\xi) = \psi$; hence $P : \mathcal{E} \to \mathbb{X}$ is an isofibration. It is not hard to show that this has the universal property of the equaliser in $\mathcal{K}/^{\cong}_{\mathbb{X}}$.

Finally, we prove that we have powers by **2**. We note that the map for any $\mathbb{A} \in \mathcal{K}$, the map $(s, t) : \mathbb{A}^{\mathbf{2}} \to \mathbb{A} \times \mathbb{A}$ is a 2-sided fibration, which exactly means that composing it with the projection $s : \mathbb{A}^{\mathbf{2}} \to \mathbb{A}$ gives an (iso)fibration. Hence, given any $F : \mathbb{A} \to \mathbb{X}$, the composite $Fs : \mathbb{A}^{\mathbf{2}} \to \mathbb{X}$ is an isofibration; this has the universal property of the power by **2** since 2-cells in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ are simply those in $\mathcal{K}$ between appropriate morphisms.

□

For the above proof we needed for chosen lifts to be preserved by morphisms between isofibrations.

Since isomorphisms in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ have isomorphisms in $\mathcal{K}$ as their underlying morphism, and all isomorphisms in $\mathcal{K}$ are in $\mathcal{S}$, it follows from Corollary 7.7.14 and Proposition 7.7.11 below that $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ is a pre-class $(2,1)$-category. The small discrete objects are precisely the morphisms of $\mathcal{S}$ with codomain $\mathbb{X}$. If $\mathbb{X}$ is small in $\mathcal{K}$ then the small objects are precisely the objects of $\mathcal{K}_\sigma/^{\cong}_{\mathbb{X}}$.

By definition $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ trivially satisfies (S1) and (S8), since these properties hold for the underlying morphisms in $\mathcal{K}/^{\cong}_{\mathbb{X}}$. By Lemma 7.7.7, $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies (S2). By Corollary 7.7.14, it satisfies (S9).

We give an internal characterisation of isofibrations in terms of comma objects which is reminiscent of the characterisation of monomorphisms in a 1-category being a pullback.

Lemma 7.7.8. *Let $\mathcal{K}$ be a $(2,1)$-category with comma objects and powers by **2**. Let $F : \mathbb{A} \to \mathbb{B}$ be a morphism in $\mathcal{K}$. Then F is an isofibration if and only if the following is a comma square.*

$$\begin{tikzcd} \mathbb{A}^{\mathbf{2}} \arrow[r, "s"] \arrow[d, "Ft"'] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

Proof. Since comma objects and isofibrations are defined representably, it is enough to verify this in **Gpd** in which case it gives the definition of an isofibration. □

Recall the definition of 2-extensive 2-category given in Definition 2.5.8.

Lemma 7.7.9. *Let $\mathcal{K}$ be a 2-extensive $(2,1)$-category. Then the coproduct of comma squares is a comma square. Moreover, for $\mathbb{A}, \mathbb{B} \in \mathcal{K}$, we have $(\mathbb{A}+\mathbb{B})^{\mathbf{2}} \cong \mathbb{A}^{\mathbf{2}} + \mathbb{B}^{\mathbf{2}}$.*

Proof. This follows from the definition of 2-extensive $(2,1)$-category, by both universality and disjointness of coproducts. □

Proposition 7.7.10. *Let $\mathcal{K}$ be an 2-extensive $(2,1)$-category. Then $\mathcal{K}/^{\cong}_{\mathbb{X}}$ has coproducts that $\mathcal{K}$ does, calculated in $\mathcal{K}$.*

Proof. Let $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ be isofibrations in $\mathcal{K}$. By Lemma 7.7.8, we have comma squares

$$\begin{tikzcd} \mathbb{A}^{\mathbf{2}} \arrow[r, "s"] \arrow[d, "Ft"'] & \mathbb{A} \arrow[d, "F"] \arrow[dl, Rightarrow] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, "s"] \arrow[d, "Gt"'] & \mathbb{B} \arrow[d, "G"] \arrow[dl, Rightarrow] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

and by Lemma 7.7.9, their coproduct is a comma square

$$\begin{tikzcd} \mathbb{A}^{\mathbf{2}} + \mathbb{B}^{\mathbf{2}} \arrow[r, "s+s"] \arrow[d, "Ft+Gt"'] & \mathbb{A} \arrow[d, "F+G"] \arrow[dl, Rightarrow] \\ \mathbb{X}+\mathbb{X} \arrow[r, equal] & \mathbb{X}+\mathbb{X} \end{tikzcd} \tag{7.5}$$

which, upon noting that $\mathbb{A}^{\mathbf{2}} + \mathbb{B}^{\mathbf{2}} \cong (\mathbb{A}+\mathbb{B})^{\mathbf{2}}$ (Lemma 7.7.9), is a witness of the fact that $F+G : \mathbb{A}+\mathbb{B} \to \mathbb{X}+\mathbb{X}$ is an isofibration by Lemma 7.7.8. We claim that $(\iota_{\mathbb{X}}, \iota_{\mathbb{X}}) : \mathbb{X}+\mathbb{X} \to \mathbb{X}$ is an isofibration; this follows from the fact that the diagram below is a pullback due to 2-extensivity of $\mathcal{K}$:

$$\begin{tikzcd} \mathbb{X}^{\mathbf{2}} + \mathbb{X}^{\mathbf{2}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{X}^{\mathbf{2}} \arrow[d, "{(s,t)}"] \\ (\mathbb{X}\times\mathbb{X}) + (\mathbb{X}\times\mathbb{X}) \arrow[r, "{(\iota_{\mathbb{X}+\mathbb{X}}, \iota_{\mathbb{X}+\mathbb{X}})}"'] & \mathbb{X}\times\mathbb{X}. \end{tikzcd}$$

Hence by noting that $(\mathbb{X}\times\mathbb{X}) + (\mathbb{X}\times\mathbb{X}) \cong \mathbb{X}\times(\mathbb{X}+\mathbb{X})$ and under this isomorphism $\iota_{\mathbb{X}+\mathbb{X}} \cong (\mathrm{id}_{\mathbb{X}}, (\iota_{\mathbb{X}}, \iota_{\mathbb{X}}))$, it follows that $(\iota_{\mathbb{X}}, \iota_{\mathbb{X}}) : \mathbb{X}+\mathbb{X} \to \mathbb{X}$ is an isofibration. Therefore, the composite $(F,G) = (\iota_X, \iota_{\mathbb{X}}) \circ (F+G) : \mathbb{A}+\mathbb{B} \to \mathbb{X}$ is an isofibration, as required. □

As a result, it follows that $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies closure under sums (S4). Recalling the definition of 2-extensivity given in Definition 2.5.8, we can deduce the following.

Proposition 7.7.11. *Let $\mathcal{K}$ be 2-extensive. Then, for any $\mathbb{X} \in \mathcal{K}$, $\mathcal{K}/^{\cong}_{\mathbb{X}}$ is 2-extensive.*

Proof. This follows since we have shown that coproducts, comma objects, pullbacks, powers by **2** and initial object are calculated as in $\mathcal{K}$ (proven in Lemma 7.7.10 and Lemma 7.7.7), and so the relevant diagrams given in Definition 2.5.8 are disjoint and universal in $\mathcal{K}$, and hence they are in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ too. □

Remark 7.7.12. Note that Proposition 7.7.11 does not seem to be true if we use the definition of extensivity given in Definition 4.4.1, since it is not true that $\mathcal{K}/^{\cong}_{\mathbb{X}}/(F : \mathbb{A} \to \mathbb{X})$ is equal to $\mathcal{K}/\mathbb{A}$, as the isofibrational slice is not a full subcategory of $\mathcal{K}$. This gives motivation for using Definition 2.5.8 instead of Definition 4.4.1.

For other colimits, we appeal to [Mes25b], which we can do since in a $(2,1)$-category, the isofibrational slice is the same as the lax slice.

Theorem 7.7.13 (Theorem 3.15 [Mes25b]). *Let $\mathcal{K}$ be a $(2,1)$-category. Then $\mathcal{K}/^{\cong}_{\mathbb{X}} \to \mathcal{K}$ creates 2-colimits. Hence $\mathcal{K}/^{\cong}_{\mathbb{X}}$ has all colimits that $\mathcal{K}$ does, calculated in $\mathcal{K}$.*

Therefore, we have shown that $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies (S7). We also have the following.

Corollary 7.7.14. *Let $\mathcal{K}$ be a BO-regular (resp. BO-exact) $(2,1)$-category. Then $\mathcal{K}/^{\cong}_{\mathbb{X}}$ is a BO-regular (resp. BO-exact) $(2,1)$-category.*

Proof. By Theorem 7.7.13, codescent objects of cateads exist in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ and calculated in $\mathcal{K}$; similarly, pullbacks are calculated in $\mathcal{K}$, and so all of the requirements for $\mathcal{K}/^{\cong}_{\mathbb{X}}$ to be BO-regular (resp. BO-exact) are true since $\mathcal{K}$ is BO-regular (resp. BO-exact). □

We note that $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/_{\mathbb{X}})$ satisfies the Representability axiom (S6), by the same reasoning as Proposition 7.7.5, since $\pi_{\mathbb{X}} : \mathbb{S} \times \mathbb{X} \to \mathbb{X}$ and $\pi_{\mathbb{X}} : \mathbb{S}_* \times \mathbb{X} \to \mathbb{X}$ are isofibrations.

We move onto exponentiability of maps in $\mathcal{S}/^{\cong}_{\mathbb{X}}$.

Lemma 7.7.15. *Let $\mathcal{K}$ be a $(2,1)$-category, and let $G : \mathbb{B} \to \mathbb{X}$ be an isofibration. Then $\mathcal{K}/^{\cong}_{\mathbb{X}}/^{\cong}_{G} \cong \mathcal{K}/^{\cong}_{\mathbb{A}}$.*

Proof. Let $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ be isofibrations. An isofibration $\hat{H} : F \to G$ in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ is an isofibration $H : \mathbb{A} \to \mathbb{B}$ such that $GH = F$ and such that H sends chosen lifts of F to chosen lifts of G. This is proven in [Jac99, Proposition 9.3.4].

Since isofibrations are closed under composition, given an isofibration $H : \mathbb{A} \to \mathbb{B}$, the composite $GH : \mathbb{A} \to \mathbb{B}$ is an isofibration; note that by definition of the composition of isofibrations this gives an isofibration $\hat{H} : GH \to G$ in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ with underlying isofibration H. □

Proposition 7.7.16. *Let $\mathcal{K}$ be a $(2,1)$-category and let $F : \mathbb{A} \to \mathbb{X}$ and $G : \mathbb{B} \to \mathbb{X}$ be isofibrations. Then a morphism $\hat{H} : F \to G$ is exponentiable in $\mathcal{K}/^{\cong}_{\mathbb{X}}$ if its underlying morphism $H : \mathbb{A} \to \mathbb{B}$ is exponentiable in $\mathcal{K}$.*

Proof. This follows easily from Lemma 7.7.15; the pullback functor $\hat{H}^* : \mathcal{K}/^{\cong}_{\mathbb{X}}/^{\cong}_{G} \to \mathcal{K}/^{\cong}_{\mathbb{X}}/^{\cong}_{F}$ is isomorphic to the pullback functor $H^* : \mathcal{K}/^{\cong}\mathbb{B} \to \mathcal{K}/^{\cong}\mathbb{A}$. If H is exponentiable, then since isofibrations are stable under pullback, the right adjoint to $\mathcal{K}/\mathbb{B} \to \mathcal{K}/\mathbb{A}$ restricts to the isofibrational slice. □

Hence $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies (S5) since maps in $\mathcal{S}$ are exponentiable.

Theorem 7.7.17. *Let $(\mathcal{K}, \mathcal{S})$ be a pre-class $(2,1)$-category satisfying (S1)-(S9), and let $\mathbb{X} \in \mathcal{K}$. Then $(\mathcal{K}/^{\cong}_{\mathbb{X}}, \mathcal{S}/^{\cong}_{\mathbb{X}})$ satisfies (S1)-(S9).*

Not all of our axioms for a class $(2,1)$-category are stable under isofibrational slicing.

Proposition 7.7.18. *The axiom (Ex) is not stable under isofibrational slice.*

Proof. Consider in **Gpd** the following groupoids:

$$\mathcal{I} := \begin{tikzcd} 0 \arrow[r, "\sim"] & 1 \end{tikzcd}$$

$$\mathcal{N} := \begin{tikzcd} & 2 \\ 0 \arrow[ur, "\sim"] \arrow[r, "\sim"] & 1 \end{tikzcd}$$

$$\mathcal{M} := \begin{tikzcd} & 2 \\ 0 \arrow[ur, "\sim"] & 1 \end{tikzcd}$$

and consider the discrete opfibration $\mathrm{id}_{\mathcal{I}}$ as well as the isofibrations $F : \mathcal{N} \to \mathcal{I}$ and $G : \mathcal{M} \to \mathcal{I}$ defined on objects by the rules $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$ and defined on morphisms functorially. These are clearly all isofibrations over $\mathcal{I}$. Moreover, the inclusion $Q : \mathcal{M} \hookrightarrow \mathcal{N}$ is bijective on objects and is therefore a codescent morphism in **Gpd** and hence by Theorem 7.7.13 in $\mathbf{Gpd}/^{\cong}_{\mathcal{I}}$. Define the inclusion $H : \mathcal{I} \hookrightarrow \mathcal{N}$. Notice that $FH = \mathrm{id}_{\mathcal{I}}$ and $FQ = G$ so that we have morphisms $\hat{H} : \mathrm{id}_{\mathcal{I}} \to F$ and $\hat{Q} : G \to F$ in $\mathbf{Gpd}/^{\cong}_{\mathcal{I}}$.

$$\begin{tikzcd} & G \arrow[d, two heads, "\hat{Q}"] \\ \mathrm{id}_{\mathcal{I}} \arrow[r, "\hat{H}"'] & F \end{tikzcd}$$

However, this diagram has no lift since there is no arrow from 0 to 1 in $\mathcal{M}$; hence $\mathcal{I}$ is not BO-projective, despite it being a discrete object in $\mathbf{Gpd}/^{\cong}_{\mathcal{I}}$.

□

7.7.3 Slicing over a discrete object

Whilst Proposition 7.7.18 tells us that the isofibrational slice cannot be a class $(2,1)$-category in general, in this subsection, we prove that in the case that we are slicing over a discrete object, the class $(2,1)$-category structure does descend.

Let $X \in \mathbf{Disc}(\mathcal{K})$. We note that any morphism $\mathbb{A} \to X$ is an isofibration since the only invertible 2-cells into X are identities. Hence $\mathcal{K}/^{\cong}_{X} \cong \mathcal{K}/_{X}$, where $\mathcal{K}/_{X}$ is the usual slice.

Lemma 7.7.19. *Let $X \in \mathbf{Disc}(\mathcal{K})$; then $\mathbf{Disc}(\mathcal{K}/_{X}) \cong \mathbf{Disc}(\mathcal{K})/_{X}$.*

Proof. Let $F : \mathbb{A} \to X$ be a discrete object in $\mathcal{K}/X$. Let $\alpha : H \to H' : \mathbb{B} \to \mathbb{A}$. We want to show that $H = H'$ and $\alpha = \mathrm{id}_H$. Consider the whiskerings $F.\alpha : \mathbb{B} \to X$. Since X is discrete, it follows that $FH = FH'$, and so H and H' constitute the underlying morphisms of a parallel pair of morphisms $\hat{H}, \hat{H}' : FH \to F$ in $\mathcal{K}/X$, and α is the underlying a 2-cell between them. Since $F : \mathbb{A} \to \mathbb{X}$ was assumed to be discrete, it follows that $\hat{H} = \hat{H}'$ and so $H = H'$ and $\alpha = \mathrm{id}_H$ as required. Conversely, it is easy to see that any discrete object in $\mathcal{K}$ gives a discrete object in $\mathcal{K}/X$. □

Hence, we have the following:

Lemma 7.7.20. *Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category, and let $X \in$ **Disc**$(\mathcal{K})$. Then every small discrete object in $\mathcal{K}/_X$ is BO-projective.*

Proof. This follows directly from the fact that small discrete objects in $\mathcal{K}$ are BO-projective, and that discrete objects in $\mathcal{K}/_X$ are morphisms in **Disc**$(\mathcal{K})$ with target X. $\square$

Theorem 7.7.21. *(Stability under slicing over a discrete object) Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category and let $X \in$ **Disc**$(\mathcal{K})$. Then $(\mathcal{K}/_X, \mathcal{S}/_X)$ is a class $(2,1)$-category.*

*In particular, we have $(\mathcal{K}/_X)_\sigma \simeq$ **Gpd**$($**Disc**$(\mathcal{K}_\sigma)/_X$.*

7.8 Examples

In this section, we explore some examples of class $(2,1)$-categories.

7.8.1 Large groupoids

The category of large groupoids **GPD** together with the class of small discrete opfibrations as defined in Definition 7.3.5 gives an example of a class $(2,1)$-category, as we explain below.

Firstly, **GPD** is BO-regular and lextensive by Proposition 7.3.2 and Proposition 7.3.3, so by letting $\mathcal{S}$ be the class of small discrete opfibrations, $(\mathbf{CAT}, \mathcal{S})$ is a pre-class-$(2,1)$-category. A discrete object in $(\mathbf{CAT}, \mathcal{S})$ is small if it is (isomorphic to) a set; thus a small object in $(\mathbf{CAT}, \mathcal{S})$ is a small category. It satisfies (S1) by Proposition 7.3.10 and Proposition 7.3.11. It satisfies (S2) by Proposition 7.3.13, (S4) by Proposition 7.3.14, (S5) by Proposition 7.3.15 and (S6) by Proposition 7.3.8. It is clear that (S3) is satisfied since 0 and $\mathbf{1} + \mathbf{1}$ are sets by (NBG3) and (NBG11). It satisfies (S7) by Proposition 7.3.4, (S9) by Proposition 7.3.16, (S9) since it is BO-exact and (Ex) since it is of the form $\mathbf{Gpd}(\mathbf{Class})$ for the 1-category **Class**. This proves that is a Class $(2,1)$-category.

It has additional features: because **Class** is well-pointed (Proposition 7.2.17), it follows that **GPD** is 2-well-pointed. If we additionally assume the Axiom of Choice in our metatheory, then it follows that **GPD** satisfies the Categorified Axiom of Choice, and so satisfies the extra axioms in Section 7.6 showing that NBG+ with Choice allows us to construct models of class $(2,1)$-categories with these extra axioms. This proves that this theory is consistent relative to NBG+.

We note that Mon$(\underline{\mathbf{Set}}) \hookrightarrow \underline{\mathbf{Set}}$ is a small setoidal opfibration, and so we also have shown that $\underline{\mathbf{Set}} \in \mathbf{GPD}$ is an internal elementary topos.

7.8.2 Relating class categories and class $(2,1)$-categories

This example relates class $(2,1)$-categories with class categories. In particular, we show that given a class $(2,1)$-category, we can build an associated class category out of its discrete objects. Conversely, we explain how given a class category, we can build a class $(2,1)$-category by considering groupoids internal to it. First, we recall the specific definition of class category which we will use, which roughly follows the formulation used in [Awo+14].

Let $\mathbb{C}$ be a Heyting 1-category and let $\mathcal{S}$ be a class of morphisms of $\mathbb{C}$ that includes all identities. Suppose that:

(A1) (Replacement) Small maps are closed under composition.

(A2) (Stability) In any pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, "Q"'] & Y \end{tikzcd} \tag{7.6}$$

if f belongs to $\mathcal{S}$ then g belongs to $\mathcal{S}$.

(A3) (Finiteness) The maps $0 \to \mathbf{1}$ and $\mathbf{1}+\mathbf{1} \to \mathbf{1}$ belong to $\mathcal{S}$.

(A4) (Sums) If $X \to Y$ and $X' \to Y'$ belong to $\mathcal{S}$ then so does $X + X' \to Y + Y'$.

(A5) (Exponentiability) Every map in $\mathcal{S}$ is exponentiable.

(A6) (Representability) There exists a morphism $\pi : E \to V$ in $\mathcal{S}$ such that for any small map $f : X \to Y$, there exists a (not necessarily unique) map $\chi_f : X \to V$ such that the following diagram is a pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ Y \arrow[r, "\chi_f"'] & V. \end{tikzcd}$$

(A7) (Diagonals) Every diagonal $\Delta = (\mathrm{id}_C, \mathrm{id}_C) : C \to C \times C$ is small.

(A8) (Quotients) If $f \circ e$ is small and e is a regular epimorphism, then f is small.

(A9) (Infinity) There exists a natural numbers object $N \in \mathbb{C}$, such that $N \to \mathbf{1}$ is in $\mathcal{S}$.

Definition 7.8.1. We call $(\mathbb{C}, \mathcal{S})$ satisfying axioms $(A1) - (A7)$ a *class category*. We call $X \in \mathbb{C}$ *small* if $X \to \mathbf{1}$ is in $\mathcal{S}$. We denote the full subcategory of small objects by $\mathbb{C}_\sigma$.

We note these axioms prove the following:

Lemma 7.8.2. *[Awo08, §2.1] Let $(\mathbb{C}, \mathcal{S})$ be a class category. Then*

- *(Separation) Every regular monomorphism is small.*
- *(Cancellation) If $g \circ f$ is small, then f is small.*
- *$\mathbb{C}_\sigma$ is an elementary topos with natural numbers object.*

The discrete class category

From a class $(2,1)$-category satisfying some extra axioms, we can obtain a class category.

For a $(2,1)$-category $\mathcal{K}$ and an object $\mathbb{X} \in \mathcal{K}$, we denote by $\mathbf{Sub}(\mathbb{X})$ the full sub-$(2,1)$-category of $\mathcal{K}/_{\mathbb{X}}$ on those monomorphisms $m : \mathbb{A} \hookrightarrow \mathbb{X}$.

Definition 7.8.3. Let $\mathcal{K}$ be a SO-regular and extensive $(2,1)$-category. If, for any $f : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$, the pullback functor $f^* : \mathbf{Sub}(\mathbb{Y}) \to \mathbf{Sub}(\mathbb{X})$ has a right adjoint, we call $\mathcal{K}$ a *SO*-Heyting $(2,1)$-category.

Let $(\mathcal{K}, \mathcal{S})$ be a SO-Heyting class $(2,1)$-category such that $\mathcal{K}$ is BO-exact, all discrete objects are BO-projective and there are enough BO-projectives— note that these extra conditions allow us to show that $\mathcal{K} \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$ (and not just $\mathcal{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$), which is a property that has not been needed up until now. Suppose additionally that it satisfies Strong Separation in the sense of Definition 7.6.6.

We show that there is a sensible way in which $\mathbf{Disc}(\mathcal{K})$ can be equipped with the structure of a class 1-category, so that the class $(2,1)$-category is similar to groupoids internal to a class 1-category.

Write $\mathbf{Disc}(\mathcal{S}) \subset \mathcal{S}$ for the class of morphisms in $\mathcal{S}$ between discrete objects in $\mathcal{K}$. We show that $(\mathbf{Disc}(\mathcal{K}), \mathbf{Disc}(\mathcal{S}))$ is a class category.

Lemma 7.8.4. *Let $\mathcal{K}$ be a SO-Heyting $(2,1)$-category. Then $\mathbf{Disc}(\mathcal{K})$ is a Heyting 1-category.*

Proof. By Lemma 2.5.10, 2-extensivity of $\mathcal{K}$ implies that $\mathbf{Disc}(\mathcal{K})$ is extensive. If $\mathcal{K}$ is SO-regular, then the proof of Proposition 5.3.4 can be adapted to show that $\mathbf{Disc}(\mathcal{K})$ is regular, essentially because SO-quotients of discrete objects are simply coequalisers. Finally, for discrete objects X, $\mathbf{Sub}(X)$ is a locally discrete $(2,1)$-category, and so can be equivalently thought of as a 1-category without loss of information. Therefore, for any $f : X \to Y$ in $\mathbf{Disc}(\mathcal{K})$, the $(2,1)$-adjunction $f^* \vdash \Pi_f$ is actually a 1-adjunction, showing that $\mathbf{Disc}(\mathcal{K})$ is Heyting. $\square$

Axioms (A1)-(A4) for $(\mathbf{Disc}(\mathcal{K}), \mathbf{Disc}(\mathcal{S}))$ have been proven in Section 7.5.1. Moreover, (A9) was shown to hold in Proposition 7.5.18. Exponentiability (A5) follows from definition of maps in $\mathbf{Disc}(\mathcal{S})$ being in $\mathcal{S}$, and hence exponentiable there.

Lemma 7.8.5. *Let $C \in \mathbf{Disc}(\mathcal{K})$. Then $\Delta_C : C \to C \times C$ is in $\mathbf{Disc}(\mathcal{S})$. Therefore, $(\mathbf{Disc}(\mathcal{K}), \mathbf{Disc}(\mathcal{S}))$ has small diagonals (A7).*

Proof. Since Δ_C is a monomorphism between discrete objects, then it is a monomorphism that is a discrete opfibration in $\mathcal{K}$. Hence, by Strong Separation, it is in $\mathcal{S}$ and is thus in $\mathbf{Disc}(\mathcal{S})$. $\square$

Lemma 7.8.6. *$(\mathbf{Disc}(\mathcal{K}), \mathbf{Disc}(\mathcal{S}))$ has small quotients (A8).*

Proof. This follows from the Colimits Axiom (S7). Let $e : X \to Y$ be a regular epimorphism and $f : Y \to Z$ be any map and suppose that $fe : X \to Z$ is in $\mathcal{S}$. We work in the slice $(2,1)$-category $(\mathcal{K}/_Z, \mathcal{S}/_Z)$, noting that fe is a small object here and e remains a regular epimorphism here. By (S7), the object $f : Y \to Z$ is a small object in $(\mathcal{K}/_Z, \mathcal{S}/_Z)$, and so is a smal map in $\mathcal{S}$, as required. $\square$

The hard axiom to show is that of representability; that discrete small maps have a classifier which is itself discrete.

Consider the classifier $p : \mathbb{S}_* \to \mathbb{S}$: since $\mathcal{K} \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$, $\mathbb{S}$ is covered by a codescent morphism $q : V \twoheadrightarrow \mathbb{S}$ with V discrete; similarly, $\mathbb{S}_*$ is covered by a codescent morphism $r : E \twoheadrightarrow \mathbb{S}_*$ in which E is discrete, as exhibited below by the following pullback square:

$$\begin{tikzcd} E \arrow[r, two heads, "r"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \arrow[d, "p"] \\ V \arrow[r, two heads, "q"'] & \mathbb{S}. \end{tikzcd}$$

From the above pullback square, $p_0 : E \to V$ is in $\mathbf{Disc}(\mathcal{S})$. Our claim is the $p_0 : E \to V$ is the classifier for small discrete morphisms.

Proposition 7.8.7. *The morphism $p_0 : E \to V$ classifies morphisms in* $\mathbf{Disc}(\mathcal{S})$*; for any morphism $f : X \to Y$ in* $\mathbf{Disc}(\mathcal{S})$*, there exists a morphism $(\chi_f)_0 : Y \to V$ such that $p_0^*((\chi_f)_0) \cong X$ and the following is a pullback square in* $\mathbf{Disc}(\mathcal{K})$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p_0"] \\ Y \arrow[r, "(\chi_f)_0"'] & V \end{tikzcd}$$

Proof. We work in $\mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$; the discrete object are therefore equivalent to those in the image of $\mathbf{disc} : \mathbf{Disc}(\mathcal{K}) \to \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$.

Let $f : X \to Y$ be in $\mathbf{Disc}(\mathcal{S})$. As $f : X \to Y$ is in $\mathcal{S}$, by (S6), there is a classifying map $\chi_f : Y \to \mathbb{S}$ giving us pullback square in $\mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$:

$$\begin{tikzcd} X \arrow[r, "p^*(\chi_f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_* \arrow[d, "p"] \\ Y \arrow[r, "\chi_f"'] & \mathbb{S}. \end{tikzcd}$$

Since $(-)_0 : \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K})) \to \mathbf{Disc}(\mathcal{K})$ is right adjoint to $\mathbf{disc} : \mathbf{Disc}(\mathcal{K}) \to \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}))$, it preserves limits and therefore pullbacks, giving us the pullback square required by noting that $p_0 : (\mathbb{S}_*)_0 \to \mathbb{S}_0 = p_0 : E \to V$.

$\square$

Theorem 7.8.8. *Let $(\mathcal{K}, \mathcal{S})$ be a class $(2,1)$-category that is additionally BO-exact and has enough discrete objects. Then* $(\mathbf{Disc}(\mathcal{K}), \mathbf{Disc}(\mathcal{S}))$ *forms a class 1-category.*

Groupoids internal to a class category

Let $(\mathbb{C}, \mathcal{S})$ be a class 1-category, and let $\pi : E \to V$ be the classifier for small maps in $\mathcal{S}$. Using the internal language of $\mathbb{C}$, we can define the object of "set isomorphisms" as a class of subsets of the product $V \times V$, to form an internal groupoid $\mathbb{V} \in \mathbf{Gpd}(\mathbb{C})$. Let $\mathbf{pt} : \mathbf{1} \to \mathbb{V}$ be the morphism in $\mathbf{Gpd}(\mathbb{C})$ which picks out the terminal set; this can be done since V forms a model of ZF. We perform a similar construction on E to get $\mathbb{E} \in \mathbf{Gpd}(\mathbb{C})$, and obtain an internal functor $p : \mathbb{E} \to \mathbb{V}$ defined by $\pi : E \to V$ on objects and the restriction of $\pi \times \pi : E \times E \to V \times V$ along the inclusion $E_1 \hookrightarrow E \times E$ on morphisms; this lands in V_1. The internal functor $p : \mathbb{E} \to \mathbb{V}$ is moreover a discrete opfibration.

Define a class of maps $\mathcal{S}'$ in $\mathbf{Gpd}(\mathbb{C})$ by those which are pullbacks of $p : \mathbb{E} \to \mathbb{V}$. The pair $(\mathbf{Gpd}(\mathbb{C}), \mathcal{S}')$ forms a class $(2,1)$-category. Most of the axioms are trivial to show and either follow from the definition of being the pullback of a map or by the axioms for a class 1-category; the only difficult one is exponentiability of maps in $\mathcal{S}'$. We explain how to prove this.

Since $\pi : E \to V$ was a small map, it is exponentiable in $\mathbb{C}$; since the inclusion $E_1 \hookrightarrow E \times E$ is a monomorphism and therefore a small map, and small maps are closed under products, so $\pi_1 \times \pi_1 : E \times E \to V \times V$ is a small map, and small maps are closed under composition, it follows that $p_1 : E_1 \to V_1$ is a small map. Therefore, it is exponentiable. By [NP19, Theorem 4.5], since p is a levelwise exponentiable (discrete op)fibration, the map p is therefore exponentiable. As any pullback of an exponentiable map is itself exponentiable, it follows that maps in $\mathcal{S}'$ are exponentiable, as required.

7.8.3 Prestacks on a locally small 1-category

Let $\mathbb{C}$ be a small 1-category and consider the $(2,1)$-category $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$. Note that this is well-defined when working in NBG+. Let $\mathcal{S}$ be the class of discrete opfibrations which are pointwise small discrete opfibrations. Note that this agrees with the terminology in [CM24, Definition 3.7]. We show that $([\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}], \mathcal{S})$ forms a class $(2,1)$-category. Firstly, $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}] \cong \mathbf{Gpd}([\mathbb{C}^{\mathrm{op}}, \mathbf{Class}])$ and so it is BO-exact and therefore BO-regular by [BG14, Proposition 60], since $[\mathbb{C}^{\mathrm{op}}, \mathbf{Class}]$ is finitely complete because $\mathbf{Class}$ is. It is 2-extensive since $\mathbf{Class}$ is extensive and hence so is $[\mathbb{C}^{\mathrm{op}}, \mathbf{Class}]$, and so the result follows from Proposition 2.5.9. Hence $([\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}], \mathcal{S})$ is a pre-class $(2,1)$-category.

All of the class $(2,1)$-category axioms except for (S5) and (S6) are satisfied because the relevant notions are calculated pointwise in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ and they hold in **GPD**— that is isomorphism, composition, limits, colimits, discrete objects, cateads.

For (S5), we can apply [SW73, Theorem 2.12], which states that in a functor 2-category $[\mathbb{C}, \mathbb{D}]$ in which $\mathbb{D}$ is finitely complete and has products indexed by the underlying set of morphisms of $\mathbb{C}$, then a morphism $p : E \to B$ is exponentiable if for all $c \in \mathbb{C}$ the map $p(c) : E(c) \to B(c)$ is exponentiable. Therefore, in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Gpd}]$, the small discrete opfibrations are exponentiable since they are pointwise small and hence pointwise exponentiable.

It remains to check is that we have a classifier for small maps. It is proven that there is a classifier for the small maps we have defined in [Mes25a, Theorem 4.14] which is an extension of the universes constructed in [HS97] for presheaves to the 2-dimensional setting. In [HS97], they pose the question of how their universe relates to class categories; this example answers the question in the 2-dimensional setting, although it is answered in the 1-dimensional setting in [Awo24]. For completeness, we describe the classifier below.

Definition 7.8.9. Define $\tilde{\Omega} : \mathbb{C}^{\mathrm{op}} \to \mathbf{GPD}$ by letting

$$c \mapsto [(\mathbb{C}/_c)^{\mathrm{op}}, \mathbf{Set}]$$
$$f : c \leftarrow d \mapsto - \circ (f \circ =) : [(\mathbb{C}/_c)^{\mathrm{op}}, \mathbf{Set}] \to [(\mathbb{C}/d)^{\mathrm{op}}, \mathbf{Set}]$$

Note that the target of this is a large groupoid since $\mathbb{C}$ is small and therefore $\mathbb{C}/_c$ is too, and so $[(\mathbb{C}/c)^{op}, \mathbf{Set}]$ is in **GPD** and similarly f is a set (since c and d are) so the image $- \circ (f \circ =)$ is a set by Replacement, and so the collection of all such functions is a class, which can be defined formally by the Class Existence Theorem.

Let $\mathbf{pt} : \mathbf{1} \to \tilde{\Omega}$ be the morphism in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ which for each $c \in \mathbb{C}$ picks out the constant prestack on the terminal in $[(\mathbb{C}/_c)^{\mathrm{op}}, \mathbf{Set}]$. Define $\tilde{\Omega}_*$ by the pullback

$$\begin{tikzcd}
\tilde{\Omega}_* \arrow[r] \arrow[d, "\pi_1"'] & \mathbf{1} \arrow[d, "\mathbf{pt}"] \\
\tilde{\Omega}^{\mathbf{I}} \arrow[r, "d_0"'] & \tilde{\Omega}.
\end{tikzcd}$$

and define $p := d_1\pi_1 : \tilde{\Omega}_* \to \tilde{\Omega}$. This is our classifying small discrete opfibration.

Proposition 7.8.10. *[Mes25a, Theorem 4.14] The map $p : \tilde{\Omega}_* \to \tilde{\Omega}$ is a classifier for maps in $\mathcal{S}$.*

Remark 7.8.11. Note that in the context of $(2,1)$-categories, this is the same classifier as is given in [Hel24, §7.1].

7.8.4 Semi-strict Stacks

We turn our attention to $(2,1)$-categories of semi-strict stacks, which we define below.

Definition 7.8.12 ([Str82])**.** Let $\mathbb{C}$ be a small 1-category.

Define $\mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ to be the $(2,1)$-category of pseudofunctors $\mathbb{C} \to \mathbf{GPD}$, pseudonatural transformations and invertible modifications.

A $(2,1)$-category $\mathcal{K}_{\mathbf{Ps}}$ is called a $(2,1)$*-category of stacks* if it is a full, bireflective sub-$(2,1)$-category of $\mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$.

For a $(2,1)$-category of stacks $\mathcal{K}$, denote by $i : \mathcal{K} \to \mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ the inclusion and $(-)^{\mathrm{sh}} : \mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}] \to \mathcal{K}$ the reflector, so that $i \vdash_{\mathrm{bi}} (-)^{\mathrm{sh}}$, i is fully faithful and $(-)^{\mathrm{sh}}$ preserves finite bi-limits.

$$\begin{tikzcd}
\mathcal{K}_{\mathbf{Ps}} \arrow[r, bend right, "i"'] \arrow[r, phantom, "\perp_{\mathrm{bi}}"] & \mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]. \arrow[l, bend right, "(-)^{\mathrm{sh}}"']
\end{tikzcd}$$

Definition 7.8.13. Let $\mathcal{K}_{\mathbf{Ps}}$ be a $(2,1)$-category of stacks. We define the $(2,1)$*-category of semi-strict stacks* $\mathcal{K}$ to be the full subcategory on strict $(2,1)$-functors:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, hook, "i"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbb{C}, \mathbf{GPD}]} \arrow[d, hook] \\
\mathcal{K}_{\mathbf{Ps}} \arrow[r, hook, "i"'] & \mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}].
\end{tikzcd}$$

We note that 2-categories of semi-strict stacks are the objects of study of [Mes25a] which gives an explicit description of the objects of a 2-category of stacks— this is the same as the description we have given here by the 2-Giraud theorem [Str06]. The main result of [Mes25a] is to provide a classifier for small discrete opfibrations in 2-categories of semi-strict stacks, which will be essential to this example. When restricting to $(2,1)$-categories of semi-strict stacks, this remains a discrete opfibration classifier.

Since **GPD** has all bicolimits, so does $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$, and hence by the description of $(-)^{\mathrm{sh}} : \mathbf{Ps}[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}] \to \mathcal{K}_{\mathbf{Ps}}$ given in [Str82] as a bicolimit, this restricts to a biadjunction:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend right, "i"'] \arrow[r, phantom, "\perp_{\mathrm{bi}}"] & {[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]}. \arrow[l, bend right, "(-)^{\mathrm{sh}}"']
\end{tikzcd}$$

in which i is injective on objects and fully faithful. Note that this means that given any $\mathbb{X} \in \mathcal{K}$ and $F \in [\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ there is an equivalence of hom-$(2,1)$-categories:

$$\mathcal{K}(F^{\mathrm{Sh}}, \mathbb{X}) \simeq [\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}](F, i\mathbb{X})$$

However, this is not a strict $(2,1)$-adjunction, which would require the equation above to be an isomorphism of $(2,1)$-categories, natural in F and $\mathbb{X}$. Since reflective left biadjoints preserve bilimits and bicolimits, this means that we can deduce that $\mathcal{K}$ has finite bi(co)limits, calculated by working out the bi(co)limit in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ and applying $(-)^{\mathrm{sh}}$ to the result. However, we cannot apply the same reasoning to all strict 2-colimits, just those which agree with the bicolimits. This includes important classes of strict 2-(co)limits such as flexible and therefore PIE (co)limits. This is recorded below.

Proposition 7.8.14. *[Mes25a, Proposition 5.1] Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then $\mathcal{K}$ has all flexible limits. In particular it has comma objects, a terminal object, and pullbacks along discrete opfibrations calculated pointwise in* $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$.

Similarly, we have a result for colimits.

Proposition 7.8.15. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then $\mathcal{K}$ has finite bicolimits, calculated by applying $(-)^{sh}$ to its pointwise bicolimit in* $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$. *In particular, it has cocomma objects, pushouts along discrete (op)fibrations and codescent objects.*

Proof. This follows easily from the fact that $(-)^{\mathrm{sh}}$ is a left biadjoint, which preserves bicolimits, and $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ has finite bicolimits calculated pointwise, since **GPD** does. The second statement follows because cocomma objects and pushouts along discrete opfibrations and codescent objects are PIE-colimits [PR91] which are in particular flexible colimits, which are in particular finite bicolimits. $\square$

Definition 7.8.16. Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. We call a discrete opfibration $f : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$ *small* if $i(f) : i(\mathbb{X}) \to i(\mathbb{Y})$ is a small discrete opfibration in $[\mathbb{C}^{\text{op}}, \textbf{GPD}]$.

Firstly, this is well-defined by [Mes25a, Proposition 5.2]. Define $\mathcal{S}'$ to be the class of small discrete opfibrations. Notice this agrees with [Mes25a, Definition 5.4]. For $\mathcal{K}$ a $(2,1)$-category of semi-strict stacks, we explore which axioms $(\mathcal{K}, \mathcal{S}')$ satisfy.

Axioms (S1), (S2) and (S8) follow simply since pullbacks along discrete fibrations are calculated in $[\mathbb{C}^{\text{op}}, \textbf{GPD}]$ and isomorphisms and composition is preserved by functors.

Note the following, which is not easily seen from the description of a $(2,1)$-category of semi-strict stacks via their matching families and descent data [Mes25a, Definition 2.3.4].

Lemma 7.8.17. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then the discrete objects form a category of sheaves.*

Proof. Let $X \in \mathcal{K}$ be discrete and $F : \mathbb{C} \to \textbf{GPD}$ be a strict 2-functor. Then, by the universal property of the biadjunction, there is an equivalence of categories:

$$\mathcal{K}(F^{\text{Sh}}, X) \simeq [\mathbb{C}^{\text{op}}, \textbf{GPD}](F, jX).$$

Since X is discrete and j preserves discrete objects as it is fully faithful, both of these categories are actually sets; hence the equivalence must be an isomorphism. Therefore, the biadjunction $i \vdash_{\text{bi}} (-)^{\text{sh}}$ restricts to a 1-adjunction

$$\begin{tikzcd} \textbf{Disc}(\mathcal{K}) \arrow[r, bend right, "i"'] \arrow[r, phantom, "\perp"] & \textbf{Disc}([\mathbb{C}^{\text{op}}, \textbf{GPD}]) \arrow[l, bend right, "(-)^{\text{sh}}"'] \end{tikzcd}.$$

from which the result follows by noting that $\textbf{Disc}([\mathbb{C}^{\text{op}}, \textbf{GPD}]) = [\mathbb{C}^{\text{op}}, \textbf{Set}]$. ☐

Remark 7.8.18. This is not the same as saying that the underlying set of a semi-strict stack is a sheaf, but rather that sheaves are semi-strict stacks that do not have any non-trivial 2-dimensional data.

Corollary 7.8.19. *Let $\mathcal{K}$ be a category of semi-strict stacks. Then $\mathcal{K}$ has colimits of discrete objects, calculated by first calculating the colimit in $[\mathbb{C}^{\text{op}}, \textbf{GPD}]$ and applying $(-)^{sh}$.*

Proof. By Lemma 7.8.17, the discrete objects form a category of sheaves; these have all finite colimits which are calculated by sheafifying the colimit calculated in presheaves [MM94]. ☐

It follows that categories of semi-strict stacks satisfy (S3), since this is true in categories of sheaves [BM12] and **0** and **1** are discrete objects.

We can prove the existence of some more colimits.

Proposition 7.8.20. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then $\mathcal{K}$ has conical colimits of diagrams of discrete opfibrations, calculated by applying the reflector to the calculation in $[\mathbb{C}^{\text{op}}, \textbf{GPD}]$.*

Proof. This follows by noting that semi-strict stacks are closed under isofibrational slicing; if $\mathcal{K}$ is a category of semi-strict stacks on $\mathbb{C}$ and $\mathbb{X} \in \mathcal{K}$, then $\mathcal{K}/^{\cong}_{\mathbb{X}}$ is a category of semi-strict stacks on the slice prestack

$$[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]/_{i(\mathbb{X})} \simeq [\mathbb{C}/_{i(\mathbb{X})}, \mathbf{GPD}],$$

where $\mathbb{C}/_{i(\mathbb{X})}$ is the category of elements of $i(\mathbb{X}) : \mathbb{C} \to \mathbf{GPD}$. The indicated equivalence is the result of the Grothendieck construction. Therefore, $\mathcal{K}/^{\cong}_{\mathbb{X}}$ is a category of semi-strict stacks.

In this category, the discrete objects are precisely the discrete opfibrations in $\mathcal{K}$. By Lemma 7.8.19, $\mathcal{K}/_{\mathbb{X}}$ has colimits of discrete objects; in $\mathcal{K}$, this give the colimit of the diagram of discrete opfibrations as required. $\square$

As a result of this, we can deduce that $(2, 1)$-categories of semi-strict stacks satisfy Axiom (S4) since small discrete objects are closed under coproducts in categories of sheaves [BM12] and small discrete opfibrations are the small discrete objects in the isofibrational slice categories.

Similarly, we can prove that small discrete opfibrations. First, recall the following.

Definition 7.8.21. Let $i : \mathbb{A} \hookrightarrow \mathbb{B}$ be an inclusion of categories. $\mathbb{A}$ is called an *exponential ideal* if for any $X \in \mathbb{A}$ and $Y \in \mathbb{B}$, if X^Y exists in $\mathbb{B}$, then it exists in $\mathbb{A}$.

Lemma 7.8.22. *[Joh02b, Proposition A4.3.1] If $i : \mathbb{A} \hookrightarrow \mathbb{B}$ is a full reflective subcategory with reflector $r : \mathbb{B} \to \mathbb{A}$, then $\mathbb{A}$ is an exponential ideal if and only if r preserves finite products.*

Therefore for any $\mathbb{X} \in \mathcal{K}$, the category $\mathbf{Disc}(\mathcal{K}/_{\mathbb{X}})$ is an exponential ideal of $[\mathbb{C}/_{i(\mathbb{X})}, \mathbf{Set}]$; hence any small discrete opfibration $F : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{K}$ is exponentiable as it is exponentiable in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ as shown in Section 7.8.3 and hence is an exponentiable object in $[\mathbb{C}/_{\mathbb{Y}}, \mathbf{Set}]$, and therefore in $\mathbf{Disc}(\mathcal{K}/_{\mathbb{Y}})$ too; we conclude that F is exponentiable and so $\mathcal{K}$ satisfies (S5).

Turning our attention to (S6), again, the classifier for small maps is given by the Hofmann-Streicher universe described in [Mes25a, Theorem 5.11].

Define $\Omega : \mathbb{C} \to \mathbf{GPD}$ by the rules:

$$c \mapsto \mathbf{Sh}(\mathbb{C}/_c, J)$$
$$f : c \leftarrow d \mapsto - \circ (f \circ =) : \mathbf{Sh}(\mathbb{C}/_c, J) \to \mathbf{Sh}(\mathbb{C}/_d, J)$$

Let $\mathbf{pt} : \mathbf{1} \to \Omega$ be the morphism in $\mathcal{K}$ which for each $c \in \mathbb{C}$ picks out the constant prestack on the terminal in $[(\mathbb{C}/_c)^{\mathrm{op}}, \mathbf{Set}]$; this is trivially a stack. Define Ω_* by the pullback

$$\begin{tikzcd} \Omega_* \arrow[r] \arrow[d, "\pi_1"'] & \mathbf{1} \arrow[d, "\mathbf{pt}"] \\ \Omega^{\mathbf{I}} \arrow[r, "d_0"'] & \Omega. \end{tikzcd}$$

and define $p := d_1\pi_1 : \Omega_* \to \Omega$. This is our classifying small discrete opfibration, and so $(\mathcal{K}, \mathcal{S}')$ satisfies (S6).

Proposition 7.8.23. *[Mes25a, Theorem 5.11] The map $p : \Omega_* \to \Omega$ is a classifier for maps in $\mathcal{S}'$.*

We prove that $(2,1)$-categories of semi-strict stacks are BO-exact (Definition 2.4.13). We have shown in Proposition 7.8.15 that codescent objects of cateads exist in $(2,1)$-categories of semi-strict stacks; it remains to show that codescent morphisms are stable under pullback, that if $f : \mathbb{A} \to \mathbb{B}$ is a codescent morphism, so is $\delta_f : \mathbb{A} \to \mathbb{A} \times_{\mathbb{B}} \mathbb{A}$ and that cateads are effective.

Lemma 7.8.24. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then codescent morphisms in $\mathcal{K}$ are stable under pullback.*

Proof. Consider a pullback square

$$\begin{tikzcd} f^*(\mathbb{B}) \arrow[r, "q^*f"] \arrow[d, "f^*q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A} \arrow[d, "q"] \\ \mathbb{C} \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

In which $q : \mathbb{A} \to \mathbb{B}$ is a codescent morphism. We want to show that $f^*q : f^*(\mathbb{B}) \to \mathbb{C}$ is a codescent morphism. Consider the higher kernel of q, displayed below.

$$q \downarrow q \downarrow q \underset{p_1}{\overset{p_0}{\underset{m}{\rightrightarrows}}} q \downarrow q \underset{d_0}{\overset{d_1}{\underset{\leftarrow i -}{\rightrightarrows}}} \mathbb{A}$$

By applying $i : \mathcal{K} \hookrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$, it remains a catead since discrete opfibrations and therefore 2-sided discrete opfibrations in a $(2,1)$-category are preserved under the inclusion [Mes25a, Proposition 5.2]. Codescent objects of cateads exist in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$; we denote the codescent object and morphism by $q' : \mathbb{A} \twoheadrightarrow Qi(\mathbf{K}(q))$. Note that this induces a unique arrow $a : Qi(\mathbf{K}(q)) \to \mathbb{B}$ such that $aq' = q$ by the universal property of the codescent object and the fact that $\mathbb{B}$ is a codescent object of the same catead in $\mathcal{K}$. Take the pullback of a along f and then the pullback of q' along $f^*(a)$ as displayed below:

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A} \arrow[d, two heads, "q'"] \\ \mathbb{D} \arrow[r, "f^*(a)"] \arrow[d, "a^*(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Qi(\mathbf{K}(q)) \arrow[d, "a"] \\ \mathbb{C} \arrow[r, "f"'] & \mathbb{B}. \end{tikzcd} \tag{7.7}$$

We note that by the 2-pullback lemma, the outer rectangle is a pullback and so $\mathbb{E} \cong f^*(\mathbb{B})$ since $q = aq'$. As codescent morphisms are stable under pullback in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$, it follows that $f^*(\mathbb{B}) \twoheadrightarrow \mathbb{D}$ is a codescent morphism. We apply $(-)^{\mathrm{sh}}$ to Diagram (7.7) to obtain the diagram below, noting that this is how codescent objects are calculated in $\mathcal{K}$ by Proposition 7.8.15.

$$
\begin{tikzcd}
f^*(\mathbb{B}) \arrow[rr] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbb{A} \arrow[d, two heads, "q'"] \\
(\mathbb{D})^{\mathrm{Sh}} \arrow[rr, "f^*(a)"] \arrow[d, "a^*(f)"'] \arrow[drr, phantom, "\lrcorner", very near start] & & (Qi(\mathbf{K}(q)))^{\mathrm{Sh}} \arrow[d, "\cong"] \\
\mathbb{C} \arrow[rr, "f"'] & & \mathbb{B}.
\end{tikzcd}
\tag{7.8}
$$

But then, since isomorphisms are stable under pullback, $a^*(f)$ is an isomorphism, and so $\mathbb{C}$ is isomorphic to the codescent object $(\mathbb{D})^{\mathrm{Sh}}$ and therefore f^*q is a codescent morphism in $\mathcal{K}$, as required.

□

Lemma 7.8.25. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks and $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ a codescent morphism. Then, the map $\delta_f : \mathbb{A} \to \mathbb{A} \times_{\mathbb{B}} \mathbb{A}$ is also a codescent morphism.*

Proof. Let $q : \mathbb{A} \to \mathbb{B}$ be a codescent morphism. Consider its higher kernel in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ and take its codescent morphism $q' : \mathbb{A} \to Qi(\mathbf{K}(q))$ as in Lemma 7.8.24. Since $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ is BO-exact, $\delta_{q'} : \mathbb{A} \to \mathbb{A} \times_{Qi(\mathbf{K}(q))} \mathbb{A}$ is a codescent morphism. The result follows then by noting that $\delta_q = (\delta_{q'})^{\mathrm{Sh}}$ since applying $(-)^{\mathrm{sh}}$ to the pullback square defining $\delta_{q'}$ gives exactly the pullback square defining δ_q as $(-)^{\mathrm{sh}}$ preserves finite limits. □

Effectivity of cateads (Definition 2.4.9) in a $(2,1)$-category of semi-strict stacks follows from the effectivity of cateads in prestacks.

Lemma 7.8.26. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Cateads in $\mathcal{K}$ are effective.*

Proof. Since the inclusion $i : \mathcal{K} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ preserves discrete opfibrations, it preserves cateads, and so any catead in $\mathcal{K}$ is equivalently a catead in $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$. Now, since $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$ is BO-exact, cateads there are effective, the result follows since $(-)^{\mathrm{sh}}$ preserves finite limits and therefore higher kernels. □

We have therefore proven the following result, which we believe is of independent interest.

Proposition 7.8.27. *Let $\mathcal{K}$ be a $(2,1)$-category of semi-strict stacks. Then $\mathcal{K}$ is BO-exact.*

Therefore $(2,1)$-categories of semi-strict stacks satisfy axioms (S1)-(S6) and (S8)-(S9) of being a class $(2,1)$-category; in addition, it satisfies a slightly weaker version of (S7):

(S7') $\mathcal{K}$ has small bicolimits of small objects.

We note that our theory works exactly the same if we replace (S7) with (S7') since bicolimits of discrete objects are still 1-colimits and all of our examples have finite bicolimits. However, in the setting of class $(2,1)$-categories, the assumption that we have finite bicolimits implies that we have finite 2-colimits; assuming bicolimits, we can still prove that $\mathbf{Disc}(\mathcal{K})$ is a locally cartesian closed, lextensive category with coequalisers, and so given that $\mathcal{K}_\sigma \simeq \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$, we can still apply Theorem 3.5.2 and deduce that $\mathcal{K}_\sigma$ has finite 2-colimits. Conversely, we expect that for the same conditions on $\mathcal{E}$ given in Chapter 3, we can prove that $\mathbf{Gpd}(\mathcal{E})$ has finite bicolimits; in $\mathbf{Gpd}(\mathcal{E})$, the bicoproduct is simply the coproduct and the bicoequaliser is given by a pushout, which both exist if $\mathbf{Gpd}(\mathcal{E})$ has finite 2-limits. We do not at the present time know if there is an easy construction of the bicotensor by $\mathbf{2}$; see [Agw25, Lemma 6.1.11] for this result worked out for a specific example of $\mathcal{E} = \mathbf{Asm}_A$. Since there is perhaps not much of a difference in these two axioms in this setting, we present class $(2,1)$-categories with the axiom (S7) in order to distinguish it from a bicategorical theory.

Returning to the issue of $(2,1)$-categories of semi-strict stacks, we do not, in general, expect them to satisfy axiom (Ex). If they did then $\mathcal{K}_\sigma \cong \mathbf{Gpd}(\mathbf{Disc}(\mathcal{K}_\sigma))$ by Proposition 7.5.18 with $\mathbf{Disc}(\mathcal{K}_\sigma)$ a Grothendieck topos by Lemma 7.8.17. By Proposition 7.8.7, considering the discrete class-category associated to the class $(2,1)$-category of semi-strict stacks, we would obtain a universe for sheaves. This makes it seem unlikely that $(2,1)$-categories of semi-strict stacks satisfy (Ex) in general, as the semi-strict stack Ω does not have a sheaf of objects— this was the problem originally noted in [HS97] which motivated the work of [Mes25a]; the solution found was to move to the setting of semi-strict stacks since it did not seem like a Hofmann–Streicher universe was possible for sheaves.

However, in the case in which $\mathbb{C}$ is itself a Grothendieck topos and we are considering J to be the regular epimorphism topology, we prove that it satisfies this axioms.

We note the following, which is proven using a slightly different definition of stacks in [Awo97, Chapter V, Lemma 4]; however the proof works exactly the same in our setting.

Proposition 7.8.28. *Let $\mathbb{C}$ be a small Grothendieck topos, and J the finite-epimorphism topology. Then every stack is equivalent to a groupoid internal to sheaves on $\mathbb{C}$ with respect to the finite-epimorphism topology.*

Corollary 7.8.29. *Let $\mathbb{C}$ be a small Grothendieck topos and let J be the finite-epimorphism topology. Then we have a $(2,1)$-equivalence $\mathbf{St}(\mathbb{C}, J) \simeq \mathbf{Gpd}(\mathbf{Sh}(\mathbb{C}, J))$.*

Proof. Since both $\mathbf{St}(\mathbb{C}, J)$ and $\mathbf{Gpd}(\mathbf{Sh}(\mathbb{C}, J))$ are full subcategories of $[\mathbb{C}^{\mathrm{op}}, \mathbf{GPD}]$, this follows from Proposition 7.8.28 since every object is equivalent. □

Proposition 7.8.30. *Let $\mathbb{C}$ be a small Grothendieck topos and let J be the finite-epimorphism topology. Then $\mathbf{St}(\mathbb{C}, J)$ is a class $(2,1)$-category.*

Proof. This follows since any $(2,1)$-category of semi-strict stacks satisfies (S1)-(S6) and (S8)-(S9), and $\mathbf{Gpd}(\mathbf{Sh}(\mathbb{C}, J))$ has finite 2-colimits by Theorem 3.5.2. In addition, $\mathbf{Gpd}(\mathcal{E})$ has the property that discrete objects are BO-projective for any $\mathcal{E}$, in particular for $\mathcal{E} = \mathbf{Sh}(\mathbb{C}, J)$. □

Chapter 8

Conclusions

Returning to the objectives of this thesis, we successfully gave $(2,1)$-categorical axioms which give a first order axiomatisation of the category of groupoids in a generalised set theory i.e. internal to an arithmetic Π-pretopos (Theorem 5.3.8), and have shown that this provides a model of Martin-Löf type theory without UIP (Theorem 6.6.5). We have shown that the discrete objects of such a $(2,1)$-category forms an arithmetic Π-pretopos, allowing us to translate between the $(2,1)$-categorical approach and the 1-categorical approach. This required first understanding when categories of internal groupoids have colimits (Theorem 3.6.12).

We gave additional $(2,1)$-categorical axioms that axiomatise the $(2,1)$-category of groupoids in BZFC (Theorem 4.8.2) and in a bounded version of CZF (Theorem 5.4.2).

We finished by giving a categorification of the notion of a class category (Definition 7.4.4) and showed that it had many nice properties, including that the small objects formed a model of the elementary theory of the $(2,1)$-category of small abstract groupoids, therefore providing both a model of groupoid theory and a model of MLTT without UIP (Proposition 7.5.18 and Theorem 7.5.28). By adding some additional axioms, we are able to give an axiomatisation of $(2,1)$-category of large groupoids in NBG (Theorem 7.6.5).

We conclude with some possible different research directions which hope to build upon the work of this thesis.

8.1 Future work

8.1.1 Colimits

The axioms placed on $\mathcal{E}$ that are studied in Chapter 3 for the existence of colimits in $\mathbf{Cat}(\mathcal{E})$ are not satisfied by the 1-category of small categories $\mathcal{E} = \mathbf{Cat}_1$, despite the fact that it is a locos and so by Theorem 3.7.7, the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint. The issue lies in the fact that coequalisers in **Cat** are not stable under pullback. An example that shows this is is given on [Shua]. However, **Cat** is locally finitely presentable, so we could apply Proposition 3.1.2 and conclude that $\mathbf{Cat}(\mathbf{Cat})$, the 2-category of double categories has finite colimits.

In future work, we look to extend the method of this paper to prove this without using local finite presentability, and more generally for $\mathcal{E} = \mathbf{Cat}(\mathcal{C})_1$ the 1-category of categories internal to $\mathcal{C}$ a an extensive 1-category $\mathcal{E}$ with pullbacks and pullback-stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint to conclude that $\mathbf{Cat}(\mathbf{Cat}(\mathcal{E}))$, the category of double internal categories has finite 2-colimits. Part of this work proves that **Cat** has lax-pullback-stable coequalisers, a result of independent interest.

In relation to this, we also hope to work on extending this methodology to show: the existence of finite 2-colimits in the 2-category of pseudocategories internal to a suitable 2-category $\mathcal{K}$, $\mathbf{PsCat}(\mathcal{K})$ after a suggestion by Bryce Clarke; the existence of finite 2-colimits in T-multicategories after a suggestion by Nathanael Arkor; the existence of finite 2-colimits in the 2-category of internal models of an essentially algebraic theory after a suggestion by Peter LeFanu Lumsdaine. For the case that $\mathcal{E} = \mathbf{Set}$, the latter is shown by a similar argument to Proposition 3.1.2 as these are equivalent to locally presentable categories, so this would provide a general way of working with internally locally presentable categories.

8.1.2 Models of type theory

In future work with Matteo Spadetto, we aim to show that the syntactic $(2, 1)$-category of a Martin-Löf type theory without UIP [Spa24] provides a model for ETCSAG, completing Figure 1.6.

In future work with Matteo Spadetto and Fernando Chu, we aim to show that the 2-category of categories internal to an arithmetic Π-pretopos forms a model of directed type theory. We aim to give a notion of *directed* model categories which take into account fibrations as well as opfibrations.

Inspired the proof strategy outlined in Chapter 6, we conjecture that for $\mathcal{E}$ a locally cartesian closed locos with coequalisers, the effective model structure of [Gam+22] can be upgraded into a type theoretic algebraic weak factorisation system for simplicial objects in $\mathcal{E}$. This could give interesting variations of models of homotopy type theory in a similar way to the method used here. Indeed, it is possible to show that for such an $\mathcal{E}$, the effective model structure is cofibrantly generated and hence algebraic. Research in this direction could shed light on the open question of whether Voevodsky's proof that the category of simplicial sets provides a model of homotopy type theory can be adapted to work in a constructive setting. Whilst a constructively valid model of homotopy type theory given by cubical sets has been proven [Coh+15; Awo+24; Awo26], this remains an open problem for simplicial sets. Perhaps an interesting direction for future study would be to adapt the work of [Gam+22] to the setting of cubical objects in $\mathcal{E}$ and then use the strategy in Chapter 6.

8.1.3 Class $(2, 1)$-categories

In future joint work with Sam Speight, we aim to explore groupoids internal to the category of assemblies for a partial combinatorial algebra A. We argue that it is a candidate for a $(2, 1)$-dimensional version of the effective topos, since it is a kind of $(2, 1)$-exact completion of the category of partitioned assemblies over A, and the effective topos is a 1-exact completion of it. We show that by taking $\mathcal{S}$ to be the class of discrete opfibrations with 'modest' fibres, this forms an example of a class $(2, 1)$-category. We also explore some of it's other interesting properties as a model of $(2, 1)$-dimensional computation. We will compare it to other proposed candidates [AE25].

We hope to expand the list of examples of class $(2, 1)$-categories given in Section 7.8 by showing that these arguments can be taken internally to any class $(2, 1)$-category, showing that these results are iterative.

Finally, we hope to be able to work out a way to get rid of the axiom (Ex), which is troublesome both in that it is not stable under slicing and in that the $(2, 1)$-category of semi-strict stacks does not satisfy it. We still want it to be true that the small objects form a model of type theory; to do this we would have to find some other way of proving that isofibrations are exponentiable from the axioms that does not go through the fact that the small objects forma $(2,1)$-category of internal groupoids. One thing we could potentially do is axiomatise a 2-category with a class of "small isofibrations" with an exponentiable universe of isofibrations, mirroring the case of large groupoids together with the forgetful functor $\mathbf{Gpd}_* \to \mathbf{Gpd}$. However, we believe that the universal property of this classifier is 3-categorical, so the theory would be significantly more difficult. We believe that related topics are being studied by Joseph Hua.

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Appendices

Appendix A

A proof of associativity in Proposition 3.3.1

We define C_3 as the following pullback.

$$
\begin{tikzcd}
C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_2 \arrow[d, "\pi_1"] \\
C_2 \arrow[r, "\pi_0"'] & C_1.
\end{tikzcd}
$$

To show associativity, we must show that the following diagram commutes

$$
\begin{tikzcd}
C_3 \arrow[r, "m\times C_1"] \arrow[d, "\sigma"'] & C_2 \arrow[dd, "m"] \\
C_1 \times_{B_0} C_2 \arrow[d, "C_1\times m"'] & \\
C_2 \arrow[r, "m"'] & C_1.
\end{tikzcd}
\tag{A.1}
$$

Construct $Q_3 : B_3 \to C_3$ by the universal property of C_3 as a pullback as in the following diagram

$$
\begin{tikzcd}
B_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] \arrow[dr, dashed, "Q_3"] & B_2 \arrow[dr, "Q_2"] & \\
B_2 \arrow[dr, "Q_2"'] & C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_2 \arrow[d, "\pi_1"] \\
& C_2 \arrow[r, "\pi_0"'] & C_1
\end{tikzcd}
$$

in which Q_3 exists by the commutativity of the following diagram:

$$
\begin{tikzcd}
B_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d, "\pi_2"] \arrow[dr, "Q_2"] & \\
B_2 \arrow[r, "\pi_2"'] \arrow[dr, "Q_2"'] & B_1 \arrow[dr, "Q_1"] & C_2 \arrow[d, "\pi_1"] \\
& C_2 \arrow[r, "\pi_0"'] & C_1
\end{tikzcd}
$$

Since coequalisers are assumed to be stable under pullback in $\mathcal{E}$, it follows that Q_3 coequalises the pair of parallel arrow $L \times_{B_0} L \times_{B_0} L \to B_3$

$$m^2 \cdot \widetilde{F} \times_{B_0} m^2 \cdot \widetilde{F} \times_{B_0} m^2 \cdot \widetilde{F}, \qquad m^2 \cdot \widetilde{G} \times_{B_0} m^2 \cdot \widetilde{G} \times_{B_0} m^2 \cdot \widetilde{G}.$$

Hence we can appeal to the universal property of the coequaliser: to show that Diagram (A.1) commutes, it is enough to show that the diagram commutes when precomposed with Q_3. This is witnessed by the following diagram.

$$\begin{tikzcd}
B_3 \arrow[rrr, "Q_3"] \arrow[d, "Q_3"'] \arrow[dr, "\sigma"] \arrow[drrr, "m\times_{B_0}B_1"] & & & C_3 \arrow[r, "m\times_{B_0}C_1"] \arrow[d, phantom, "\mathbf{A}"] & C_2 \arrow[ddd, "m"] \\
C_3 \arrow[d, "\sigma"'] & B_1 \times_{B_0} B_2 \arrow[dr, "B_1\times m"] & & B_2 \arrow[ur, "Q_2"'] \arrow[d, "m"] & \\
C_1 \times_{B_0} C_2 \arrow[d, "1\times_{B_0}m"'] \arrow[rr, phantom, "\mathbf{B}"] & & B_2 \arrow[r, "m"'] \arrow[dll, "Q_2"] & B_1 \arrow[dr, "Q_1"] & \\
C_2 \arrow[rrrr, "m"'] & & & & C_1
\end{tikzcd}$$

In the above, the regions labelled **A** and **B** are shown to commute by appealing to the universal property of C_2 as a pullback of $\pi_0, \pi_1 : C_2 \to C_1$, and showing that the regions commute after postcomposing with these projections.

The commutativity of the region **A** is shown by the following pair of commutative diagrams.

$$\begin{tikzcd}
B_3 \arrow[rr, "Q_3"] \arrow[dr, "\pi_{3,0}"] \arrow[ddd, "m\times_{B_0}B_1"'] & & C_3 \arrow[r, "m\times_{B_0}C_1"] \arrow[d, "\pi_{3,0}"] & C_2 \arrow[ddd, "\pi_0"] \\
& B_2 \arrow[r, "Q_2"] \arrow[d, "m"] & C_2 \arrow[ddr, "m"] & \\
& B_1 \arrow[drr, "Q_1"] & & \\
B_2 \arrow[ur, "\pi_0"] \arrow[rr, "Q_2"'] & & C_2 \arrow[r, "\pi_0"'] & C_1
\end{tikzcd}$$

$$\begin{tikzcd}
B_3 \arrow[rr, "Q_3"] \arrow[dr, "\pi_{3,1}"] \arrow[ddd, "m\times_{B_0}B_1"'] & & C_3 \arrow[r, "m\times_{B_0}C_1"] \arrow[d, "\pi_{3,1}"] & C_2 \arrow[ddd, "\pi_1"] \\
& B_2 \arrow[r, "Q_2"] \arrow[d, "\pi_1"] & C_2 \arrow[ddr, "\pi_1"] & \\
& B_1 \arrow[drr, "Q_1"] & & \\
B_2 \arrow[ur, "\pi_1"] \arrow[rr, "Q_2"'] & & C_2 \arrow[r, "\pi_1"'] & C_1
\end{tikzcd}$$

The commutativity of the region **B** is shown by the following pair of commutative diagrams.

$$
\begin{tikzcd}
B_3 \arrow[r, "\sigma"] \arrow[dr, "\pi_{3,0}"] \arrow[dd, "Q_3"'] & B_1 \times_{B_0} B_2 \arrow[r, "B_1 \times_{B_0} m"] & B_2 \arrow[r, "Q_2"] \arrow[d, "\pi_0"] & C_2 \arrow[ddd, "\pi_0"] \\
 & B_2 \arrow[r, "\pi_0"] \arrow[d, "Q_2"] & B_1 \arrow[ddr, "Q_1"] & \\
C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\sigma"'] & C_2 \arrow[drr, "\pi_0"] & & \\
C_1 \times_{B_0} C_2 \arrow[rr, "C_1 \times_{B_0} m"'] & & C_2 \arrow[r, "\pi_0"'] & C_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_3 \arrow[r, "\sigma"] \arrow[dr, "\pi_{3,1}"] \arrow[dd, "Q_3"'] & B_1 \times_{B_0} B_2 \arrow[r, "B_1 \times_{B_0} m"] \arrow[d, "\pi_{B_2}"] & B_2 \arrow[r, "Q_2"] \arrow[d, "\pi_0"] & C_2 \arrow[ddd, "\pi_0"] \\
 & B_2 \arrow[r, "m"] \arrow[d, "Q_2"] & B_1 \arrow[ddr, "Q_1"] & \\
C_3 \arrow[r, "\pi_{3,1}"] \arrow[d, "\sigma"'] & C_2 \arrow[drr, "m"] & & \\
C_1 \times_{B_0} C_2 \arrow[ur, "\pi_{C_2}"] \arrow[rr, "C_1 \times_{B_0} m"'] & & C_2 \arrow[r, "\pi_0"'] & C_1
\end{tikzcd}
$$

Putting all the above steps together, we have shown that associativity holds.

Appendix B

A second proof of Theorem 3.5.2 using monadicity

In this section, we give a second proof of Theorem 3.5.2 using the methodology described in [JW78].

Assume that $\mathcal{E}$ is a lextensive category $\mathcal{E}$ with pullbacks and pullback stable coequalisers in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ has left adjoint. We wish to show that $\mathbf{Cat}(\mathcal{E})$ has coequalisers of reflexive pairs. To do this, it is enough to show that $\mathcal{U} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{Gph}(\mathcal{E})$ is monadic as monadic functors create all colimits that exist in their codomain and are preserved by the monad. The category $\mathbf{Gph}(\mathcal{E})$ is an $\mathcal{E}$-valued presheaf category on the parallel arrow category, and so has all colimits that $\mathcal{E}$ does. By assumption, this includes coequalisers.

We wish to apply the crude monadicity theorem to the functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$. The conditions for the crude monadicity theorem are the following.

1. $\mathcal{U}$ has a left adjoint.
2. $\mathcal{U}$ reflects isomorphisms.
3. $\mathbf{Cat}(\mathcal{E})$ has coequalisers of reflexive pairs.
4. $\mathcal{U}$ preserves coequalisers of reflexive pairs.

By assumption, (1) holds. By definition, it is clear that (2) holds. Therefore, it remains to show (3) and (4). To this end, we note the following which follows from Proposition 3.3.1.

Lemma B.0.1. *Let $\mathcal{E}$ be a category with pullback stable coequalisers. Any reflexive pair $F, G : \mathbb{A} \to \mathbb{B}$ of internal functors between internal categories such that $A_0 = B_0$ and $F_0 = G_0 = 1_{A_0}$ has a reflexive coequaliser in* $\mathbf{Cat}(\mathcal{E})$.

Remark B.0.2. In this case, the coequaliser is simpler. It is enough to run the argument of Proposition 3.3.1 with L there replaced by the coequaliser of $F_1, G_1 : A_1 \to B_1$. This is because we are not causing any newly composable arrows since the categories are already equal on objects.

The following is evident in the proof of Proposition 3.3.1.

Corollary B.0.3. *The forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ creates coequalisers of reflexive pairs $F, G : \mathbb{A} \to \mathbb{B}$ such that $A_0 = B_0$ and $F_0 = G_0 = 1_{A_0}$.*

The argument in [JW78] concludes by saying that both Linton's theorem and the crude monadicity theorem only require these kinds of reflexive coequalisers in their argument. In order to be vaguely self-contained, we spell out the details of this below.

Since we are in an abstract setting, we must prove the following.

Lemma B.0.4. *Let $\mathcal{E}$ be a category with pullback stable coequalisers and terminal object in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint $\mathbb{F} : \mathbf{Gph}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})$. Then for any $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, $(\mathbb{F}\mathcal{U}\mathbb{X})_0 = X_0$.*

Proof. We first note that the functor $(-)_0 : \mathbf{Gph}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})$ has right adjoint **indisc** $: \mathcal{E} \to \mathbf{Gph}(\mathcal{E})$ which is equal to the composite **indisc** $\circ\, \mathcal{U} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$. Since left adjoints are unique, it follows that $(-)_0 \circ \mathbb{F} = (-)_0$ as required. □

We will denote the counit of the adjunction $\epsilon : \mathbb{F}\mathcal{U} \Rightarrow 1$.

Next we note that given $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, the following is a reflexive coequaliser diagram that is identity on objects.

$$\mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}\mathbb{A} \underset{\mathbb{F}\mathcal{U}(\epsilon_{\mathbb{A}})}{\overset{\epsilon_{\mathbb{F}\mathcal{U}\mathbb{A}}}{\rightrightarrows}} \mathbb{F}\mathcal{U}\mathbb{A} \xrightarrow{\epsilon_{\mathbb{A}}} \mathbb{A}.$$

Given $F, G : \mathbb{A} \to \mathbb{B}$, we spell out how to construct its reflexive coequaliser below. Let $Q : \mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B} \to E$ denote the coequaliser in $\mathbf{Gph}(\mathcal{E})$ of $\mathcal{U}F, \mathcal{U}G : \mathcal{U}\mathbb{A} \to \mathcal{U}\mathbb{B}$ and $Q' : \mathcal{U}\mathbb{B} \to E'$ denote the reflexive coequaliser in $\mathbf{Gph}(\mathcal{E})$ of

$$\mathcal{U}\mathbb{F}\mathcal{U}F, \mathcal{U}\mathbb{F}\mathcal{U}G : \mathcal{U}\mathbb{F}\mathcal{U}\mathbb{A} \to \mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B}.$$

Note that there is a pair of maps induced by

$$\epsilon_{\mathbb{F}\mathcal{U}\mathbb{B}}\mathbb{F}Q : \mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B} \to \mathbb{F}E$$

and

$$\mathbb{F}\mathcal{U}(\epsilon_{\mathbb{B}})\mathbb{F}Q : \mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B} \to \mathbb{F}E.$$

Calculate this pair's reflexive coequaliser:

$$\mathbb{F}E' \rightrightarrows \mathbb{F}E \dashrightarrow \mathbb{C}$$

which exists by as it is a reflexive coequaliser of internal functors that are identity-on-objects by Lemma B.0.4 which exist by Lemma B.0.1.

Construct the following diagram.

$$\begin{tikzcd}
\mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}\mathbb{A} \arrow[r, shift left, "\mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}F"] \arrow[r, shift right, "\mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}G"'] \arrow[d, shift right, "\mathbb{F}\mathcal{U}(\epsilon_{\mathbb{A}})"'] \arrow[d, shift left, "\epsilon_{\mathbb{F}\mathcal{U}\mathbb{A}}"] & \mathbb{F}\mathcal{U}\mathbb{F}\mathcal{U}\mathbb{B} \arrow[r] \arrow[d, shift right, "\mathbb{F}\mathcal{U}(\epsilon_{\mathbb{B}})"'] \arrow[d, shift left, "\epsilon_{\mathbb{F}\mathcal{U}\mathbb{B}}"] & \mathbb{F}E' \arrow[d, shift right] \arrow[d, shift left] \\
\mathbb{F}\mathcal{U}\mathbb{A} \arrow[r, shift left, "\mathbb{F}\mathcal{U}F"] \arrow[r, shift right, "\mathbb{F}\mathcal{U}G"'] \arrow[d, "\epsilon_{\mathbb{A}}"'] & \mathbb{F}\mathcal{U}\mathbb{B} \arrow[r] \arrow[d, "\epsilon_{\mathbb{B}}"] & \mathbb{F}E \arrow[d] \\
\mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[r, dashed] & \mathbb{C}
\end{tikzcd}$$

In any such diagram in which the top two rows are except reflexive coequalisers, and every column is a reflexive coequaliser, then there exists an arrow on the row making it a coequaliser diagram [Wol74, Proposition 2.11]. Hence we have calculated the coequaliser and shown the following result.

Proposition B.0.5. *The functor* $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ *is monadic.*

Corollary B.0.6. $\mathbf{Cat}(\mathcal{E})$ *has finite* 2*-colimits.*

Appendix C

Proof of Proposition 4.7.7

Notation C.0.1. For this section, we use the following notational convention:

1. For s, f morphisms in a 2-category $\mathcal{K}$, write $s \perp_1 f$ if for any commutative square in $\mathcal{K}$ as depicted below left, there is a unique morphism $u : B \to X$ satisfying $fu = q$ and $us = p$.

2. Write $s \perp f$ if $s \perp_1 f$ and moreover for any commutative pair of 2-cells as depicted below right, with $u : B \to X$ and $u' : B \to X$ the corresponding morphisms induced by $fp = qs$ and $fp' = q's$ respectively, there is a unique 2-cell $\gamma : u \Rightarrow u'$ satisfying $f.\gamma = \beta$ and $\gamma.s = \alpha$.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ B \arrow[r, "q"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "p"] \arrow[r, bend right, "p'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ B \arrow[r, bend left, "q"] \arrow[r, bend right, "q'"'] \arrow[r, phantom, "\Downarrow\beta"] & Y \end{tikzcd}$$

3. For $\mathcal{M}_1, \mathcal{M}_2$ classes of morphisms in a category $\mathcal{C}$, write $\mathcal{M}_1 \perp_1 \mathcal{M}_2$ if for every $s \in \mathcal{M}_1$ and $f \in \mathcal{M}_2$, we have $s \perp_1 f$.

4. For $\mathcal{M}_1, \mathcal{M}_2$ classes of morphisms in a 2-category $\mathcal{K}$, write $\mathcal{M}_1 \perp \mathcal{M}_2$ if for every $s \in \mathcal{M}_1$ and $f \in \mathcal{M}_2$, we have $s \perp f$.

5. For $(\mathcal{L}, \mathcal{R})$ an orthogonal factorisation system on $\mathcal{E}$, let $\mathcal{L}'$ denote the class of internal functors which are $\mathcal{L}$-on-objects, and let $\mathcal{R}'$ denote the class of internal functors which are $\mathcal{R}$-on-objects and fully faithful.

It is clear that both $\mathcal{L}'$ and $\mathcal{R}'$ contain all isomorphisms of internal categories and are closed under composition, since these properties hold for the classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ in $\mathcal{E}$, and for the class of fully faithful functors in $\mathbf{Cat}(\mathcal{E})$. By Lemma 2.2 of [Bou77], it therefore suffices to show that the following properties hold to establish that $(\mathcal{L}', \mathcal{R}')$ is an orthogonal factorisation system on the category $\mathbf{Cat}(\mathcal{E})_1$.

- $\mathcal{L}' \perp_1 \mathcal{R}'$.
- Any internal functor $f : \mathbb{X} \to \mathbb{Y}$ admits a factorisation $f = rl$ with $l \in \mathcal{L}'$ and $r \in \mathcal{R}'$.

If moreover $\mathcal{L}' \perp \mathcal{R}'$, then $(\mathcal{L}', \mathcal{R}')$ is an orthogonal factorisation system on the 2-category $\mathbf{Cat}(\mathcal{E})$. We prove $\mathcal{L}' \perp \mathcal{R}'$ in Lemma C.0.2, and the existence of an appropriate factorisation in Lemma C.0.3.

Lemma C.0.2. $\mathcal{L}' \perp \mathcal{R}'$.

Proof. We first prove the one-dimensional aspect of orthogonality. Consider a diagram in $\mathbf{Cat}(\mathcal{E})$ as depicted below, in which $s \in \mathcal{L}'$ and $f \in \mathcal{R}'$.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "p"] \arrow[d, "s"'] & \mathbb{X} \arrow[d, "f"] \\
\mathbb{B} \arrow[r, "q"'] & \mathbb{Y}
\end{tikzcd}$$

Apply the functor $(-)_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ to get a commutative square as depicted below left in $\mathcal{E}$, in which the unique lift exists as $s_0 \in \mathcal{L}$ and $f_0 \in \mathcal{R}$. We define $u_1 : B_1 \to X_1$ by the universal property of X_1, as depicted below right.

$$\begin{tikzcd}
A_0 \arrow[r, "p_0"] \arrow[d, "s_0"'] & X_0 \arrow[d, "f_0"] \\
B_0 \arrow[r, "q_0"'] \arrow[ur, dashed, "\exists ! u_0"] & Y_0
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[d, "{(d_0,d_1)}"'] \arrow[dr, dashed, "\exists ! u_1"] \arrow[drr, bend left, "q_1"] & & \\
B_0 \times B_0 \arrow[dr, "u_0 \times u_0"'] & X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
 & X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

By construction, $u := (u_0, u_1) : \mathbb{B} \to \mathbb{X}$ is a morphism of graphs. We show that $u := (u_0, u_1) : \mathbb{B} \to \mathbb{X}$ is a functor. Fix $k \in \{0, 1\}$. Then $u : \mathbb{B} \to \mathbb{X}$ respects identities by the universal property of X_1, as witnessed by the following commutative diagrams.

$$\begin{tikzcd}
B_0 \arrow[r, "u_0"] \arrow[d, "i"'] \arrow[dr, "q_0"] & X_0 \arrow[r, "i"] \arrow[d, "f_0"] & X_1 \arrow[d, "f_1"] \\
 & X_0 \arrow[dr, "i"] & \\
B_1 \arrow[r, "u_1"'] \arrow[rr, bend left, "q_1"] & X_1 \arrow[r, "f_1"'] & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, "u_0"] \arrow[d, "i"'] \arrow[dr, "1_{B_0}"] & X_0 \arrow[r, "i"] \arrow[ddr, "1_{X_0}"] & X_1 \arrow[dd, "d_k"] \\
 & B_0 \arrow[dr, "u_0"] & \\
B_1 \arrow[r, "u_1"'] \arrow[ur, "d_k"] & X_1 \arrow[r, "d_k"'] & X_0
\end{tikzcd}$$

Similarly, the following commutative diagrams show that it respects composition.

$$\begin{tikzcd}
B_2 \arrow[r, "u_2"] \arrow[dd, "m"'] \arrow[dr, "q_2"] & X_2 \arrow[r, "m"] \arrow[d, "f_2"] & Y_2 \arrow[dd, "f_1"] \\
 & X_0 \arrow[dr, "m"] & \\
B_1 \arrow[r, "u_1"'] \arrow[rr, bend left, "q_1"] & X_1 \arrow[r, "f_1"'] & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[rr, "u_2"] \arrow[ddd, "m"'] \arrow[dr, "\pi_k"] & & X_2 \arrow[r, "m"] \arrow[d, "\pi_k"] & X_1 \arrow[ddd, "d_k"] \\
 & B_1 \arrow[r, "u_1"] \arrow[d, "d_k"] & X_1 \arrow[ddr, "d_k"] & \\
 & B_0 \arrow[drr, "u_0"] & & \\
B_1 \arrow[rr, "u_1"'] \arrow[ur, "d_k"] & & X_1 \arrow[r, "d_k"'] & X_0
\end{tikzcd}$$

Hence $u : \mathbb{B} \to \mathbb{X}$ is an internal functor. But observe that $u_0 : B_0 \to X_0$ is the unique morphism satisfying $f_0 u_0 = q_0$ and $u_0 s_0 = p_0$, since $\mathcal{L} \perp_1 \mathcal{R}$ in $\mathcal{E}$. Moreover, it is clear by the construction of u_1 via the pullback that $(u_0, u_1) :$ $\mathbb{B} \to \mathbb{X}$ is the unique morphism of graphs providing a factorisation $fu = q$. But also $u_1 s_1 = p_1$, as per the following calculations using the universal property of X_1.

$$
\begin{tikzcd}
A_1 \arrow[r, "s_1"] \arrow[d, "p_1"'] & B_1 \arrow[r, "u_1"] \arrow[dr, "q_1"] & X_1 \arrow[d, "f_1"] \\
X_1 \arrow[rr, "f_1"'] & & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[rr, "s_1"] \arrow[dr, "d_k"] \arrow[dd, "p_1"'] & & B_1 \arrow[r, "u_1"] \arrow[d, "d_k"] & X_1 \arrow[dd, "d_k"] \\
& A_0 \arrow[r, "s_0"] \arrow[drr, "p_0"'] & B_0 \arrow[dr, "u_0"] & \\
X_1 \arrow[rrr, "d_k"'] & & & X_0
\end{tikzcd}
$$

Thus $\mathcal{L}' \perp_1 \mathcal{R}'$. For the two-dimensional aspect of orthogonality, let $fp^0 = q^0 s$, $fp^1 = q^1 s$, $\overline{\alpha} : p^0 \Rightarrow p^1$ and $\overline{\beta} : q^0 \Rightarrow q^1$ be internal natural transformations satisfying $f.\overline{\alpha} = \overline{\beta}.s$, and let $u^0 : \mathbb{B} \to \mathbb{X}$ and $u^1 : \mathbb{B} \to \mathbb{X}$ be the uniquely induced maps from the one-dimensional aspect of orthogonality. Then by fully faithfulness of $\mathbf{Cat}(\mathcal{E})(\mathbb{B}, f) : \mathbf{Cat}(\mathcal{E})(\mathbb{B}, \mathbb{X}) \to \mathbf{Cat}(\mathcal{E})(\mathbb{B}, \mathbb{Y})$, there is a unique internal natural transformation $\overline{\gamma} : u^0 \Rightarrow u^1$ satisfying $f.\overline{\gamma} = \overline{\beta}$. As such, the components assigner for $\overline{\gamma}$ is induced by the universal property of X_1 as displayed below.

$$
\begin{tikzcd}
B_0 \arrow[dr, dashed, "\exists ! \gamma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "(u_0^0, u_0^1)"'] & & \\
& X_1 \arrow[r, "f_1"] \arrow[d, "(d_0, d_1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "(d_0, d_1)"] \\
& X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}
$$

Finally, the following diagrams for $k \in \{0, 1\}$ verify that $\overline{\gamma}.s = \overline{\alpha}$, completing the proof.

$$
\begin{tikzcd}
A_0 \arrow[r, "s_0"] \arrow[d, "\alpha"'] \arrow[drr, "p_0^k"] & B_0 \arrow[r, "\gamma"] \arrow[dr, "u_0^k"] & X_1 \arrow[d, "d_k"] \\
X_1 \arrow[rr, "d_k"'] & & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[r, "s_0"] \arrow[d, "\alpha"'] & B_0 \arrow[r, "\gamma"] \arrow[dr, "\beta"] & X_1 \arrow[d, "f_1"] \\
X_1 \arrow[rr, "f_1"'] & & Y_1
\end{tikzcd}
$$

□

Lemma C.0.3. *Any internal functor $f : \mathbb{X} \to \mathbb{Y}$ may be factorised as $f = rl$ with $l \in \mathcal{L}'$ and $r \in \mathcal{R}'$.*

Proof. Let $f : \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Cat}(\mathcal{E})$. Using the $(\mathcal{L}, \mathcal{R})$ orthogonal factorisation system, we obtain a unique factorisation of f_0 in $\mathcal{E}$ depicted below left. We construct C_1 and maps $r_1 : C_1 \to Y_1$ and $(d_0, d_1) : C_1 \to C_0 \times C_0$ via the pullback in $\mathcal{E}$ depicted below right.

$$
\begin{tikzcd}
X_0 \arrow[rr, "f_0"] \arrow[dr, "l_0"'] & & Y_0 \\
& C_0 \arrow[ur, "r_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "r_1"] \arrow[d, "(d_0, d_1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "(d_0, d_1)"] \\
C_0 \times C_0 \arrow[r, "r_0 \times r_0"'] & Y_0 \times Y_0
\end{tikzcd}
\tag{C.1}
$$

Define $l_1 : X_1 \to C_1$ by the universal property of this pullback, as depicted below.

$$
\begin{tikzcd}
X_1 \arrow[d, "{(d_0,d_1)}"'] \arrow[dr, dashed, "\exists ! l_1"] \arrow[drr, bend left, "f_1"] & & \\
X_0 \times X_0 \arrow[dr, "l_0 \times l_0"'] & C_1 \arrow[r, "r_1"] \arrow[d, "{(d_0,d_1)}"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
& C_0 \times C_0 \arrow[r, "r_0 \times r_0"'] & Y_0 \times Y_0
\end{tikzcd}
$$

Then $(f_0, f_1) = (r_0, r_1) \circ (l_0, l_1)$ is clearly a factorisation at the level of morphisms of graphs. It remains to give an internal category structure to the graph in $\mathcal{E}$ displayed below, and to show that these morphisms of graphs are well-defined as internal functors. Once we have shown this, it will follow by construction that $l \in \mathcal{L}'$ and $r \in \mathcal{R}'$.

$$
\mathbb{C} := \; C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0
$$

Define the identity assigner $i : C_0 \to C_1$ for $\mathbb{C}$ using the universal property of C_1, as depicted below left. Then construct $C_2 \in \mathcal{E}$ as the pullback depicted below right.

$$
\begin{tikzcd}
C_0 \arrow[r, "r_0"] \arrow[dr, dashed, "i"] \arrow[ddr, bend right, "{(1_{C_0},1_{C_0})}"'] & Y_0 \arrow[dr, "i"] & \\
& C_1 \arrow[r, "r_1"] \arrow[d, "{(d_0,d_1)}"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
& C_0 \times C_0 \arrow[r, "r_0 \times r_0"'] & Y_0 \times Y_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_1"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
\tag{C.2}
$$

Define $r_2 : C_2 \to Y_2$ by the universal property of Y_2, as described in Remark 2.2.4. Then define $m : C_2 \to C_1$ by the universal property of C_1 as depicted below left, given the commutativity of the diagram depicted below right.

$$
\begin{tikzcd}
C_2 \arrow[r, "r_2"] \arrow[d, "{(\pi_0,\pi_1)}"'] \arrow[dr, dashed, "m"] & Y_2 \arrow[dr, "m"] & \\
C_1 \times C_1 \arrow[dr, "d_0 \times d_1"'] & C_1 \arrow[r, "r_1"] \arrow[d, "{(d_0,d_1)}"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
& C_0 \times C_0 \arrow[r, "r_0 \times r_0"'] & Y_0 \times Y_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[r, "r_2"] \arrow[d, "{(\pi_0,\pi_1)}"'] & Y_2 \arrow[d, "{(\pi_0,\pi_1)}"'] \arrow[dr, "m"] & \\
C_1 \times C_1 \arrow[r, "r_1 \times r_1"] \arrow[dr, "d_0 \times d_1"'] & Y_1 \times Y_1 \arrow[dr, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
& C_0 \times C_0 \arrow[r, "r_0 \times r_0"'] & Y_0 \times Y_0
\end{tikzcd}
\tag{C.3}
$$

We now consider the internal category axioms for $\mathbb{C}$. Sources and targets for identities and composites hold by construction. This allows us to define the maps $\pi_{1,3}, m_1, m_0, \pi_{0,3} : C_3 \to C_2$ and $i_0, i_1 : C_1 \to C_2$ as in Remark 2.2.2. Furthermore, define $r_3 : C_3 \to Y_3$ in the obvious way, using the universal property of Y_3. It remains to check the associativity law and the left and right unit laws.

To check associativity, we use the universal property of C_1, and the defining properties of m and the relevant pullbacks.

For $k \in \{0, 1\}$ and $j = k + 1 \bmod 2$, we have:

$$
\begin{tikzcd}
C_3 \arrow[rr, "m_j"] \arrow[d, "m_k"'] \arrow[dr, "\pi_{3,j}"] & & C_2 \arrow[r, "m"] \arrow[d, "\pi_k"] & C_1 \arrow[ddd, "d_k"] \\
C_2 \arrow[ddd, "m"'] \arrow[dr, "\pi_k"'] & C_2 \arrow[r, "m"] \arrow[d, "\pi_k"] & C_1 \arrow[ddr, "d_k"] & \\
& C_1 \arrow[drr, "d_k"] & & \\
C_1 \arrow[rrr, "d_k"'] & & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[rr, "m_1"] \arrow[d, "m_0"'] \arrow[dr, "r_3"] & & C_2 \arrow[r, "m"] \arrow[d, "r_2"] & C_1 \arrow[ddd, "r_1"] \\
C_2 \arrow[dd, "m"'] \arrow[dr, "r_2"'] & Y_3 \arrow[r, "m_1"] \arrow[d, "m_0"] & Y_2 \arrow[ddr, "m"] & \\
& Y_2 \arrow[drr, "m"] & & \\
C_1 \arrow[rrr, "r_1"'] & & & Y_1
\end{tikzcd}
$$

We now consider the unit laws. We first note that the equations $r_2.i_k^{\mathbb{C}} = i_k^{\mathbb{Y}}.r_1$ for $k \in \{0, 1\}$ hold by the universal property of Y_2, as per the following calculations.

$$
\begin{tikzcd}
C_1 \arrow[rr, "i_k"] \arrow[dr, "d_k"] \arrow[dd, "r_1"'] & & C_2 \arrow[r, "r_2"] \arrow[d, "\pi_k"] & Y_2 \arrow[ddd, "\pi_k"] \\
& C_0 \arrow[r, "i"] \arrow[d, "r_0"] & C_1 \arrow[ddr, "r_1"] & \\
Y_1 \arrow[r, "d_k"'] \arrow[d, "i_k"'] & Y_0 \arrow[drr, "i"] & & \\
Y_2 \arrow[rrr, "\pi_k"'] & & & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "i_k"] \arrow[d, "r_1"'] \arrow[dr, "1_{C_1}"] & C_2 \arrow[r, "r_2"] \arrow[d, "\pi_j"] & Y_2 \arrow[dd, "\pi_j"] \\
Y_1 \arrow[d, "i_k"'] \arrow[drr, "1_{C_1}"] & C_1 \arrow[dr, "r_1"] & \\
Y_2 \arrow[rr, "\pi_j"'] & & Y_1
\end{tikzcd}
$$

The left and right unit laws for $\mathbb{C}$ hence follow from the universal property of C_1, given the calculations displayed below where $j = k + 1 \bmod 2$.

$$
\begin{tikzcd}
C_1 \arrow[rr, "i_k"] \arrow[dr, "d_k"] \arrow[dd, "1_{C_1}"'] & & C_2 \arrow[r, "m"] \arrow[d, "\pi_k"] & C_1 \arrow[dd, "d_k"] \\
& C_0 \arrow[r, "i"] \arrow[drr, "1_{C_0}"'] & C_1 \arrow[dr, "d_k"] & \\
C_1 \arrow[rrr, "d_k"'] & & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "i_k"] \arrow[dr, "r_1"] \arrow[dd, "1_{C_1}"'] & & C_2 \arrow[r, "m"] \arrow[d, "r_2"] & C_1 \arrow[dd, "r_1"] \\
& Y_1 \arrow[r, "i_k"] \arrow[drr, "1_{Y_1}"'] & Y_2 \arrow[dr, "m"] & \\
C_1 \arrow[rrr, "r_1"'] & & & Y_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_1 \arrow[r, "i_k"] \arrow[dr, "1_{C_1}"] \arrow[dd, "1_{C_1}"'] & C_2 \arrow[r, "m"] \arrow[d, "\pi_j"] & C_1 \arrow[dd, "d_j"] \\
& C_1 \arrow[dr, "d_j"] & \\
C_1 \arrow[rr, "d_j"'] & & C_0
\end{tikzcd}
$$

So $\mathbb{C}$ is a category internal to $\mathcal{E}$. It is clear from the construction of the identity assigner in (C.2) and composition in (C.3) that the morphism of graphs $r := (r_0, r_1) : \mathbb{C} \to \mathbb{Y}$ is well-defined as an internal functor, which is moreover evidently fully faithful and $\mathcal{R}$-on-objects as per (C.1).

It remains to show that the morphism of graphs $l := (l_0, l_1) : \mathbb{X} \to \mathbb{C}$ is well-defined as an internal functor. Once again, we do this using the universal property of C_1 as a pullback. Define $l_2 : X_2 \to C_2$ by the universal property of

C_2, as described in Remark 2.2.4. Fix $k \in \{0,1\}$ as above. Respect for identities for $l : \mathbb{X} \to \mathbb{C}$ is exhibited by the commutativity of the diagrams in $\mathcal{E}$ displayed below.

$$
\begin{tikzcd}
X_0 \arrow[r, "l_0"] \arrow[d, "i"'] \arrow[dr, "1_{X_0}"] & C_0 \arrow[r, "i"] \arrow[ddr, "1_{C_0}"] & C_1 \arrow[dd, "d_k"] \\
X_1 \arrow[r, "d_k"] \arrow[d, "l_1"'] & X_0 \arrow[dr, "l_0"'] & \\
C_1 \arrow[rr, "d_k"'] & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r, "l_0"] \arrow[d, "i"'] \arrow[dr, "f_0"] & C_0 \arrow[r, "i"] \arrow[d, "r_0"] & C_1 \arrow[dd, "r_1"] \\
X_1 \arrow[drr, "f_1"'] \arrow[d, "l_1"'] & Y_0 \arrow[dr, "i"] & \\
C_1 \arrow[rr, "r_1"'] & & Y_1
\end{tikzcd}
$$

Finally, respect for composition for $l : \mathbb{X} \to \mathbb{C}$ follows from the commutativity of the diagrams in $\mathcal{E}$ displayed below. This completes the proof.

$$
\begin{tikzcd}
X_2 \arrow[rr, "l_2"] \arrow[dr, "\pi_k"] \arrow[dd, "m"'] & & C_2 \arrow[r, "m"] \arrow[d, "\pi_k"] & C_1 \arrow[ddd, "d_k"] \\
& X_1 \arrow[r, "l_1"'] \arrow[d, "d_k"] & C_1 \arrow[ddr, "d_k"] & \\
X_1 \arrow[r, "d_k"] \arrow[d, "l_1"'] & X_0 \arrow[drr, "l_0"] & & \\
C_1 \arrow[rrr, "d_k"'] & & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "l_2"] \arrow[d, "m"'] \arrow[dr, "f_2"] & C_2 \arrow[r, "m"] \arrow[d, "r_2"] & C_1 \arrow[dd, "r_1"] \\
X_1 \arrow[drr, "f_1"] \arrow[d, "l_1"'] & Y_2 \arrow[dr, "m"] & \\
C_1 \arrow[rr, "r_1"'] & & Y_1
\end{tikzcd}
$$

□