The elementary theory of the 2-category of small categories

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Lawvere's Elementary Theory of the Category of Sets (ETCS) $\rightsquigarrow$ sets, functions, composition.
The Elementary Theory of the 2-Category of Small Categories (ET2CSC) $\rightsquigarrow$ categories, functors, natural transformations, composition, whiskering, vertical composition.

ETCS

$\mathcal{E} \vDash$ ETCS if:

- (1) ☆ finite limits
- ♣ cartesian closed
- (2) subobject classifier
- (3) well-pointed
- (4) natural numbers object
- (5) axiom of choice

Internal categories

Let $\mathcal{E}$ be a category with pullbacks. A *category internal to* $\mathcal{E}$ is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \underset{p_2}{\overset{p_1,\ m}{\rightrightarrows}} C_1 \underset{d_0}{\overset{d_1,\ i}{\rightleftarrows}} C_0$$

where $C_0, C_1 \in \mathcal{E}$.
$\rightsquigarrow$ 2-category $\mathbf{Cat}(\mathcal{E})$.

A 2-category $\mathcal{K} \vDash$ ET2CSC if:

- (1) Bourke's axioms
- ☆ terminal object
- ♣ cartesian closed
- (2) full subobject classifier
- (3) 2-well-pointed
- (4) 2-natural numbers object
- (5) categorified axiom of choice

(1) Bourke's Theorem:

If $\mathcal{E}$ is a category with pullbacks then the 2-category $\mathcal{K} := \mathbf{Cat}(\mathcal{E})$ satisfies Bourke's exactness axioms. Conversely, if $\mathcal{K}$ satisfies Bourke's exactness axioms, then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{Cat}\,(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}\,(\mathcal{K})$.

This characterises 2-categories $\mathcal{K}$ which are of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$.

(2) Full subobject classifier:

a full monomorphism $\underline{\top} : \underline{\mathbf{1}} \to \underline{\Omega}$ such that:

$$\begin{array}{ccc} A & \xrightarrow{\ !\ } & \underline{\mathbf{1}} \\ {\scriptstyle \forall i \in \mathbf{FullMono}}\downarrow & \lrcorner & \downarrow{\scriptstyle \underline{\top}} \\ B & \xrightarrow[\exists! \chi_i]{} & \underline{\Omega}. \end{array}$$

(3) 2-well-pointedness:

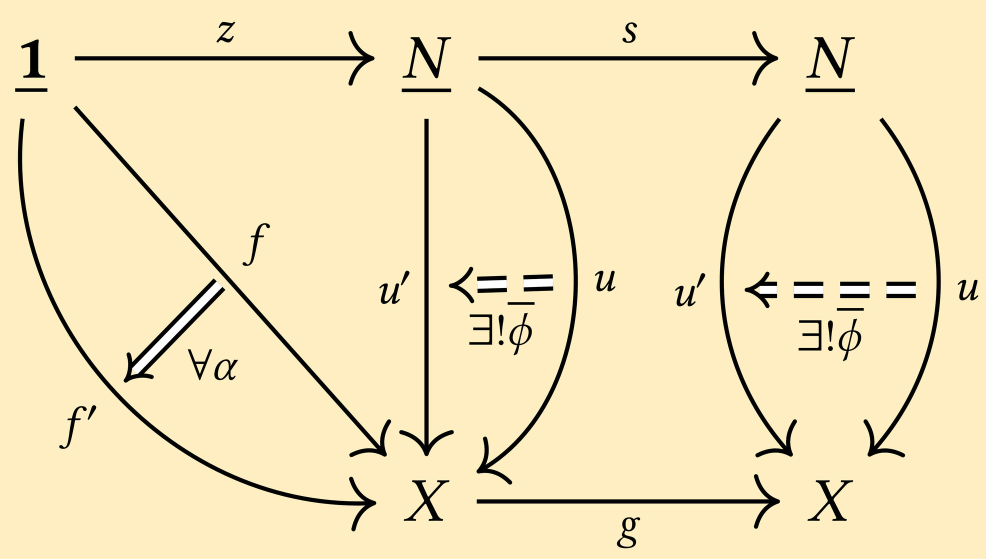
1. $\mathcal{K}$ has a terminal object $\underline{\mathbf{1}}$.
2. The copower $\mathbf{2} \odot \underline{\mathbf{1}}$ exists in $\mathcal{K}$.
3. The family containing just $\mathbf{2} \odot \underline{\mathbf{1}}$ is a generator for the 2-category $\mathcal{K}$.

$$\mathbf{2} \xrightarrow{\forall f} \mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$$

(4) 2-natural numbers object:

$$\underline{\mathbf{1}} \xrightarrow{z} \underline{N} \xrightarrow{s} \underline{N}$$

that is a natural numbers object for the underlying 1-category of $\mathcal{K}$ and:



(5) The categorified axiom of choice:

Any acute fully faithful morphism has a section.

$$\mathbf{Acute} := {}^{\pitchfork}(\mathbf{FullMono})$$

In $\mathbf{Cat}(\mathcal{E})$, these are epi-on-objects.

Main Theorem

1. Let $\mathcal{E}$ be a category. Then $\mathcal{E} \vDash$ ETCS if and only if $\mathbf{Cat}(\mathcal{E}) \vDash$ ET2CSC, and in this case $\mathcal{E} \simeq \mathbf{Disc}(\mathbf{Cat}(\mathcal{E}))$.
2. Let $\mathcal{K}$ be a 2-category. Then $\mathcal{K} \vDash$ ET2CSC if and only if $\mathbf{Disc}\,(\mathcal{K}) \vDash$ ETCS, and in this case $\mathcal{K} \simeq \mathbf{Cat}(\mathbf{Disc}\,(\mathcal{K}))$.
3. This extends to a biequivalence

$$\mathbf{ETCS} \underset{\mathbf{Cat}(-)}{\overset{\mathbf{Disc}(-)}{\rightleftarrows}} \mathbf{ET2CSC} \quad (\sim)$$

2-Categories of Categories

By considering a 2-category $\mathcal{K}$ equipped with a discrete opfibration classifier $p : S_* \to S$, we define the notion of a *2-category of categories*. This models the situation:

$$\mathbf{Cat} \hookrightarrow \mathbf{CAT}$$

adding the axiom of replacement.

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